Jacobians

If u and v are the functions of two independent variables x and y then

$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of $u\,,v$ with respect to x and y. it is denoted by

$\frac{\partial(u,v)}{\partial(x,y)} = J$

Note:

1. $J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

2. The Jacobian of $u\,,v\,,w$ with respect to $x\,,y\,,z$ is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties of Jacobian:

Property:1

If u and v are the functions of x and y then $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

Proof:

Let $J = \frac{\partial(u,v)}{\partial(x,y)}$, $J' = \frac{\partial(x,y)}{\partial(u,v)}$

$$J\,J' = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix} \; \ldots (1)$$

$$\frac{\partial u}{\partial u} = 1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial u}{\partial v} = 0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial v} = 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v}$$

$$\frac{\partial v}{\partial u} = 0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u}$$

$$(1) \Rightarrow \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\therefore \ JJ' = 1$

Example:

If $x = u(1-v)$, $y = uv$ find J and J' and prove that $JJ' = 1$

Solution:

To prove $JJ' = 1$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \ldots (1)$$

Given $x = u(1-v)$, $y = uv$

$$\frac{\partial x}{\partial u} = 1 - v \ , \frac{\partial y}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = -u \quad , \quad \frac{\partial x}{\partial u} = u$$

$$(1) \Rightarrow \quad J = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u - uv + uv = u$$

$$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \ldots (2)$$

Given $x = u - uv$, $y = uv$

$$x = u - y, \ v = \frac{y}{u}$$

$$u = x + y, \ v = \frac{y}{x+y}$$

$$\frac{\partial u}{\partial x} = 1 \ , \frac{\partial v}{\partial x} = y\left(-\frac{1}{(x+y)^2}\right) \times 1 = -\frac{y}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = 1 \ , \frac{\partial v}{\partial y} = \frac{(x+y)\times 1 - y \times 1}{(x+y)^2} = \frac{x}{(x+y)^2}$$

$$(2) \Rightarrow \quad J' = \begin{vmatrix} 1 & 1 \\ -\frac{y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{1}{x+y}$$

$$= \frac{1}{u}$$

here $J = u$ and $J' = \frac{1}{u}$

$$\therefore \ JJ' = u \times \frac{1}{u} = 1$$

Hence proved.

Example:

If $x = uv, y = \frac{u}{v}$ prove $\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = 1$

Solution:

Let $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$ ... (1)

Given $x = uv, y = \frac{u}{v}$

$\frac{\partial x}{\partial u} = v, \frac{\partial y}{\partial u} = \frac{1}{v}$

$\frac{\partial x}{\partial v} = u \quad , \quad \frac{\partial x}{\partial u} = \frac{-u}{v^2}$

$(1) \Rightarrow \quad J = \begin{vmatrix} v & u \\ \frac{1}{v} & \frac{-u}{v^2} \end{vmatrix}$

$= \frac{-uv}{v^2} - \frac{u}{v} = \frac{-2u}{v}$

$J' = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ ... (2)

$x = uv, \; y = \frac{u}{v} \quad \Rightarrow \quad u = vy$

$x = vyv \;\Rightarrow\; x = v^2y \;\Rightarrow v^2 = \frac{x}{y} \;\Rightarrow v = \frac{\sqrt{x}}{\sqrt{y}}$

$x = uv \;\Rightarrow\; u = \frac{x}{v} \;\Rightarrow\; u = \frac{x\sqrt{y}}{\sqrt{x}} = \sqrt{x}\sqrt{y}$

$Now \Rightarrow \quad u = \sqrt{x}\sqrt{y}, \; v = \frac{\sqrt{x}}{\sqrt{y}}$

$\frac{\partial u}{\partial x} = \frac{\sqrt{y}}{2\sqrt{x}}, \frac{\partial v}{\partial x} = \frac{1}{\sqrt{y}} \frac{1}{2\sqrt{x}}$

$\frac{\partial u}{\partial y} = \frac{\sqrt{x}}{2\sqrt{y}}, \frac{\partial v}{\partial y} = \sqrt{x}\left(\frac{-1}{y}\right)\frac{1}{2\sqrt{y}}$

$(2) \Rightarrow J' = \begin{vmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ \frac{1}{2\sqrt{y}\sqrt{x}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix}$

$= -\frac{1}{4y} - \frac{1}{4y} \quad = -\frac{2}{4y} \quad = -\frac{1}{2y}$

$= -\frac{v}{2u} \qquad\qquad (\because y = \frac{u}{v})$

here $J = \frac{-2u}{v}$ and $J' = -\frac{v}{2u}$

$\therefore \; JJ' = \frac{-2u}{v} \times (-\frac{v}{2u}) = 1$

Hence proved.

Property: 2

If u and v are the functions of r , s where r , s are the functions of x and y then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$

Proof:

$$\frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \times \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} & \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial(u,v)}{\partial(x,y)}$$

Similarly $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial(u,v,w)}{\partial(r,s,t)} \times \frac{\partial(r,s,t)}{\partial(x,y,z)}$

Example:

If $u = 2xy$, $v = x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$. Evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ without actual substitution

Solution:

By property: 2

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \quad \ldots (1)$$

Given $u = 2xy$, $v = x^2 - y^2$, $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial u}{\partial x} = 2y , \frac{\partial v}{\partial x} = 2x , \quad \frac{\partial x}{\partial r} = \cos\theta , \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial u}{\partial y} = 2x , \frac{\partial v}{\partial y} = -2y , \quad \frac{\partial x}{\partial \theta} = -r\sin\theta , \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$(1) \Rightarrow \frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-4y^2 - x^2)(r\cos^2\theta + r\sin^2\theta)$$

$$= -4(x^2 + y^2) \times r \qquad \left(\because \begin{matrix} x^2 = r^2\cos^2\theta \\ y^2 = r^2\sin^2\theta \end{matrix}\right)$$

$$= -4r^2 \times r$$

$$= -4r^3$$

Example:

If $x = a(u+v)$, $y = b(u-v)$ and $u = r^2cos2\theta$, $v = r^2sin2\theta$ then evaluate $\frac{\partial(x,y)}{\partial(r,\theta)}$

Solution:

By Property: 2

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \times \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix} \dots (1)$$

Given $x = a(u+v)$, $y = b(u-v)$, $u = r^2cos2\theta$, $v = r^2sin2\theta$

$$\frac{\partial x}{\partial u} = a \ , \ \frac{\partial y}{\partial u} = b \ , \quad \frac{\partial u}{\partial r} = 2rcos2\theta \ , \frac{\partial v}{\partial r} = 2rsin\theta$$

$$\frac{\partial x}{\partial v} = a \ , \ \frac{\partial y}{\partial v} = -b \ , \quad \frac{\partial u}{\partial \theta} = -2r^2sin2\theta \ , \ \frac{\partial v}{\partial \theta} = 2r^2cos2\theta$$

$$(1) \Rightarrow \ = \begin{vmatrix} a & a \\ b & -b \end{vmatrix} \times \begin{vmatrix} 2rcos2\theta & -2r^2sin2\theta \\ 2rsin2\theta & 2r^2cos\theta \end{vmatrix}$$

$$= (-ab - ab)(4r^3cos^2 2\theta + 4r^3sin^2 2\theta)$$

$$= -2ab \times 4r^3$$

$$= -8r^3ab$$

Property: 3

***If* u , v , w are functionally dependent of three independent variables x , y , z then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$**

Proof:

As u , v , w are not independent then $f(u,v,w) = 0 \dots (1)$

Differentiating equation (1) with respect to x , y , z we get

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = 0 \dots (2)$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y} = 0 \dots (3)$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z} = 0 \dots (4)$$

Eliminating f derivatives from (2) ,(3) and (4) we have

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

On interchanging rows and columns, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = 0$$

i.e) $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

Example:

If $p = 3x + 2y - z$, $q = x - 2y = z$, $r = x + 2y - z$ prove that p , q , r are functionally dependent.

Solution:

To prove p , q , r are functionally dependent.

i.e) To prove $\frac{\partial(p,q,r)}{\partial(x,y,z)} = 0$

Consider $\frac{\partial(p,q,r)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \\ \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

= 3(2-2) – 2(-1 -1) -1(2+2)

= 4 – 4

= 0

$\therefore p , q , r$ are functionally dependent.

Example:

If $x + y + z = u$, $y + z = uv$, $z = uvw$ prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

Solution:

Given $x + y + z = u$, $y + z = uv$, $z = uvw$

$x + uv = u$, $y + uvw = uv$, $z = uvw$

$x = u - uv$, $y = uv - uvw$, $z = uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial z} \end{vmatrix} \dots (1)$$

$x = u - uv , \; y = uv - uvw , \; z = uvw$

$\frac{\partial x}{\partial u} = 1 - v \, , \; \frac{\partial y}{\partial u} = v - vw \, , \; \frac{\partial z}{\partial u} = vw$

$\frac{\partial x}{\partial v} = -u \, , \; \frac{\partial y}{\partial v} = u - uw \, , \; \frac{\partial z}{\partial v} = uw$

$\frac{\partial x}{\partial w} = 0 \, , \; \frac{\partial y}{\partial w} = -uv, \; \frac{\partial z}{\partial w} = uv$

$$(1) \Rightarrow \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v - vw & u - uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)(u^2v - u^2vw + u^2vw) + u(uv^2 - uv^2w + uv^2w) +$$

$$0(uvw - uvw^2 - uvw + uvw^2$$

$$= (1-v)u^2v + u^2v^2$$

$$= u^2v - u^2v^2 + u^2v^2$$

$$= u^2v$$

Hence proved.

Example:

Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}$ of the transformation $x = rsin\theta cos\varphi$, $y = rsin\theta sin\varphi$, $z = \mathrm{r}\cos\theta$

Solution:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} \dots (1)$$

$x = rsin\theta cos\varphi \, , y = rsin\theta sin\varphi \, , z = rcos\theta$

$\frac{\partial x}{\partial r} = \sin\theta cos\varphi \, , \; \frac{\partial y}{\partial r} = \sin\theta sin\varphi \, , \; \frac{\partial z}{\partial r} = \cos\theta$

$\frac{\partial x}{\partial \theta} = r\cos\theta cos\varphi \, , \; \frac{\partial y}{\partial \theta} = r\cos\theta sin\varphi \, , \; \frac{\partial z}{\partial \theta} = -\mathrm{r}\sin\theta$

$\frac{\partial x}{\partial \varphi} = -r\sin\theta sin\varphi \, , \; \frac{\partial y}{\partial \varphi} = r\sin\theta cos\varphi, \; \frac{\partial z}{\partial \varphi} = 0$

$$(1) \Rightarrow \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{vmatrix} \sin\theta cos\varphi & r\cos\theta cos\varphi & -r\sin\theta sin\varphi \\ \sin\theta sin\varphi & r\cos\theta sin\varphi & r\sin\theta cos\varphi \\ \cos\theta & -\mathrm{r}\sin\theta & 0 \end{vmatrix}$$

$$= sin\theta cos\varphi(0 + r^2 sin^2\theta cos\varphi) -$$
$$rcos\theta cos\varphi(-rsin\theta cos\theta cos\varphi) -$$
$$r(sin\theta sin\varphi(-rsin^2\theta sin\varphi - rcos^2\theta sin\varphi)$$
$$= r^2 sin^3\theta cos^2\varphi + r^2 cos^2\theta cos^2\varphi sin\theta + r^2 sin^3\theta sin^2\varphi +$$
$$r^2 cos^2\theta sin^2\varphi sin\theta$$
$$= r^2 sin^3\theta + r^2 cos^2\theta sin\theta$$
$$= r^2 sin\theta(sin^2\theta + cos^2\theta)$$
$$= r^2 sin\theta$$

Example:

If $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$ **show that** $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

Solution:

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \dots (1)$$

$$u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}$$

$$\frac{\partial u}{\partial x} = -\frac{yz}{x^2}, \quad \frac{\partial v}{\partial x} = \frac{z}{y}, \quad \frac{\partial w}{\partial x} = \frac{y}{z}$$

$$\frac{\partial u}{\partial y} = \frac{z}{x}, \quad \frac{\partial v}{\partial y} = -\frac{zx}{y^2}, \quad \frac{\partial w}{\partial y} = \frac{x}{z}$$

$$\frac{\partial u}{\partial z} = \frac{y}{z}, \quad \frac{\partial v}{\partial z} = \frac{x}{y}, \quad \frac{\partial w}{\partial z} = -\frac{xy}{z^2}$$

$$(1) \Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= -\frac{yz}{x^2}\left[\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[-\frac{xyz}{yz^2} - \frac{xy}{yz}\right] + \frac{y}{x}\left[\frac{zx}{yz} + \frac{zxy}{zy^2}\right]$$

$$= -\frac{x^2y^2z^2}{x^2y^2z^2} + \frac{yzx^2}{x^2yz} + \frac{xyz^2}{xyz^2} + \frac{xyz}{xyz} + \frac{yzx}{xyz} + \frac{zxy^2}{xzy^2}$$

$$= -1 + 1 + 1 + 1 + 1 + 1 = 4$$

Hence proved.

Example:

If $x = acosh\varphi cos\theta, y = asinh\varphi sin\theta$ **show that** $\frac{\partial(x,y)}{\partial(\varphi,\theta)} = \frac{a^2}{2}(cosh2\varphi - cos2\theta)$

Solution:

$$\frac{\partial(x,y)}{\partial(\varphi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \theta} \end{vmatrix} \ldots (1)$$

Given $x = acosh\varphi cos\theta \, , y = asinh\varphi sin\theta$

$$\frac{\partial x}{\partial \varphi} = a\cos\theta sinh\varphi \, , \quad \frac{\partial y}{\partial \varphi} = a\sin\theta cosh\varphi$$

$$\frac{\partial x}{\partial \theta} = -a\sin\theta cosh\varphi \, , \quad \frac{\partial y}{\partial \theta} = a\cos\theta sinh\varphi$$

$$(1) \Rightarrow \frac{\partial(x,y)}{\partial(\varphi,\theta)} = \begin{vmatrix} a\cos\theta sinh\varphi & -a\sin\theta cosh\varphi \\ a\sin\theta cosh\varphi & a\cos\theta sinh\varphi \end{vmatrix}$$

$$= a^2 sinh^2\varphi cos^2\theta + a^2 cosh^2\varphi sin^2\theta$$

$$= a^2[sinh^2\varphi(1 - sin^2\theta) + (1 + sinh^2\varphi)sin^2\theta]$$

$$= a^2[sinh^2\varphi - sinh^2\varphi sin^2\theta + sin^2\theta + sinh^2\varphi sin^2\theta]$$

$$= a^2[sinh^2\varphi + sin^2\theta]$$

$$= a^2[\frac{cosh2\varphi - 1}{2} + \frac{1 - cos2\theta}{2}]$$

$$= a^2[\frac{cosh2\varphi - 1 + 1 - cos2\theta}{2}]$$

$$= \frac{a^2}{2}(\cos 2\varphi - cos2\theta)$$

Hence proved.

Jacobian of Implicit Functions

If $f_1(u, v, w, x, y, z) = 0$

$f_2(u, v, w, x, y, z) = 0$

and $f_3(u, v, w, x, y, z) = 0$ are three implicit functions, we can consider u, v, w as implicit function x, y, z then it can be proved

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

Also $\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}$

Example:

If $f_1 = u - x - y - z = 0$, $f_2 = uv - y - z = 0$, $f_3 = uvw - z = 0$, prove that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

Solution:

Given $f_1 = u - x - y - z$, $f_2 = uv - y - z$, $f_3 = uvw - z$

$$\frac{\partial f_1}{\partial x} = -1 \, , \frac{\partial f_1}{\partial u} = 1 \, , \quad \frac{\partial f_2}{\partial x} = 0 \, , \frac{\partial f_2}{\partial u} = v \, , \quad \frac{\partial f_3}{\partial x} = 0 \, , \frac{\partial f_3}{\partial u} = vw$$

$$\frac{\partial f_1}{\partial y} = -1 \,, \frac{\partial f_1}{\partial v} = 0 \,, \frac{\partial f_2}{\partial y} = -1 \,, \frac{\partial f_2}{\partial v} = u \,, \quad \frac{\partial f_3}{\partial y} = 0 \,, \frac{\partial f_3}{\partial v} = uw$$

$$\frac{\partial f_1}{\partial z} = -1 \,, \frac{\partial f_1}{\partial w} = 0 \,, \frac{\partial f_2}{\partial z} = -1 \,, \frac{\partial f_2}{\partial w} = 0 \,, \quad \frac{\partial f_3}{\partial z} = -1 \,, \frac{\partial f_3}{\partial w} = uv$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}$$

$$= -\frac{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial 3}{\partial w} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & 0 & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix}}{\begin{vmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{vmatrix}} = -\frac{u^2 v}{-1 \times 1}$$

$$= u^2 v$$

Hence proved.