2.5. CONVOLUTION

The convolution property of the Z Transform makes it convenient to obtain the Z Transform for the convolution of two sequences as the product of their respective Z Transforms, then the Z Transform of the convolution of the two sequences x1(n) and x2(n) is the product of their corresponding Z transforms.

Consider a LTI system characterized with the impulse response function h[n]

Given an input signal x[n] the output of the system equals y[n] =x[n] ∗h[n]

$$y(n)=\sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

The Z transform of the output is defined as:

$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n}$

$=\sum_{n=-\infty}^{\infty}(\sum_{k=-\infty}^{\infty} x[n-k]h[k])z^{-n}$

$=\sum_{k=-\infty}^{\infty} h[k]\sum_{n=-\infty}^{\infty} x[n-k]z^{-n}$

$= X(z)\sum_{k=-\infty}^{\infty} h[k]z^{-k}$

$= X(z)\,H(z)$

Therefore:

$$x(n)*h(n) \xrightarrow{z} X(z)\,H(z)$$

Example-1

Find the inverse z-transform of $\frac{z^2}{(z-1)(2z-1)}$ using convolution theorem.

Let $U(z)=Z\{u_n\}=\frac{z}{z-1}$ and $V(z)=Z\{v_n\}=\frac{z}{(2z-1)} = \frac{1}{2}(\frac{z}{z-\frac{1}{2}})$

Clearly $u_n = (1)^n$ and $u_n = \frac{1}{2}(\frac{1}{z})^n$

$Z^{-1}[\frac{z}{z-a}] = a^n$

Now by convolution theorem $z^{-1}[U(z).V(z)] = u_n * v_n$

$=> z^{-1}[\frac{z^2}{(z-1)(2z-1)}] = (1)^n * (\frac{1}{2})^{n+1}$

We know that $u_n * v_n = \sum_{m=0}^{n} u_m\, v_{n-m}$

$= \sum_{m=0}^{n}(1)^m\,(\frac{1}{2})^{n+1-m}$

$=(\frac{1}{2})^{n+1}+(\frac{1}{2})^{n}+(\frac{1}{2})^{n-1}+\ldots..+(\frac{1}{2})$

$=\frac{1}{2}[1+\frac{1}{2}+(\frac{1}{2})^2+\ldots\ldots+(\frac{1}{2})^n$

$S_n = \frac{a}{1-r}(1-r^n)$

$= \frac{1}{2}[2(1-\left(\frac{1}{2}\right)^{n+1})]$

$= 1- \left(\frac{1}{2}\right)^{n+1}$