SINGULARITIES

Zeros of an analytic function

If a function $f(z)$ is analytic in a region R, is zero at a point $z = z_0$ in R, then z_0 is called a zero of $f(z)$.

Simple zero

If $f(z_0) = 0$ and $f'(z_0) \neq 0$, then $z = z_0$ is called a simple zero of $f(z)$ or a zero of the first order.

Zero of order n

If $f(z_0) = f'(z_0) = \cdots = f^{n-1}(z_0) = 0$ and $f^n(z_0) \neq 0$, then z_0 is called zero of order.

Example: Find the zeros of $f(z) = \frac{z^2+1}{1-z^2}$

Solution:

The zeros of $f(z)$are given by $f(z) = 0$

$$(i.e.) f(z) = \frac{z^2+1}{1-z^2} = \frac{(z+i)(z-i)}{1-z^2} = 0$$

$$\Rightarrow (z+i)(z-i) = 0$$

$\Rightarrow z = i$ and $-i$ are simple zero.

Example: Find the zeros of $f(z) = \sin\frac{1}{z-a}$

Solution:

The zeros are given by $f(z) = 0$

$$(i.e.) \sin\frac{1}{z-a} = 0$$

$$\Rightarrow \frac{1}{z-a} = n\pi, n = \pm1, \pm2, \ldots$$

$$\Rightarrow (z-a)n\pi = 1$$

$\therefore$ The zeros are $z = a + \frac{1}{n\pi}, n = \pm1, \pm2, ,,$

Example: Find the zeros of $f(z) = \frac{\sin z - z}{z^3}$

Solution:

The zeros are given by $f(z) = 0$

$$(i.e.) \frac{\sin z - z}{z^3} = 0$$

$$\Rightarrow \frac{\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots\right]}{z^3} - z = 0$$

$$\Rightarrow \frac{-\frac{z^3}{3!}+\frac{z^5}{5!}}{z^3} \dots . = 0$$

$$\Rightarrow -\frac{1}{3!}+\frac{z^2}{5!} \dots = 0$$

But $\lim_{z\to 0}\frac{\sin z - z}{z^3} = -\frac{1}{3!} + 0$

$\therefore f(z)$ has no zeros.

Singular points

A point $z = z_0$ at which a function $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$.

Example: Consider $f(z) = \frac{1}{z-5}$

Here, $z = 5$, is a singular point of $f(z)$

Types of singularity

A point $z = z_0$ is said to be isolated singularity of $f(z)$ if

(i) $f(z)$ is not analytic at $z = z_0$

(ii) There exists a neighbourhood of $z = z_0$ containing no other singularity

Example: $f(z) = \frac{z}{z^2-1}$

This function is analytic everywhere except at $z = 1, -1$

$\therefore z = 1, -1$ are two isolated singular points.

When $z = z_0$ is an isolated singular point of $f(z)$, it can expand $f(z)$ as a Laurent's series about $z = z_0$

Thus

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} b_n (z - z_0)^{-n}$$

Note: If $z = z_0$ is an isolated singular point of a function $f(z)$, then the singularity is called

(i) a removable singularity (or)

(ii) a pole (or)

(iii) an essential singularity

According as the Laurent's series about $z = z_0$ of $f(z)$ has

(i) no negative powers (or)

(ii) a finite number of negative powers (or)

(iii) an infinite number of negative powers

Removable singularity

If the principal part of $f(z)$ in Laurent's series expansion contains no term $(i.e.)b_n = 0$ for all n, then the singularity $z = z_0$ is known as the removable singularity of $f(z)$

$$\therefore f(z) = \sum_{n=0}^{\infty} a_n (z - z_o)^n$$

(OR)

A singular point $z = z_0$ is called a removable singularity of $f(z)$, if $\lim_{z \to z_0} f(z)$ ezists finitely

Example: $f(z) = \frac{\sin z}{z}$

$$\frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!}$$

There is no negative powers of z.

$\therefore z = 0$ is a removable singularity of $f(z)$.

Poles

If we can find the positive integer n such that $\lim_{z \to z_0} (z - z_0)^n f(z) \neq 0$, then $z = z_0$ is called a pole of order n for $f(z)$.

(or)

If $\lim_{z \to z_0} f(z) = \infty, then\ z = z_0$ is a pole of $f(z)$

Simple pole

A pole of order one is called a simple pole.

Example: $f(z) = \frac{1}{(z-1)^2(z+2)}$

Here $z = 1$ is a pole of order 2

$z = 2$ is a pole of order 1.

Essential singularity

If the principal part of $f(z)$ in Laurent's series expansion contains an infinite number of non zero terms, then $z = z_0$ is known as an essential singularity.

Example: $f(z) = e^{1/z} = 1 + \frac{\frac{1}{z}}{1!} + \frac{\left(\frac{1}{z}\right)^2}{\mathbf{2}!} + \cdots$ has $z = 0$ as an essential singularity since, $f(z)$ is an infinite series of negative powers of z.

$f(z) = e^{\frac{1}{z}-4}$ has $z = 4$ an essential singularity

Note: The removable singularity and the poles are isolated singularities. But, the essential singularity is either an isolated or non-isolated singularity.

Entire function (or) Integral function

A function $f(z)$ which is analytic everywhere in the finite plane (except at infinity) is called an entire function or an integral function.

Example: $e^z, \sin z, \cos z$ are all entire functions.

Example: What is the nature of the singularity $z = 0$ of the function $f(z) = \frac{sinz - z}{z^3}$

Solution:

Given $f(z) = \frac{sinz - z}{z^3}$

The function $f(z)$is not defined at $z = 0$

By L' Hospital's rule.

$$\lim_{z \to 0} \frac{sinz - z}{z^3} = \lim_{z \to 0} \frac{cosz - 1}{3z^2}$$

$$= \lim_{z \to 0} \frac{-sinz}{6z}$$

$$= \lim_{z \to 0} -\frac{cosz}{6z} = \frac{-1}{6}$$

Since, the limit exists and is finite, the singularity at $z = 0$ is a removable singularity.

Example: Classify the singularities for the function$f(z) = \frac{z - sinz}{z}$

Solution:

Given $f(z) = \frac{z - sinz}{z}$

The function $f(z)$is not defined at $z = 0$

But by L' Hospital's rule.

$$\lim_{z \to 0} \frac{z - sinz}{z} = \lim_{z \to 0} 1 - cosz \ = 1 - 1 = 0$$

Since, the limit exists and is finite, the singularity at $z = 0$ is a removable singularity.

Example: Find the singularity of$f(z) = \frac{e^{1/z}}{(z-a)^2}$

Solution:

Given $f(z) = \frac{e^{1/z}}{(z-a)^2}$

Poles of $f(z)$ are obtained by equating the denominator to zero.

$$(i.e.)(z - a)^2 = 0$$

$\Rightarrow z = a$ is a pole of order 2.

Now, Zeros of $f(z)$

$$\lim_{z \to 0} \frac{e^{1/z}}{(z-a)^2} = \frac{\infty}{a^2} = \infty \neq 0$$

$\Rightarrow z = 0$is a removable singularity.

$\therefore\ f(z)$ has no zeros.

Example: Find the kind of singularity of the function $f(z) = \frac{cot\pi z}{(z-a)^2}$

Solution:

$$\text{Given } f(z) = \frac{cot\pi z}{(z-a)^2}$$

$$= \frac{\cos\pi z}{sin\pi z(z-a)^2}$$

Singular points are poles, are given by

$$\Rightarrow sin\pi z(z-a)^2 = 0$$

$$(i.e.)sin\pi z = 0, (z-a)^2 = 0$$

$\pi z = n\pi$, where $n = 0, \pm1, \pm2, \ldots$

$$(i.e.)z = n$$

$z = a$ is a pole of order 2

Since $z = n, n = 0, \pm1, \pm2, \ldots$

$z = \infty$ is a limit of these poles.

$\therefore z = \infty$ is non- isolated singularity.

Example: Find the singular point of the function $f(z) = sinz\frac{1}{z-a}$. State nature of singularity.

Solution:

$$\text{Given } f(z) = sinz\frac{1}{z-a}$$

$z = a$ is the only singular point in the finite plane.

$$sinz\frac{1}{z-a} = \frac{1}{z-a} - \frac{1}{3!(z-a)^3} + \frac{1}{5!(z-a)^5} - \cdots$$

$z = a$ is an essential singularity

It is an isolated singularity.

Example: Identify the type of singularity of the function $f(z) = sin\left(\frac{1}{1-z}\right)$.

Solution:

$z = 1$ is the only singular point in the finite plane.

$z = 1$ is an essential singularity

It is an isolated singularity.

Example: Find the singular points of the function $f(z) = \left(\frac{1}{sin\frac{1}{z-a}}\right)$, state their nature.

Solution:

$f(z)$has an infinite number of poles which are given by

$$\frac{1}{z-a} = n\pi, n = \pm 1, \pm 2, \ldots$$

$(i.e.)z - a = \frac{1}{n\pi}\ ; z = a + \frac{1}{n\pi}$

But $z = a$ is also a singular point.

It is an essential singularity.

It is a limit point of the poles.

So, It is an non - isolated singularity.

Example: Classify the singularity of $f(z) = \frac{tanz}{z}$.

Solution:

Given $f(z) = \frac{tanz}{z}$

$$= \frac{z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots}{z}$$

$$= 1 + \frac{z^2}{3} + \frac{2z^4}{15} + \ldots$$

$$\lim_{z\to 0}\frac{tanz}{z} = 1 \neq 0$$

$\Rightarrow$ $z = 0$is a removable singularity of $f(z)$.