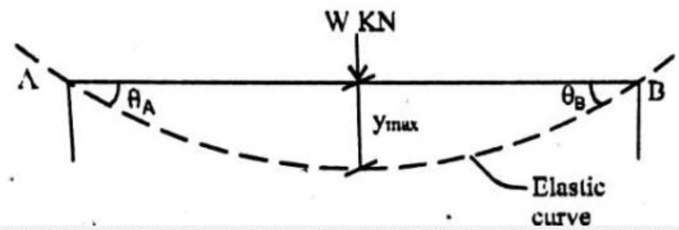
UNIT IV

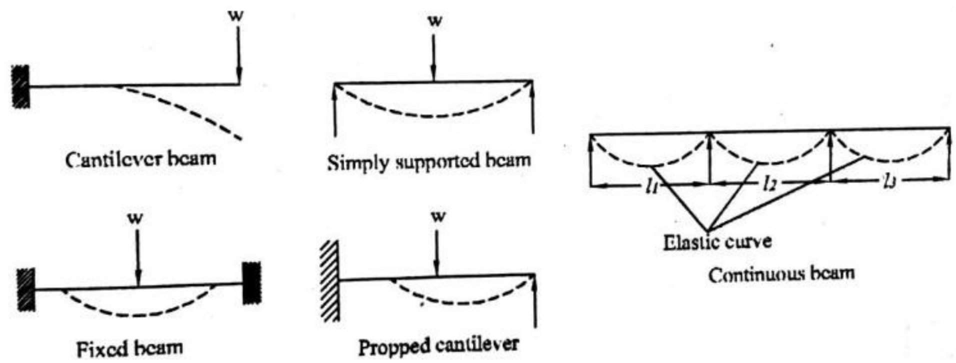
DEFLECTION OF BEAMS

4.1. ELASTIC CURVE OR DEFLECTED SHAPE

The curved shape of the longitudinal centroidal surface of a beam due to transverse loads is known as Elastic curve.



4.2. DEFLECTED SHAPES (or) ELASTIC CURVES OF BEAMS WITH DIFFERENT SUPPORT CONDITIONS



4.3. SLOPE

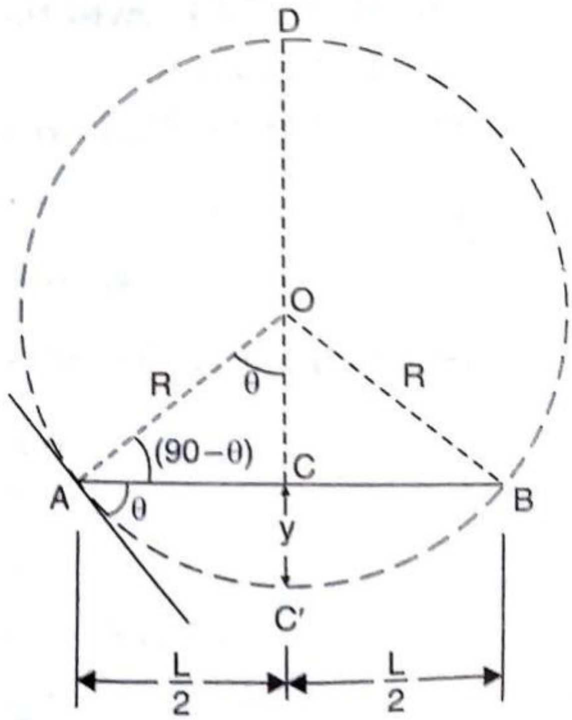
Slope is the angle formed by the tangent drawn at the Elastic curve to the original axis of the beam

4.4. DEFLECTION

Deflection is the translational movement of the beam from its original position.

4.5. DEFLECTION AND SLOPE OF A BEAM SUBJECTED TO UNIFORM BENDING MOMENT

A beam AB of length L is subjected to a uniform bending moment M as shown in Fig. As the beam is subjected to a constant bending moment, hence it will bend into a circular arc. The initial position of the beam is shown by ACB, whereas the deflected position is shown by AC'B.



Let R = Radius of curvature of the deflected beam,

y = Deflection of the beam at the centre (i.e., distance CC')

I = Moment of inertia of the beam section,

E = Young's modulus for the beam material, and

θ = Slope of the beam at the end A (i.e., the angle made by the tangent at A with the beam AB). For a practical beam the deflection y is a small Quantity.

Hence $\tan\theta = \theta$ where θ is in radians. Here θ becomes the slope

$$\frac{dy}{dx} = \tan\theta = \theta .$$

Now, $AC = BC = \frac{L}{2}$

Also from the geometry of a circle, we know that

$$AC \times CB = DC \times CC'$$

$$\frac{L}{2} \times \frac{L}{2} = (2R-y) \times y$$

$$\frac{L^2}{4} = 2Ry - y^2$$

For a practical beam, the deflection y is a small quantity. Hence the square of a small quantity will be negligible. Hence neglecting y^2 in the above equation, we get

$$\frac{L^2}{4} = 2Ry$$

$$\therefore y = \frac{L^2}{8R} \qquad \ldots(i)$$

But from bending equation, we have

$$\frac{M}{I} = \frac{E}{R}$$

Or $$R = \frac{EXI}{M} \qquad \ldots(ii)$$

Substituting the value of R in equation (i), we get

$$y = \frac{L^2}{8X\frac{EI}{M}}$$

$$y = \frac{ML^2}{8EI} \qquad \ldots(iii)$$

Equation (iii) gives the central deflection of a beam which bends in a circular arc

Value of slope(θ)

From triangle AOB, we know that

$$\sin\theta = \frac{AC}{AO} = \frac{\left[\frac{L}{2}\right]}{R} = \frac{L}{2R}$$

since the angle θ is very small, hence sin θ = θ (in radians)

$$\therefore \theta = \frac{L}{2R}$$

$$= \frac{L}{2X\frac{EI}{M}}$$

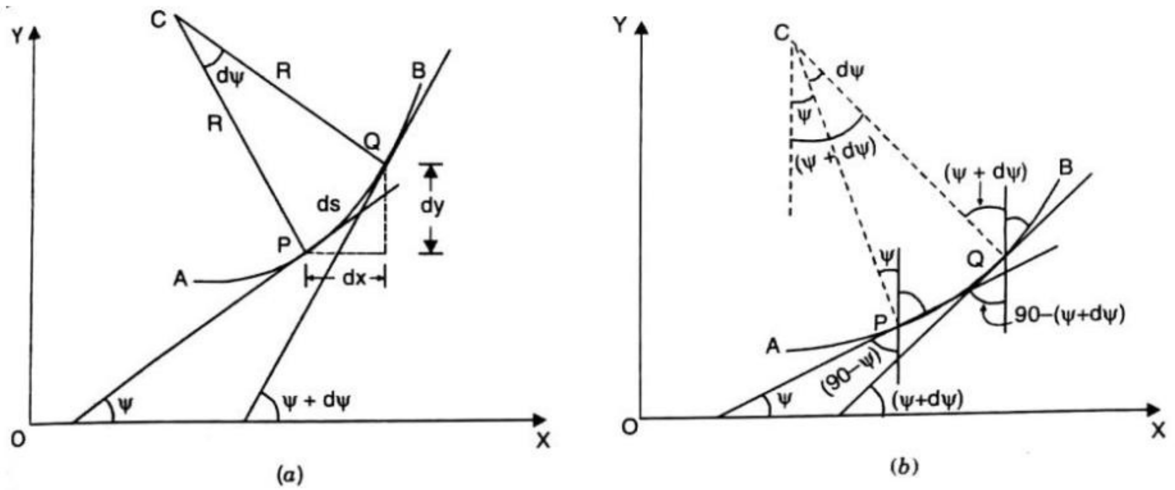
$$= \frac{MXL}{2EI} \qquad \ldots(iv)$$

Equation (iv) gives the slope of the deflected beam at A or at B

4.6. RELATION BETWEEN SLOPE, DEFLECTION AND RADIUS OF CURVATURE or DERIVATION OF DIFFERENTIAL EQUATION OF FLEXURE

Let the curve AB represents the deflection of a beam as shown in Fig. Consider a small portion PQ of this beam. Let the tangents at P and Q make angle Ψ and Ψ + d Ψ with x axis.Normal at P and Q will meet at C such that

PC = QC = R



The point C is known as the centre of curvature of the curve PQ.

Let the length of PQ is equal to ds.

From fig.3.4.b we see that

$$\text{Angle PCQ} = d\,\Psi$$

$$PQ = ds = R.d\Psi$$

$$R = \frac{ds}{d\Psi} \qquad \text{...(i)}$$

But if x and y be the coordinates of P, then

$$\tan\Psi = \frac{dy}{dx} \qquad \text{...(ii)}$$

$$\sin\Psi = \frac{dy}{ds}$$

and $$\text{Cos}\,\Psi = \frac{dx}{ds}$$

Now equatioin (i) can be written as

$$R = \frac{ds}{d\Psi} = \frac{\left[\frac{ds}{dx}\right]}{\left(\frac{d\Psi}{dx}\right)} = \frac{\left[\frac{1}{cos\Psi}\right]}{\left(\frac{d\Psi}{dx}\right)}$$

$$R = \frac{sec\Psi}{\left(\frac{d\Psi}{dx}\right)} \qquad \text{...(iii)}$$

Differentiating equation (ii) w.r.t.x, we get

$$\text{Sec}^2\,\Psi\,\frac{d\Psi}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{d\Psi}{dx} = \frac{\frac{d^2y}{dx^2}}{sec^2\Psi}$$

Substituting this value of $\frac{d\Psi}{dx}$ in equation (iii), we get

$$R = \frac{sec\Psi}{\left[\frac{\frac{d^2y}{dx^2}}{sec^2\Psi}\right]} = \frac{sec\Psi sec^2\Psi}{\frac{d^2y}{dx^2}} = \frac{sec^3\Psi}{\frac{d^2y}{dx^2}}$$

Taking the reciprocal to both sides, we get

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{sec^3\Psi} = \frac{\frac{d^2y}{dx^2}}{(sec^2\Psi)^{3/2}}$$

$$= \frac{\frac{d^2y}{dx^2}}{(1+tan^2\Psi)^{3/2}}$$

For a practical beam, the slope tanΨ at any point is a small quantity. Hence $tan^2\Psi$ can be neglected.

$$\therefore \frac{1}{R} = \frac{d^2y}{dx^2} \qquad \text{...(iv)}$$

From the bending equation, we have

$$\frac{M}{I} = \frac{E}{R}$$

Or $$\frac{1}{R} = \frac{M}{EI} \qquad \text{...(v)}$$

Equating equations (iv) and (v), we get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\therefore \qquad M = EI\frac{d^2y}{dx^2} \qquad \text{...(vi)}$$

Differentiating the above equation w.r.t.x, we get

$$\frac{dM}{dx} = EI\frac{d^3y}{dx^3}$$

But $\frac{dM}{dx}$ = F shear force

$$\therefore F = EI\frac{d^3y}{dx^3} \qquad \text{...(vii)}$$

Differentiating equation (vii) w.r.t.x., we get

$$\frac{dF}{dx} = EI\frac{d^4y}{dx^4}$$

But $\frac{dF}{dx}$ = w the rate of loading

∴ $w = EI\frac{d^4y}{dx^4}$

Hence, the relation between curvature, slope, deflection etc. at a section is given by

Deflection = y

Slope $= \frac{dy}{dx}$

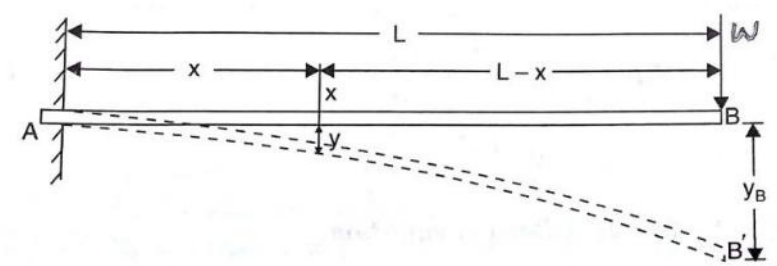
Bending moment $= EI\frac{d^2y}{dx^2}$

Shear Force $= EI\frac{d^3y}{dx^3}$

The rate of loading $= EI\frac{d^4y}{dx^4}$

4.7. DEFLECTION OF CANTILEVER WITH A POINT LOAD AT THE FREE END BY DOUBLE INTEGRATION METHOD

A cantilever AB of length L fixed at the point A and free end at the point B and carrying a point load at the free end B as shown in fig. AB shows the position of cantilever before any load is applied whereas AB' shows the position of cantilever after loading.



Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

$M_x = -W(L-x)$ (minus sign due to hogging)

But B.M at any section is also given by

$$M = EI\frac{d^3y}{dx^2}$$

Equating the two values of B.M., we get

$$EI\frac{d^3y}{dx^2} = -W(L-x) = -WL + Wx$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2} + C1 \qquad ...(i)$$

Integrating again, we get

$$EIy = -\frac{WLx^2}{2} + \frac{W}{2} X \frac{x^3}{3} + C1x + C2 \qquad ...(ii)$$

Where C1 and C2 are constant of integration. Their values are obtained from boundary conditions, which are: (i) at x=0, y=0 (ii) x=0, $\frac{dy}{dx} = 0$

(At the fixed end, deflection and slopes are zero)

(i) By substituting x=0, y=0 in equation (ii), we get

0=0+0+0+C2 ∴ C2 = 0

By substituting x=0, $\frac{dy}{dx}$ =0 in equation (i), we get

0=0+0+C1 ∴ C1 = 0

Substituting the value of C1 in equation (i), we get

$$EI\frac{dy}{dx} = -WLx + \frac{Wx^2}{2}$$

$$= -W\left[Lx - \frac{x^2}{2}\right] \qquad ...(iii)$$

Equation (iii) is known as slope equation. We can find the slope at any point on the cantilever by substituting the value of x. The slope and deflection are maximum at the free end. These can be determined by substituting x=L in these equations.

Substituting the values of C1 and C2 in equation (ii), we get

$$EIy = -WL\frac{x^2}{2} + \frac{Wx^3}{6}$$

$$= -W\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right] \qquad ...(iv)$$

Equation (iv) is known as deflection equation.

Let θ_B = slope at free end B

y_B =Deflection at the free end B

Substituting θ_B for $\frac{dy}{dx}$ and x=L in equation (iii), we get

$$EI\theta_B = -W\left[L.L - \frac{L^2}{2}\right] \quad = -W\frac{L^2}{2}$$

$$\therefore \theta_B = -\frac{WL^2}{2EI}$$

Negative sign shows the tangent at B makes an angle in the anti-clockwise direction with AB.

$$\therefore \boldsymbol{\theta_B = \frac{WL^2}{2EI}}$$

Substituting y_B for y and x=L in equation (iv), we get

$$EIy_B = -W\left[L.\frac{L^2}{2} - \frac{L^3}{6}\right]$$

$$= -W\left[\frac{L^3}{2} - \frac{L^3}{6}\right]$$

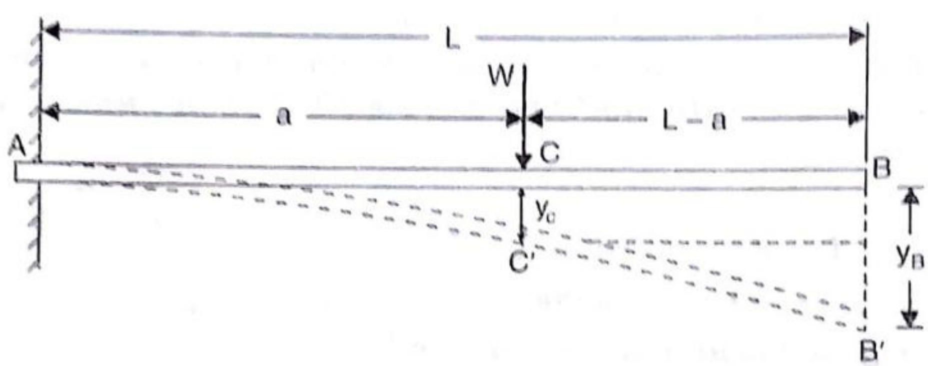
$$= -W.\frac{L^3}{3}$$

$$\therefore \quad y_B = -\frac{WL^3}{3EI}$$ (Negative sign shows that deflection is downwards)

$$\therefore \boldsymbol{y_B = \frac{WL^3}{3EI}}$$

4.8. DFFLECTION OF A CANTILEVER WITH A POINT LOAD AT A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at point B and carrying a point load W at a distance 'a' from the fixed end A, is shown in Fig.



Let θ_C = Slope at point C i.e., $\frac{dy}{dx}$ at C

y_C = Deflection at point C

y_B = Deflection at point B

The portion AC of the cantilever may be taken as similar to a cantilever in Art. (*i.e.*, load at the free end).

$$\therefore \qquad \theta_C = +\frac{Wa^2}{2EI}$$

and
$$y_C = \frac{Wa^3}{3EI}$$

The beam will bend only between A and C, but from C to B, it will remain straight since B.M. between C and B is zero.

Since the portion CB of the cantilever is straight, therefore

Slope at C = Slope at B

$$\theta_C = \theta_B = \frac{Wa^2}{2EI}$$

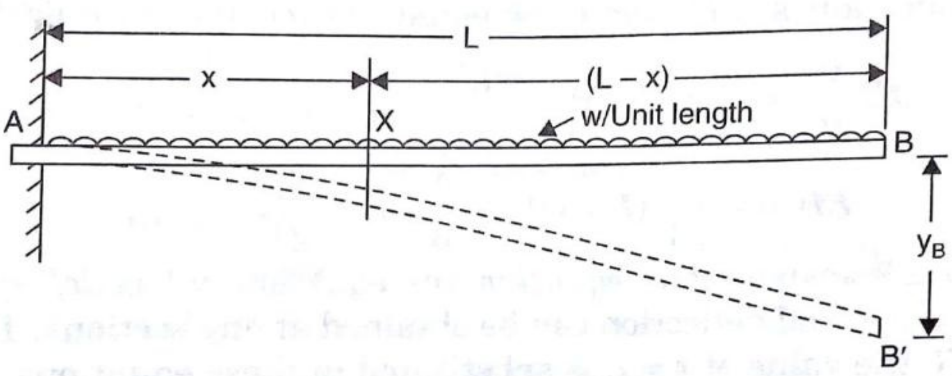
Now from Fig. we have

$$Y_B = y_C + \theta_C\,(L\text{-}a)$$

$$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L\text{-}a) \qquad \left[since, \quad \theta C = \frac{Wa^2}{2EI}\right]$$

4.9. DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w per unite length, is shown in Fig.



Consider a section X, at a distance x from the fixed end A. The B.M. at this section is given by,

$$M_x = -\ w\ (L - x).\ \frac{(L-x)}{2} \qquad \text{(Minus sign due to hogging)}$$

But B.M. at any section is also given by equation as

$$M = EI \frac{d^2y}{dx^2}$$

Equating the two values of B.M., we get

$$EI \frac{d^2y}{dx^2} = -\frac{w}{2} (L - x)^2$$

Integrating the above equation, we get

$$EI \frac{dy}{dx} = -\frac{w}{2} \frac{(L-x)^3}{3} (-1) + C_1$$

$$= \frac{w}{6} (L - x)^3 + C_1 \qquad \text{..... } (i)$$

Integrating again, we get

$$EIy = \frac{w}{6} \frac{(L-x)^4}{4} (-1) + C_1x + C_2$$

$$= -\frac{w}{24} (L - x)^4 + C_1x + C_2 \qquad \text{..... } (ii)$$

where C_1 and C_2 are constant of integrations. There values are obtained from boundary conditions, which are : (*i*) at x = 0, y = 0 and (*ii*) at x = 0, $\frac{dy}{dx} = 0$ (as the deflection and slope at fixed end A are zero).

(*i*) By substituting x = 0, y = 0 in equation (*ii*), we get

$$0 = -\frac{w}{24} (L - 0)^4 + C_1 \times 0 + C_2 = -\frac{wL^4}{24} + C_2$$

$$\therefore \qquad C_2 = \frac{wL^4}{24}$$

(*ii*) By substituting x = 0 and $\frac{dy}{dx} = 0$ in equation (*i*), we get

$$= \frac{w}{64} (L - 0)^3 + C_1 = -\frac{wL^3}{6} + C_1$$

$$\therefore \qquad C_1 = \frac{wL^3}{6}$$

Substituting the values of C_1 and C_2 in equations (*i*) and (*ii*), we get

$$EI \frac{dy}{dx} = \frac{w}{6} (L - x)^3 - \frac{wL^3}{6} \qquad \text{..... } (iii)$$

and

$$EIy = -\frac{w}{24} (L - x)^4 - \frac{wL^3}{6} x + \frac{wL^4}{24} \qquad \text{..... } (iv)$$

Equation (*iii*) is known as slope equation and equation (*iv*) as deflection equation. From these equations the slope and deflection can be obtained at any sections. To find the slope and deflection at point B, the value of x = L is substituted in these equations.

Let θ_B = Slope at the free end B *i.e.,* $\frac{dy}{dx}$ at B

y_B = Deflection at the free end B.

From equation (*iii*), we get slope at B as

$$EI.\ \theta_B = \frac{w}{6}\ (L - L)^3 - \frac{wL^3}{6} = -\frac{wL^3}{6}$$

$$\theta_B = -\frac{wL^3}{6EI} = -\frac{wL^2}{6EI} \qquad \text{(Since, W = Total load = w.L)}$$

From equation (*iv*), we get the deflection at B as

$$EI.\ y_B = -\frac{w}{24}\ (L - L)^4 - \frac{wL^3}{6} \times L + -\frac{wL^4}{24}$$

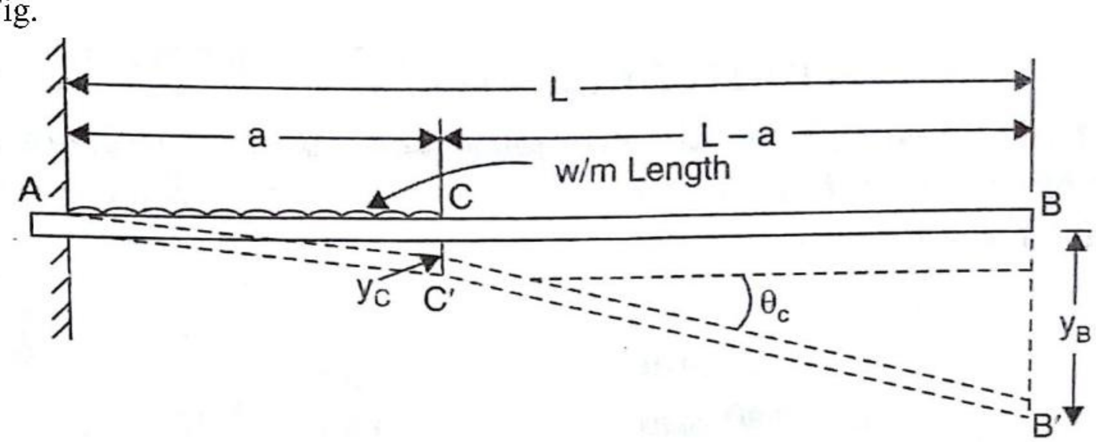
$$= -\frac{wL^4}{6} + \frac{wL^4}{24} = -\frac{3}{24}wL^4 = -\frac{wL^4}{8}$$

∴ $$y_B = -\frac{wL^4}{8EI} = -\frac{wL^3}{8EI} \qquad \text{(Since, W = w.L)}$$

∴ Downward deflection at B, $\boldsymbol{y_B = -\frac{wL^4}{8EI}}$

4.10. DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD FOR A DISTANCE 'a' FROM THE FIXED END

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w/m length for a distance 'a' from the fixed end, is shown in Fig.



The beam will bend only between A and C, but from C to B it will remain straight since B.M. between C and B is zero. The deflected shape of the cantilever is shown by AC'B' in which portion C'B' is straight.

Let θ_C = Slope at C, *i.e.,* $\frac{dy}{dx}$ at C

y_C = Deflection at point C, and

y_B = Deflection at point B.

The portion AC of the cantilever may be taken as similar to a cantilever in Art.

∴ $\theta_C = \frac{w.a^2}{6EI}$

and $y_C = \frac{w.a^4}{8EI}$

Since the portion C'B' of the cantilever is straight, therefore slope at C = Slope at B

or $\theta_C = \theta_B = \frac{wa^3}{6EI}$ ….. (1)

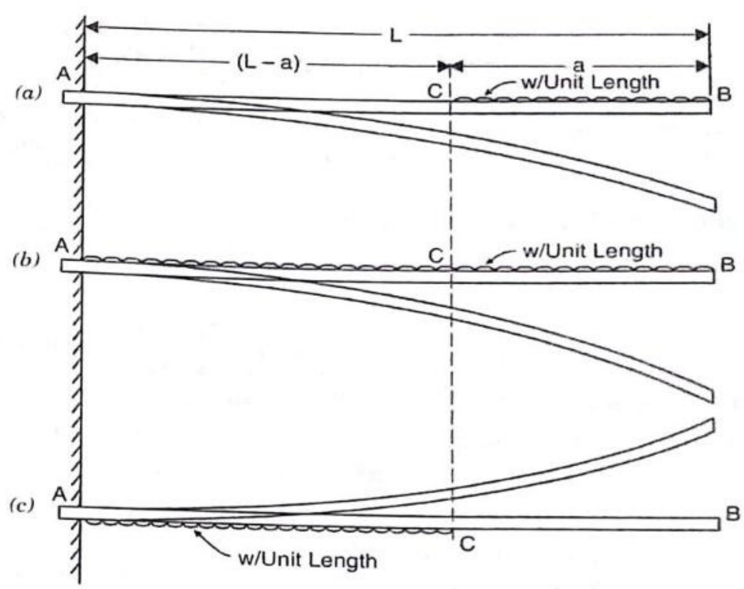
Now from Fig. we have

$y_B = y_C + \theta_C$ (L - a)

$= \frac{wa^4}{8EI} + \frac{w.a^3}{6EI}$ (L -a) ….. (2)

4.11. DEFLECTION OF A CANTILEVER WITH A UNIFORMLY DISTRIBUTED LOAD FOR A DISTANCE 'a' FROM THE FREE END

A cantilever AB of length L fixed at the point A and free at the point B and carrying a uniformly distributed load of w/m length for a distance 'a' from the free end, is shown in Fig.



The slope and deflection at the point B is determined by considering :

(i) the whole cantilever AB loaded with a uniformly distributed load of w per unite length as shown in Fig.

(ii) a part of cantilever from A to C of length (L - a) loaded with an upward uniformly distributed load of w per unit length as shown in Fig.

Then slope at B = Slope due to downward uniform load over the whole length
- Slope due to upward uniform load from A to C

and deflection at B = Deflection due to downward uniform load over the whole length
- deflection due to upward uniform load from A to C.

(a) Now slope at B due to downward uniformly distributed load over the whole length

$$= \frac{wL^3}{6EI}$$

(b) slope at B or at C due to upward uniformly distributed load over the length(L - a)

$$= \frac{w(L-a)^3}{6EI}$$

Hence net slope at B is given by,

$$\theta_B = \frac{wL^3}{6EI} - \frac{w(L-a)^3}{6EI} \qquad \text{..... (i)}$$

The downward deflection of point B due to downward distributed load over the whole length AB

$$= \frac{wL^4}{8EI}$$

The upward deflection of point B due to upward uniformly distributed load acting on the portion AC

= upward deflection of C + slope at C × CB

$$= \frac{w(L-a)^4}{8EI} + \frac{w.(L-a)^3}{6EI} \times a \qquad \text{(since CB = a)}$$

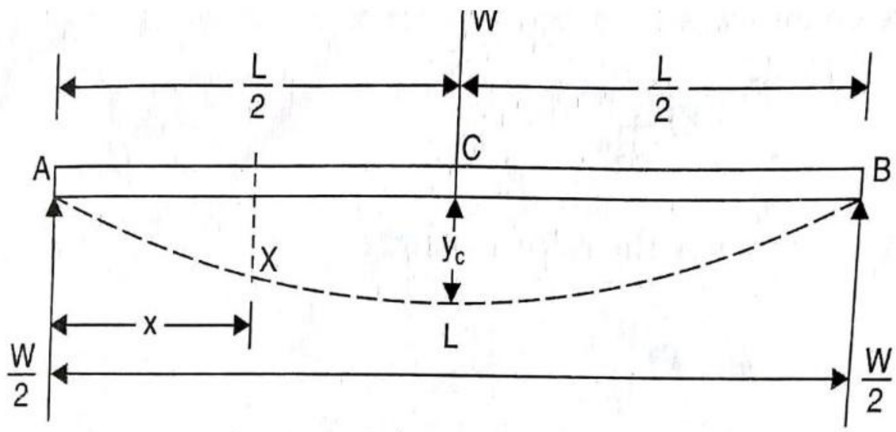
∴ Net downward deflection of the free end B is given by

$$\boldsymbol{y_B = \frac{wL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \times a\right]} \qquad \text{..... (ii)}$$

4.12. DEFLECTION OF A SIMPLY SUPPORTED BEAM CARRYING A POINT LOAD AT THE CENTRE

A simply supported beam AB of length L and carrying a point load W at the centre is shown in Fig.

As the load is symmetrically applied the reactions R_A and R_B will be equal. Also the maximum deflection will be at the centre.



Now $R_A = R_B = \frac{W}{2}$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$$M_x = R_A \times x$$

$$= \frac{W}{2} \times x$$

(Plus sign is as B.M. for left portion at X is clockwise)

But B.M. at any section is also given by equation as

$$M = EI\frac{d^2y}{dx^2}$$

Equation the two values of B.M., we get

$$EI\frac{d^2y}{dx^2} = \frac{W}{2} \times x \qquad \text{..... } (i)$$

On integration, we get

$$EI\frac{dy}{dx} = \frac{W}{2} \times \frac{x^2}{2} + C_1 \qquad \text{..... } (ii)$$

Where C_1 is the constant of integration. And its value is obtained from boundary conditions. The boundary condition is that at $x = \frac{L}{2}$, slope $\frac{dy}{dx} = 0$ (As the deflection is at the centre, hence slope at the centre will be zero). Substituting this boundary condition in equation (*ii*), we get

$$O = \frac{W}{4} \times \left(\frac{L}{2}\right)^2 + C_1$$

or $$C_1 = \frac{WL^2}{16}$$

Substituting the value of C_1 in equation (*ii*), we get

$$EI\frac{dy}{dx} = \frac{Wx^2}{4} - \frac{WL^2}{16} \qquad \text{..... } (iii)$$

The above equation is known the slope equation. We can find the slope at any point on the beam by substituting the values of x. Slope is maximum at A. At A , x = 0 and hence slope at A will be obtained by substituting x = 0 in equation *(iii).*

$$\therefore \qquad EI\frac{dy}{dx} = \frac{W}{4} \times 0 - \frac{WL^2}{16}$$

$$EI \times \theta_A = -\frac{WL^2}{16}$$

$$\therefore \qquad \theta_A = -\frac{WL^2}{16EI}$$

The slope at point B will be equal to θ_A, since the load is symmetrically applied.

$$\therefore \qquad \boldsymbol{\theta_B = \theta_A = -\frac{WL^2}{16EI}}$$

The above equation gives the slope in radians.

Deflection at any point

Deflection at any point is obtained by integrating the slope equation *(iii)*. Hence integrating equation *(iii)*, we get

$$EI \times y = \frac{W}{4} \cdot \frac{x^3}{3} - \frac{WL^2}{16}x + C_2 \qquad \text{..... } (iv)$$

Where C_2 is another constant of integration. At A, x = 0 and the deflection (y) is zero.

Hence substituting these values in equation *(iv)*, we get

$$EI \times O = O - 0 + C_2$$

Or $C_2 = 0$

Substituting the values of C_2 in equation *(iv)*, we get ..... (*v*)

$$EI \times y = \frac{Wx^3}{12} - \frac{WL^2.x}{16}$$

The above equation is known as the deflection equation. We can find the deflection at any point on the beam by substituting the values of x. The deflection is maximum at centre point C, where $x = \frac{L}{2}$. Let y_c represents the deflection at C. Then substituting $x = \frac{L}{2}$ and $y = y_c$ in equation *(iv)*, we get

$$EI \times y_c = \frac{W}{12}\left(\frac{L}{2}\right)^3 - \frac{WL^2}{16} \times \left(\frac{L}{2}\right)$$

$$= \frac{WL^3}{96} - \frac{WL^3}{32} = \frac{WL^3 - 3WL^3}{96}$$

$$= -\frac{2WL^3}{96} = \frac{WL^3}{48}$$

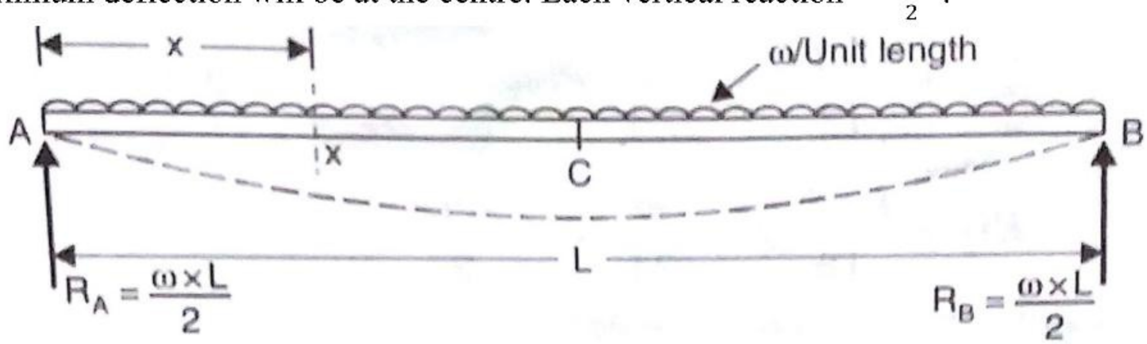
$$\therefore \quad y_c = -\frac{WL^3}{48EI}$$

(Negative sign shows that deflection is downwards)

$\therefore$ Maximum deflection, $\mathbf{y_c} = \frac{WL^3}{48EI}$

4.13. DEFLECTION OF A SIMPLY SUPPORTED BEAM WITH A UNIFORMLY DISTRIBUTED LOAD

A simply supported beam AB of length L and carrying a uniformly distributed load of w per unite length is shown in Fig. The reactions at A and B will be equal. Also the maximum deflection will be at the centre. Each vertical reaction = $\frac{w \times L}{2}$.



$$\therefore \quad R_A = R_B = \frac{w \times L}{2}$$

Consider a section X at a distance x from A. The bending moment at this section is given by,

$$M_x = R_A \times x - w \times x \times \frac{x}{2} = \frac{w\,.\,L}{2}\,.x - \frac{w.x^2}{2}$$

But B.M. at any section is also given by equation ( ), as

$$M = EI\,\frac{d^2y}{dx^2}$$

Equation the two values of B.N., we get

$$EI\,\frac{d^2y}{dx^2} = \frac{w\,.\,L}{2}\,x - \frac{w.x^2}{2}$$

Integrating the above equation, we get

$$EI\,\frac{dy}{dx} = \frac{w\,.\,L}{2}\,.\,\frac{x^2}{2} - \frac{w}{2}\,.\,\frac{x^3}{3} + C_1 \qquad .....(i)$$

where C_1 is a constant of integration.

Integrating the above equation again, we get

$$EI.y = \frac{w\,.\,L}{4}\,.\,\frac{x^3}{3} - \frac{w}{6}\,.\,\frac{x^4}{4} + C_1x + C_2 \qquad .....(ii)$$

where C_2 is a another constant of integration. Thus two constant of integration (*i.e.*, C_1 and C_2) are obtained from boundary conditions. The boundary conditions are:

(i) at x = 0, y= 0 and *(ii)* at x = L, y = 0

Substituting first boundary condition i.e., x = 0, y = 0 in equation (*ii*), we get

$$0 = 0 - 0 + 0 + C_2 \text{ or } C_2 = 0$$

Substituting the secondary boundary condition i.e., x = L, y = 0 in equation (*ii*), we get

$$0 = \frac{w\,.\,L}{4}\,.\frac{L^3}{3} - \frac{w}{6}\,.\frac{L^4}{4} + C_1.L \qquad (C_2 \text{ is already zero})$$

$$= \frac{w.L^4}{12} - \frac{w.L^4}{24} + C_1.L$$

or

$$C_1 = -\frac{wL^3}{12} + \frac{wL^3}{24} = -\frac{wL^3}{12}$$

Substituting the value of C_1 in equation (i) and (ii), we get

$$EI\,\frac{dy}{dx} = \frac{w.L}{4}\,.\,x^2 - \frac{w}{6}\,x^2 - \frac{w}{6}\,x^3 - \frac{wL^3}{24} \qquad .....(iii)$$

$$EI.y = \frac{w.L}{12}x^3 - \frac{w}{24}x^4 + \left[-\frac{wL^3}{24}\right]x + 0 \qquad (\text{since, } C_2 = 0)$$

$$EIy = \frac{w.L}{12}x^3 - \frac{w}{24}x^4 - \frac{wL^3}{24}x \qquad \text{--- (iv)}$$

Equation (iii) is known as slope equation. We can find the slope $\left[i.e., the\ value\ of\ \frac{dy}{dx}\right]$ at any point on the beam by substituting the different values of x in this equation.

Equation (iv) is known as deflection equation. We can find the deflection $[i.e., the\ value\ of\ y]$ at any point on the beam by substituting the different values of x in this equation.

Slope at the supports

Let θ_A = Slope at support A.

and θ_B = Slope at support B

At A, x = 0 and $\frac{dy}{dx} = \theta_A$,

Substituting these value in equation (iii), we get,

$$EI\ \theta A = \frac{WL}{4} \times 0 - \frac{w}{6} \times 0 - \frac{WL^3}{24}$$

$$EI \times \theta_A = -\frac{WL^3}{24}$$

$$\therefore \qquad \theta_A = -\frac{WL^3}{24EI}$$

The slope at point B will be equal to θ_A, since the load is symmetrically applied.

$$\therefore \qquad \boldsymbol{\theta_B = \theta_A = -\frac{WL^3}{24EI}}$$

The above equation gives the slope in radians.

Deflection at any point

The deflection is maximum at centre point C, where x = $\frac{L}{2}$. Let y_c represents the deflection at C. Then substituting x =$\frac{L}{2}$ and y = y_c in equation *(iv)*, we get

$$EI \times y_c = \frac{WL}{12}\left(\frac{L}{2}\right)^3 - -\frac{W}{24}\left(\frac{L}{2}\right)^4 - \frac{WL^3}{24} \times \left(\frac{L}{2}\right)$$

$$= \frac{WL^4}{96} - \frac{WL^4}{384} - \frac{WL^4}{48} = -\frac{5WL^4}{384}$$

$$\therefore \qquad y_c = -\frac{5WL^4}{384EI}$$

(Negative sign shows that deflection is downwards

∴ Maximum deflection,

$$\mathbf{y_c = \frac{5WL^4}{384EI}}$$

Example.3.1. A cantilever of length 3 m is carrying a point load of 25 kN at the free end. If the moment of inertia of the beam = 10^8 mm^4 and value of E = 2.1 × 1O^5 N/mm^2, find (*i*) slope of the cantilever at the free end and (*ii*) deflection at the free end.

Sol. Given:

Length, L = 3m = 3000 mm

Point load, W = 25kN = 25000 N

M.O.I., I =10^8 mm^4

Value of E = 2.1 × 10^5 N/mm^2

(i) Slope at the free end is given by equation.

$$\therefore \qquad \theta_B = \frac{WL^2}{2EI} = \frac{25000 \times 3000^2}{2 \times 2.5 \times 10^5 \times 10^8} = \mathbf{0.005357\ rad}. \qquad \textbf{Ans.}$$

(ii) Deflection at the free end is given by equation

$$y_B = \frac{WL^3}{3EI} = \frac{25000 \times 3000^3}{3 \times 2.1 \times 10^5 \times 10^8} = \mathbf{10.71\ mm}. \qquad \textbf{Ans.}$$

Example.3.2. A cantilever of length 3 m is carrying a point load of 50 kN at a distance of 2 m from the fixed end. If I = 10^5 mm^4 and value of E = 2 × 1O^5 N/mm^2, find (*i*) slope at the free end and (*ii*) deflection at the free end.

Sol. Given:

Length, L = 3m = 3000 mm

Point load, W = 50 kN = 50000 N

Distance between the load and the fixed end,

a = 2 m =2000 mm

M.O.I., I =10^8 mm^4

Value of $E = 2 \times 10^5$ N/mm^2

(i) Slope at the free end is given by equation as

$$\therefore \quad \theta_B = \frac{Wa^2}{2EI} = \frac{50000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8} = \mathbf{0.005\ rad.} \quad \textbf{Ans.}$$

(ii) Deflection at the free end is given by equation as

$$y_B = \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI}(L - a)$$

$$= \frac{50000 \times 2000^3}{3 \times 2 \times 10^5 \times 10^8} + \frac{50000 \times 2000^2}{2 \times 2 \times 10^5 \times 10^8}(3000 - 2000)$$

$$= 6.67 + 5.0 = \mathbf{11.67\ mm.} \quad \textbf{Ans.}$$

Example 3.3.A cantilever of length 2m carries a uniformly distributed load of 2.5kN/m run for a length of 1.25m from the fixed end and a point load of 1 kN at the free end. Find the deflection at the free end if the section is rectangular 12 cm wide and 24 cm deep and E = 1 X 10^4N/mm^2

Given Data:

Length, $L = 2m = 2000mm$

u.d.l,w $= 2.5$ kN/m $= 2.5 \times \frac{1000}{1000}$ N/mm = 2.5 N/mm run for a 'length of 'a

= 1.25m = 1250mm from the fixed end.

Point load at the free end

W = 1 kN = 1000N

Width, b = 12cm = 120mm

Depth, d = 24cm = 240mm

E = 1 X 10^4N/mm^2

To Find: The deflection at the free end

Solution:

Moment of inertia of the rectangular section

$$I = \frac{bd^3}{12} = \frac{120 \times 240^3}{12} = 13824 \times 10^4 mm^4$$

Downward deflection at the free end due to point load

$$y_1 = \frac{WL^3}{3EI} = \frac{1000 \ X \ 2000^3}{3 \ X10^4 \ \ X \ 13824 \ X \ 10^4} = 1.929 \text{ mm.}$$

Downward deflection at the free end due to uniformly distributed load run over 1.25m from the fixed end.

$$y_2 = \frac{Wa^4}{8EI} + \frac{Wa^3}{6EI} (L\text{-}a)$$

$$= \frac{2.5 \ X \ 1250^4}{8 \ X \ 10^4 \ X \ 13824 \ X \ 10^4} + \frac{2.5 \ X \ 1250^3}{6 \ X \ 10^4 \ X \ 13824 \ X \ 10^4} (2000\text{-}1250)$$

$$= 0.9934 \text{ mm}$$

∴ Total deflection at the free end due to point load and u.d.l

$= y_1 + y_2 = 1.929 + 0.9934$ **= 2.9224 mm**

Example.3.4. A cantilever of length 2m carries a uniformly distributed load 2 kN/m over a length of 1m from the free end, and a point load of 1 kN at the free end. Find the slope and deflection at the free end if E = 2.1 X 10^5N/mm^2 and I = 6.667 X 10^7mm^4.

Given Data

Length, L = 2m =2000mm

u.d.l,w = 2 kN/m = 2 X $\frac{1000}{1000}$ N/mm = 2 N/mm run for a length of

. 1m = 1000mm from the free end

Point load at the free end

W = 1 kN = 1000N

E = 2.1 X 10^5N/mm^2

I = 6.667 X 10^7mm^4

To Find: The slope and deflection at the free end

Solution:

Slope at the free end.

The slope at the free end due to a point load

$$\theta_1 = \frac{WL^2}{2EI}$$

$$= \frac{1000 \ X \ 2000^2}{2 \ X \ 2.1 \ X \ 10^5 X6.667 \ X \ 10^7}$$

= 0.0001428 radians.

The slope at the free end due to u.d.l of 2 kN/m over a length of 1m from the free end.

$$\theta_2 = \frac{WL^3}{6EI} - \frac{W(L-a)^3}{6EI}$$

$$= \frac{2 \; X \; 2000^3}{6 \; X \; 2.1 \; X \; 10^5 X 6.667 \; X \; 10^7} - \frac{2 \; X \; (2000-1000)^3}{6 \; X \; 2.1 \; X \; 10^5 X 6.667 \; X \; 10^7}$$

= 0.0001666 radians.

∴ Total slope at the free end = $\theta_1 + \theta_2$

= 0.0001428 + 0.0001666

= **0.0003094 radians**

Deflection at the free end.

The Deflection at the free end due to a point load

$$y_1 = \frac{WL^3}{3EI}$$

$$= \frac{1000 \; X \; 2000^3}{3 \; X \; 2.1 \; X \; 10^5 X 6.667 \; X \; 10^7}$$

= 0.1904mm.

The Deflection at the free end due to u.d.l of 2 kN/m over a length of 1m from the free end.

$$y_2 = \frac{WL^4}{8EI} - \left[\frac{w(L-a)^4}{8EI} + \frac{w(L-a)^3}{6EI} \; X \; a\right]$$

$$= \frac{2 \; X \; 2000^4}{8 \; X \; 2.1 \; X \; 10^5 X 6.667 \; X \; 10^7} - \frac{2 \; X \; (2000-1000)^4}{8 \; X \; 2.1 \; X \; 10^5 X 6.667 \; X \; 10^7} - \frac{2 \; X \; (2000-1000)^3 \; X \; 1000}{6 \; X \; 2.1 \; X \; 10^5 X 6.667 \; X \; 10^7}$$

= 0.244mm

∴ Total deflection at the free end = $y_1 + y_2$

= 0.1904 + 0.244mm **= 0.4344mm**

Example.3.5. A beam 6m long, simply supported at its ends, is carrying a point load of 50 kN at its centre. The moment of inertia of the beam is equal to 78 X 10^6 mm^4. If E for the material of the beam = 2.1 X 10^5N/mm^2, Calculate the slope at the supports and the deflection at the centre of the beam.

Given Data:

Length, L = 6m =6000mm

Point load, W = 50kN = 50000N

M.O.I I $= 78 \times 10^6\ mm^4$

Value of E $= 2.1 \times 10^5 N/mm^2$

To Find:

The maximum slope and Deflection.

Solution:

Maximum slope at supports

$$\theta_B = \theta_A = -\frac{WL^2}{16EI}$$

$$= \frac{WL^2}{16EI}$$

$$= \frac{50000\ X\ 6000^2}{16\ X\ 2.1\ X\ 10^5 X78\ X\ 10^6}$$

= 0.06868 radians.

Maximum deflection at centre

$$y_c = \frac{WL^3}{48EI} = \frac{50000\ X\ 6000^3}{48\ X\ 2.1\ X\ 10^5 X78\ X\ 10^6}$$ **=13.736mm.**

Example.3.6. A beam 4m long, simply supported at its ends, carries a point load W at its centre. If the slope at the ends of the beam is not to exceed 1 degree. Find the deflection at the centre of the beam

Given Data:

Length, L = 4m =4000mm

Point load at centre, = W

Slope at supports, $\theta_B = \theta_A = 1^o = \frac{1\ X\ \pi}{180} = 0.01745$ radians.

We know that

slope at supports, $\theta_A = \frac{WL^2}{16EI} = 0.01745$ radians.

Maximum deflection at centre

$$y_c = \frac{WL^3}{48EI} = \frac{WL^2}{16EI} X \frac{L}{3}$$

$$= 0.01745\ X \frac{4000}{3}$$

= 23.26mm.

Example.3.7. A beam of uniform rectangular section 200mm wide and 300mm deep is simply supported at its ends. It carries a uniformly distributed load of 9kN/m run over the entire span of 5m. If the value of E for the material is 1 X 10^4N/mm^2, Find the slope at the supports and maximum deflection.

Solution.

Moment of inertia of the rectangular section

$$I = \frac{bd^3}{12} = \frac{200\,X\,300^3}{12} = 4.5\ X\ 10^8 mm^4$$

Maximum slope at supports,

$$\theta_B = \theta_A = \frac{WL^3}{24EI} = \frac{9\,X\,5000^3}{24\,X\,1\,X\,10^4 X 4.5\,X\,10^8} = \mathbf{0.0104\ radians}$$

Maximum Deflection at centre

$$y_c = \frac{5WL^4}{384EI} = \frac{5\,X\,9\,X\,5000^4}{384\,X\,1\,X\,10^4 X 4.5\,X\,10^8} = \mathbf{16.27mm.}$$

4.14. MACAULAY'S METHOD

The procedure of finding slope and deflection for a simply supported beam with an eccentric point load is a very laborious. There is a convenient method for determining the deflection of the beam subjected to point loads.

This method was devised by Mr. M.H. Macaulay and is known as Macaulay's method. This method mainly consists in a special manner in which the bending moment at any section is expressed and in the manner in which the integrations are carried out.

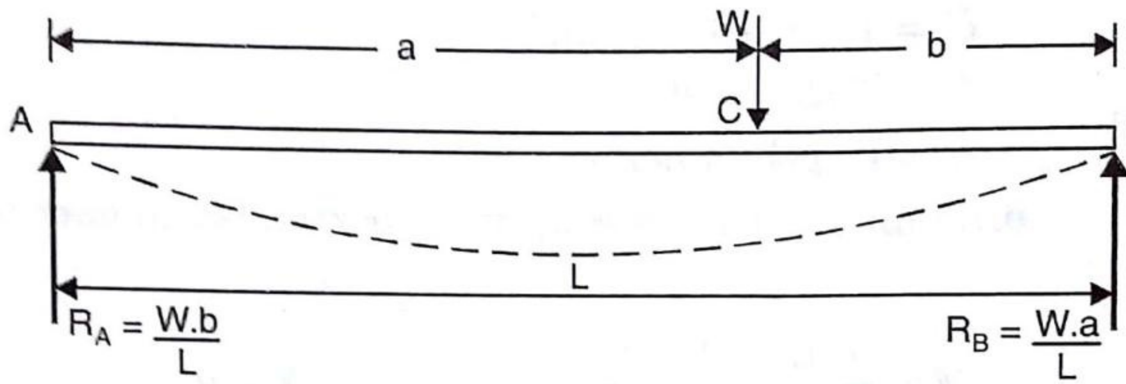
4.15. DEFLECTION OF A SIMPLY SUPPORTED LOAD WITH AN ECCENTRIC POINT LOAD

A simply supported beam AB of length L and carrying a point load W at a distance 'a' from left support and at a distance 'b' from right support is shown in Fig. The reaction at A and B are given by,

$$R_A = \frac{Wb}{L} \quad \text{and} \quad R_B = \frac{Wa}{L}$$

The bending moment at any section between A and C at a distance x from A is given by,

$$M_x = R_A\ X\ x = \frac{Wb}{L}\ X\ x$$



The above equation of B.M. holds good for the values of x between 0 and 'a'. The bending moment at any section between C and B at a distance x from A is given by,

$M_x = R_A \times x - W \times (x-a) = \frac{Wb}{L} \times x - W(x-a)$

The above equation of B.M holds good for all values of x between x = a and x = b.

The B.M for all sections of the beam can be expressed in a single equation written as

$$M_x = \frac{Wb}{L} \times x \;\vdots\; - W(x-a) \qquad \ldots (i)$$

Stop at the dotted line for any point in section AC. But for any point in section CB, add the expression beyond the dotted line also.

The B.M. at any section is also given by

$$M = EI \frac{d^2y}{dx^2} \qquad \ldots (ii)$$

Hence equating (i) and (ii), we get

$$EI \frac{d^2y}{dx^2} = \frac{Wb}{L} \times x \;\vdots\; - W(x-a) \qquad \ldots (iii)$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = \frac{Wb}{L}\frac{x^2}{2} + C_1 \;\vdots\; - \frac{W(x-a)^2}{2} \qquad \ldots (iv)$$

Where C_1 is a constant of integration. This constant of integration should be written after the first term. Also the brackets are to be integrated as a whole. Hence the integration of (x-a) will be $\frac{(x-a)^2}{2}$ and not $\frac{x^2}{2}$ - ax.

Integrating equation (iv) once again, we get

$$EIy = \frac{Wb}{2L}\frac{x^3}{3} + C_1x + C_2 \;\vdots\; - \frac{W}{2}\frac{(x-a)^3}{3} \qquad \ldots (v)$$

Where C_2 is another constant of integration. This constant is written after C_1x. The integration of $(x-a)^2$ will be $\frac{(x-a)^3}{3}$. This type of integration is justified as the constants of integrations C_1 and C_2 are valid for all values of x.

The values of C_1 and C_2 are obtained from boundary conditions. The two boundary conditions are :

(i).At x = 0, y = 0 and (ii) At x = L, y = 0. since deflection is 0 at A and B)

(i) At A, x =0 and y = 0. Substituting these values in equation (v) upto dotted line only as the point A lies in AC (i.e. at first portion), we get

$$0 = 0+0+C_2$$

$$\therefore \quad C_2 = 0$$

(ii).At B, x = L and y = 0. Substituting these values in equation (v), we get

$$0 = \frac{Wb}{2L}\frac{L^3}{3} + C_1L + \ 0 \ - \frac{W}{2}\frac{(L-a)^3}{3}$$

$$= \frac{WbL^2}{6} + C_1L - \frac{Wb^3}{6} \qquad \text{(since L-a = b)}$$

$$\therefore C_1L = \frac{Wb^3}{6} - \frac{WbL^2}{6} = -\frac{Wb(L^2-b^2)}{6}$$

$$\therefore C_1 = -\frac{Wb(L^2-b^2)}{6L} \qquad \ldots\text{(vi)}$$

Substituting the value of C_1 in equation (iv), we get

$$EI\frac{dy}{dx} = \frac{Wb}{L}\frac{x^2}{2} - \frac{Wb(L^2-b^2)}{6L} \ \vdots \ -\frac{W(x-a)^2}{2} \qquad \ldots\text{(vii)}$$

Equation (vii) gives the slope at any point in the beam. Slope is maximum at A or B. To find the slope at A, substitute x = 0 in the above equation upto dotted line as point A lies in AC.

$$EI\theta_A = \frac{Wb}{2L} X\ 0 - \frac{Wb(L^2-b^2)}{6L}$$

$$= -\frac{Wb(L^2-b^2)}{6L}$$

$$\therefore \theta_A = -\frac{Wb(L^2-b^2)}{6EIL}$$

Substituting the values of C_1 and C_2 in equation (v), we get

$$EIy = \frac{Wb}{2L}\frac{x^3}{3} - \frac{Wb(L^2 - b^2)}{6L}x + 0 \vdots - \frac{W}{2}\frac{(x-a)^3}{3} \quad \ldots(Viii)$$

Equation (viii) gives the deflection at any point in the beam. To find the deflection y_c under the load, substitute x = a in equation (viii) and consider the equation upto dotted line (as point C lies in AC). Hence, we get

$$EIy_c = \frac{Wb}{2L}\frac{a^3}{3} - \frac{Wb(L^2 - b^2)a}{6L}$$

$$= \frac{Wb}{6L}.a.(a^2 - L^2 + b^2)$$

$$= -\frac{Wab}{6L}(L^2 - a^2 - b^2)$$

$$= \frac{Wab}{6L}\left[(a+b)^2 - a^2 - b^2\right] \quad \text{(since L = a+b)}$$

$$= -\frac{Wab}{6L}(2ab)$$

$$= -\frac{Wa^2b^2}{3L}$$

$$\therefore \quad y_c = -\frac{Wa^2b^2}{3EIL}$$

Example.3.8. A horizontal beam of uniform section and 6 meters long is simply supported at its ends. Two vertical concentrated loads of 48 kN and 40 kN act at 1m and 3m respectively from the left hand support. Determine the magnitude of the deflection under the loads and maximum deflection using Macaulay's method. If E = 200GN/m^2 and I = 85 X 10^{-6} m^4

Given Data

W_C = 48 kN

W_D = 40 kN

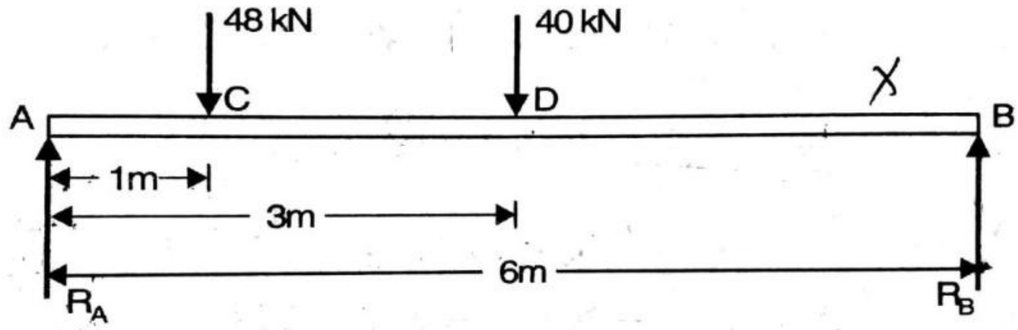
E = 200GN/m^2 = 200 X 106kN/m^2

I = 85 X 10^{-6} m^4

To Find:

The deflection under the loads and the maximum deflection

Solution:



Taking moment about A,

$$R_B \times 6 - (40 \times 3) - (48 \times 1) = 0$$

$$6R_B - 120 - 48 = 0$$

$$6R_B = 168$$

$$R_B = \frac{168}{6} = 28 \text{ KN}$$

$$R_A + R_B = 48 + 40$$

$$R_A + 28 = 88$$

$$R_A = 88 - 28 = 60 \text{ kN}$$

BM for the section X-X

$$M_x = R_A \times x \vdots - 48(x-1) \vdots -40(x-3)$$

$$= 60 \times x \vdots - 48(x-1) \vdots -40(x-3) \qquad \ldots \text{(i)}$$

The B.M. at any section is also given by

$$M = EI \frac{d^2y}{dx^2} \qquad \ldots\text{(ii)}$$

Hence equating (i) and (ii), we get

$$EI \frac{d^2y}{dx^2} = 60 \times x \vdots - 48(x-1) \vdots -40(x-3) \qquad \ldots\text{(iii)}$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = 60 \frac{x^2}{2} + C_1 \vdots - \frac{48(x-1)^2}{2} \vdots - \frac{40(x-3)^2}{2} \qquad \ldots\text{(iv)}$$

Where C_1 is a constant of integration. This constant of integration should be written after the first term. Also the brackets are to be integrated as a whole.

Integrating equation (iv) once again, we get

$$EIy = 60 \frac{x^3}{6} + C_1 x + C_2 \vdots - \frac{48(x-1)^3}{6} \vdots - \frac{40(x-3)^3}{6} \ldots\text{(v)}$$

Where C_2 is another constant of integration. This constant is written after C_1x.

The values of C_1 and C_2 are obtained from boundary conditions. The two boundary conditions are :

(i).At x = 0, y = 0 and (ii) At x = 6m, y = 0. (since deflection is 0 at A and B)

(ii) At A, x =0 and y = 0. Substituting these values in equation (v) upto the first dotted line only as the point A lies in AC (i.e. at first portion), we get

$$0 = 0+0+C_2$$

$$\therefore \quad C_2 = 0$$

(ii).At B, x = 6m and y = 0. Substituting these values in equation (v), we get

$$0 = 60 \text{ X } \frac{6^3}{3} + 6C_1 + 0 - \frac{48(6-1)^3}{3} - \frac{40(6-3)^3}{3}$$

$$6C_1 = -980$$

$$C_1 = -\frac{980}{6} = -163.33 \qquad \ldots(vi)$$

Substituting the value of C_1 in equation (iv), we get

$$EI\frac{dy}{dx} = \frac{60x^2}{2} - 163.33 \;\vdots - \frac{48\,(x-1)^2}{2} \;\vdots - \frac{40\,(x-3)^2}{6} \qquad \ldots(vii)$$

Equation (vii) gives the slope at any point in the beam.

Substituting the values of C_1 and C_2 in equation (v), we get

$$EIy = 60\frac{x^3}{6} - 163.33x \;\vdots - \frac{48(x-1)^3}{6} \;\vdots - \frac{40(x-3)^3}{6} \qquad \ldots(viii)$$

Deflection at the point C.

This is obtained by substituting x =1 in equation (viii) up to the first dotted line we get,

$$EIy_c = 10 \text{ X } 1^3 - 163.33 \text{ X } 1$$

$$= -153.33 \text{ kNm}^3$$

$$\therefore y_c = \frac{-153.33}{EI} = \frac{-153.33}{200X10^6 85X10^{-6}}$$

$$= -0.00902m$$

= -9.02mm

Deflection at the point D.

This is obtained by substituting x =3 in equation (viii) up to the Second dotted line we get,

$EIy_D = 10 \times 3^3 - 163.33 \times 3 - 8(3-1)^3$

$= -283.99 \ kNm^3$

$\therefore y_c = \frac{-283.99}{EI} = \frac{-283.99}{200X10^6 85X10^{-6}} = -0.0167m$

= - 16.7 mm

Maximum Deflection

The maximum deflection should be between C and D Where $\frac{dy}{dx} = 0$.

$\therefore$ Put $\frac{dy}{dx} = 0$. In equation (vii) up to second dotted line only

$\frac{60x^2}{2} - 163.33 - \frac{48\,(x-1)^2}{2} = 0$

Or $30x^2 - 163.33 - 24(x^2 - 2x + 1) = 0$

Or $30x^2 - 163.33 - 24x^2 + 48x - 24 = 0$

By solving We get,

X = 2.87m or x = -10.87m

X = -10.87m is not possible , so we take

X = 2.87m When x = 2.87m, $y = y_{max}$

Substituting the above condition in equation viii up to the second dotted line we get,

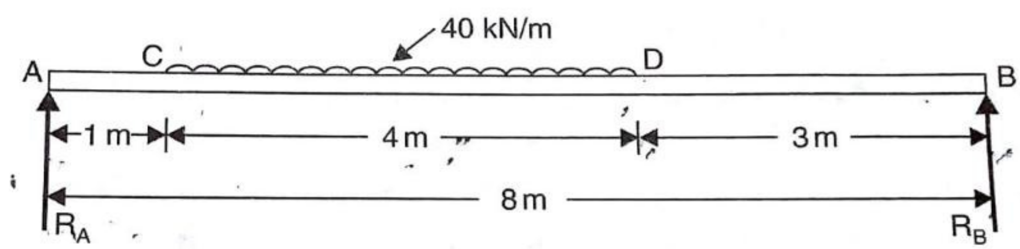
$EIy_{max} = \frac{60 X 2.87^3}{6} - (163.33 X 2.87) - \frac{48(2.87-1)^3}{6}$

$EIy_{max} = 236.399 - 468.757 - 52.31 = -284.668$

$\therefore y_{max} = \frac{-284.668}{EI} = \frac{-284.668}{200X10^6 85X10^{-6}} = -.01674m$

$\therefore$ **y_{max} = -16.74 mm**

Example.3.9. A beam of length 8m is simply supported at its ends. It carries a uniformly distributed load of 40 kN/m as shown in Fig. Determine the deflection of the beam at its mid point and also the position of maximum deflection and maximum deflection. Take $E = 2 X 10^5 N/mm^2$ and $I = 4.3 X 10^8 mm$



Given Data:

Length, $L = 8m$

u.d.l, $w = 40KN/m$

$E = 2 \text{ X } 10^5 N/mm^2$

$I = 4.3 \text{ X } 10^8 mm^4$

To find

(i) The central deflection

(ii) The position and magnitude of maximum deflection.

Solution

Taking moment about A,

$R_B \text{ X } 8 - 40 \text{ X } 4 \text{ X } \left[1 + \frac{4}{2}\right] = 0$

$8R_B - 480 = 0$

$8R_B = 480$

$\therefore \quad R_B = \frac{480}{8} = 60 \text{ KN}$

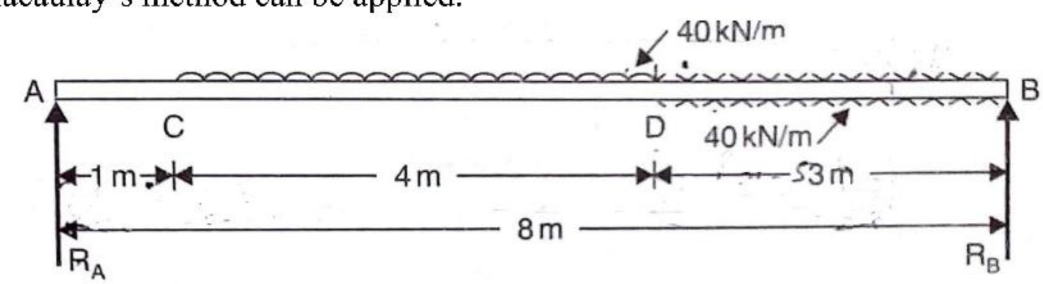
$R_A + R_B = (40 \text{ x } 4)$

$\therefore \quad R_A + 60 = 160$

$\therefore R_A = 160 - 60 = 100 \text{ kN}$

In ordet to obtain the general expression for the bending moment at a distance x from the left end A, which will apply for all values of x, it is necessary to extend the udl upto the support B, compensating with an equal upward load of 40kN/m over the span DB as shown in Fig.

Now Macaulay's method can be applied.



BM for the section X-X is given by,

$$M_x = R_A \times x \,\vdots - 40\,(x-1) \times \frac{(x-1)}{2} \,\vdots + 40\,(x-5) \times \frac{(x-5)}{2}$$

$$= 100 \times x \,\vdots - 20(x-1)^2 \,\vdots + 20(x-5)^2$$

The B.M. at any section is also given by

$$M = EI\,\frac{d^2y}{dx^2}$$

Equating the both values of B.M we get,

$$EI\,\frac{d^2y}{dx^2} = 100 \times x \,\vdots - 20(x-1)^2 \,\vdots + 20(x-5)^2$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = 100\,\frac{x^2}{2} + C_1 \,\vdots - \frac{20(x-1)^3}{3} \,\vdots + \frac{20(x-5)^3}{3} \qquad \ldots(i)$$

Where C_1 is a constant of integration. This constant of integration should be written after the first term. Also the brackets are to be integrated as a whole.

Integrating the above equation once again, we get,

$$EIy = 50\,\frac{x^3}{3} + C_1x + C_2 \,\vdots - \frac{20(x-1)^4}{3 \times 4} \,\vdots + \frac{20(x-5)^4}{3 \times 4}$$

$$= 50\frac{x^3}{3} + C_1x + C_2 \,\vdots - \frac{5(x-1)^4}{3} \vdots + \frac{5(x-5)^4}{3} \qquad \ldots(ii)$$

Where C_2 is another constant of integration. This constant is written after C_1x.

The values of C_1 and C_2 are obtained from boundary conditions. The two boundary conditions are :

(i).At x = 0, y = 0 and (ii) At x = 8m, y = 0. (since deflection is 0 at A and B)

(i) At A, x =0 and y = 0. Substituting these values in equation (ii) upto the first dotted line only as the point A lies in AC (i.e. at first portion), we get

$$0 = 0+0+C_2$$

$$\therefore C_2 = 0$$

(ii).At B, x = 8m and y = 0. Substituting these values in equation (ii), we get

$$0 = \frac{50}{3} \times 8^3 + 8C_1 + 0 - \frac{5(8-1)^4}{3} + \frac{5(8-5)^4}{3}$$

$$8C_1 = -4666.67$$

$$C_1 = -\frac{4666.67}{8} = -583.33$$

Substituting the value of C_1 in equation (ii), we get

$$EIy = 50\frac{x^3}{3} - 583.33x \;\vdots\; -\frac{5(x-1)^4}{3} \quad \vdots\; +\frac{5(x-4)^4}{3} \quad \ldots(iii)$$

Deflection at the Centre.

This is obtained by substituting x =4 in equation (iii) up to the second dotted line we get,

$$EIy = 50\frac{4^3}{3} - 583.33 \times 4 - \frac{5(4-1)^4}{3}$$

$= - 1401.66\ kNm^3$

$= - 1401.66 \times 10^{12}\ Nmm^3$

∴ $$y = \frac{-1401.66 \times 10^{12}}{EI} = \frac{-1401.66 \times 10^{12}}{2 \times 10^5 \times 4.3 \times 10^8}$$

= -16.3mm

(i) Maximum Deflection

The maximum deflection should be between C and D Where $\frac{dy}{dx} = 0$.

∴ Put $\frac{dy}{dx} = 0$. In equation (i) up to second dotted line only

$$0 = 100\frac{x^2}{2} + C_1 - \frac{20(x-1)^3}{3}$$

$$0 = 50x^2 - 583.33 - 6.667(x-1)^3 \quad \ldots(iv)$$

The above equation is solved by trial and error method.

Let x=1, then R.H.S of equation (iv)

= 50-583.33-6.667 X 0 = -533.33

Let x=2, then R.H.S = 50 X 4 – 583.33 -6.667 X 1 = -390

Let x=3, then R.H.S = 50 X 9 – 583.33 -6.667 X 8 = -136.69

Let x=4, then R.H.S = 50 X 16 – 583.33 -6.667 X 27 = +36.58

In equation (iv), when x = 3 then R.H.S is negative but when x =4 then R.H.S is positive.

Hence exact value of x lies between 3 and 4

Let x=3.82, then R.H.S = 50 X 3.82 – 583.33 – $6.667(3.82-1)^3$ = -3.22

Let x=3.83, then R.H.S = 50 X 3.83 – 583.33 – $6.667(3.83-1)^3$ = -0.99

The R.H.S is approximately zero

∴ x = 3.83m.

Hence maximum deflection will be at a distance of 3.83m from support A.

When x = 3.83m, $y = y_{max}$

Substituting the above condition in equation viii up to the second dotted line we get,

$$EIy_{max} = \frac{50 X 3.83^3}{3} - (583.33 X 3.83) - \frac{5(3.83-1)^4}{3}$$

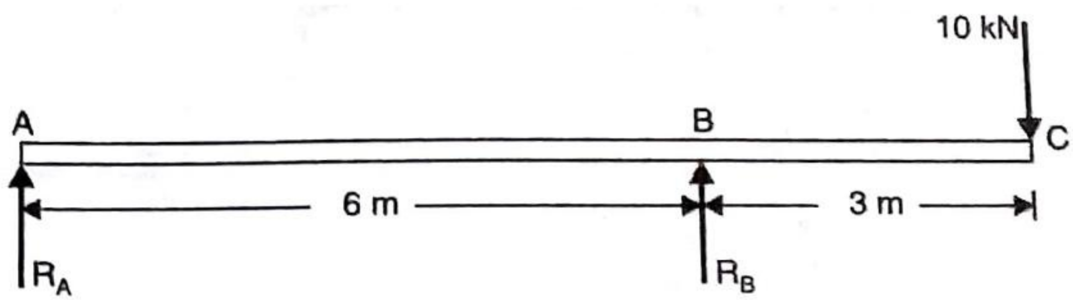
$$EIy_{max} = -1404.69 \text{ kNm}^3 = -1404.69 X 10^{12} \text{Nmm}^3$$

$$\therefore y_{max} = \frac{-1404.69 X 10^{12}}{EI} = \frac{-1404.69 X 10^{12}}{2 X 10^5 X 4.3 X 10^8}$$

$$\therefore \mathbf{y_{max} = -16.33 \ mm}$$

Example.3.10. An overhanging beam ABC is loaded as shown in Fig. Find the slopes over each support and at the right end. Find also the maximum upward deflection between supports and the deflection at the right end.

Take $E = 2 X 10^5 N/mm^2$ and $I = 5 X 10^8 mm^4$.



Sol. Given :

Point load, W = 10 KN

$E = 2 X 10^5 N/mm^2$.

$I = 5 X 10^8 mm^4$

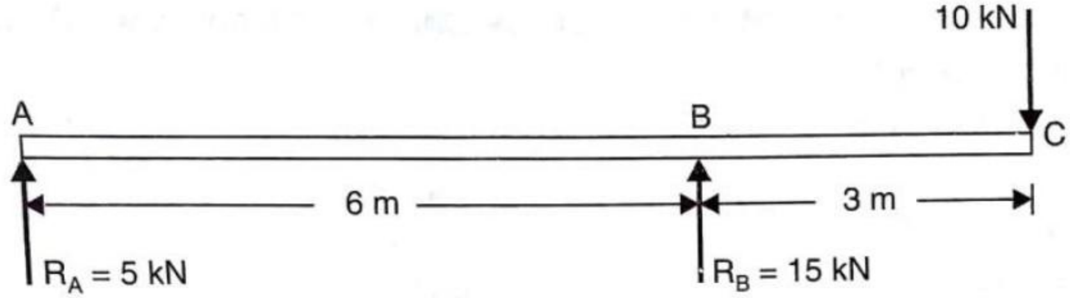
Taking moments about A, we get

$$R_B X 6 = 10 X 9$$

$$\therefore R_B = \frac{10 X 9}{6} = 15 \text{ kN}$$

$\therefore R_A$ = Total load - R_B = 10-15 = -5 kN

Hence the reactions R_A will be in the downward direction. Hence above Fig. will be modified as shown in following fig. Now write down an expression for the B.M in the last section of the beam.



BM for the section X-X is given by,

$$EI\frac{d^2y}{dx^2} = R_A X x \vdots + R_B (x-6)$$

$$= -5x \vdots + 15 (x-6)$$

Integrating the above equation, we get

$$EI\frac{dy}{dx} = -5\frac{x^2}{2} + C_1 \vdots + \frac{15(x-6)^2}{2} \quad \ldots(i)$$

Where C_1 is a constant of integration. This constant of integration should be written after the first term. Also the brackets are to be integrated as a whole.

Integrating the above equation once again, we get,

$$EIy = -\frac{5}{2}\frac{x^3}{3} + C_1x + C_2 \vdots + \frac{15(x-6)^3}{2 X 3}$$

$$= -5\frac{x^3}{6} + C_1x + C_2 \vdots + \frac{5(x-6)^3}{2} \quad \ldots (ii)$$

Where C_2 is another constant of integration. This constant is written after C_1x.

The values of C_1 and C_2 are obtained from boundary conditions. The two boundary conditions are :

(i).At x = 0, y = 0 and (ii) At x = 6m, y = 0. (since deflection is 0 at A and B)

(ii) At A, x =0 and y = 0. Substituting these values in equation (ii) upto the dotted line only as the point A lies in AB (i.e. at first portion), we get

$$0 = 0+0+C_2$$

$$\therefore C_2 = 0$$

(ii).At B, x = 6m and y = 0. Substituting these values in equation (ii), we get

$$0 = -\frac{5}{6} X\, 6^3 + 6C_1 + 0$$

$$6C_1 = 5 X 36$$

$$C_1 = \frac{5 X 36}{8} = 30$$

Substituting the value of C_1 and C_2 in equation (i) & (ii), we get,

$$EI\frac{dy}{dx} = -5\frac{x^2}{2} + 30 \;\vdots\; + \frac{15(x-6)^2}{2} \qquad \text{...(iii)}$$

And $$EI_y = -5\frac{x^3}{6} + 30x + 0 \;\vdots\; + \frac{5(x-6)^3}{2} \qquad \text{...(vi)}$$

slope over the support A

By substituting x= 0 in equation (iii) upto dotted line, we get the slope at Support A (the point x = 0 lies in the first part AB of the beam)

∴ $$EI\theta_A = -\frac{5}{2} X\, 0 + 30 = 30kNm^2 = 30 X 10^9 Nmm^2$$

$$\theta_A = \frac{30 X 10^9}{EI} = \frac{30 X 10^9}{2X\, 10^5 X 5 \times 10^8} = \mathbf{0.0003 \text{ radians.}}$$

Slope at the support B

By substituting x= 6 m in equation (iii) upto dotted line, we get the slope at Support B (the point x = 6 lies in the first part AB of the beam)

∴ $$EI\theta_B = -\frac{5}{2} X\, 6^2 + 30 = -60kNm^2 = -60 X 10^9 Nmm^2$$

$$\theta_B = \frac{-60 X 10^9}{E \times I} = \frac{-60 X 10^9}{2X\, 10^5 X\, 5 \times 10^8}$$

= - 0.0006 radians.

Slope at the right end i.e., at C

By substituting x= 9 m in equation (ii), we get the slope at C. In this case, complete equation is to be taken as the point x = 9 m lies in the last part of the beam)

∴ $$EI\theta_C = -\frac{5}{2} X\, 9^2 + 30 + \frac{15}{2}(9 - 6)^2 = -105 kNm^2$$

$$= -105 X 10^9 Nmm^2$$

$$\theta_C = \frac{-105 X 10^9}{E \times I} = \frac{-105 X 10^9}{2X\, 10^5 X 5 \times 10^8}$$

= - 0.00105 radians.

Maximum upward deflection between the supports

For the maximum deflection between the supports, $\frac{dy}{dx}$ should be zero. Hence equating the slope given by equation (iii) to be zero upto dotted line, we get

$$0 = -\frac{5}{2}x^2 + 30 = -5x^2 + 60$$

or $5x^2 = 60$ or $x = \sqrt{\frac{60}{5}} = \sqrt{12} = 3.464$ m

Now substituting x = 3.464 m in equation (iv) upto dotted line, we get maximum deflection as

$$EIy_{max} = -\frac{5}{6} \times 3.464^3 + 30 \times 3.464$$

$= 69.282$ kNm^3

$= 69.282 \times 1000 \times 10^9$ Nmm^3

$= 69.282 \times 10^{12}$ mm^3

$$y_{max} = \frac{69.282 \times 10^{12}}{2 \times 10^5 \times 5 \times 10^8}$$

$= 0.6928$ mm(upward)

Deflection at the right end i.e., at point C

By substituting x= 9 m in equation (iv), we get the deflection at point C. Here complete equation is to be taken as the point x = 9 m lies in the last part of the beam.

$$EIy_c = -\frac{5}{6} \times 9^3 + 30 \times 9 + \frac{5}{2}(9-6)^3$$

$= 270$ kNm^3 $= -270 \times 10^{12}$Nmm^3

$$y_c = \frac{-270 \times 10^{12}}{2 \times 10^5 \times 5 \times 10^8}$$

= - 2.7 mm(downward)

4.16. MOMENT AREA METHOD

Fig. shows a beam AB carrying some type of loading, and hence subjected to bending moment as shown in Fig.3.18. Let the beam bent into AP_1Q_1B and due to the load acting on the beam A be a point of zero slope and zero deflection.

Consider an element PQ of small length dx at a distance of x from B. The corresponding points on the deflected bear are P_1Q_1.

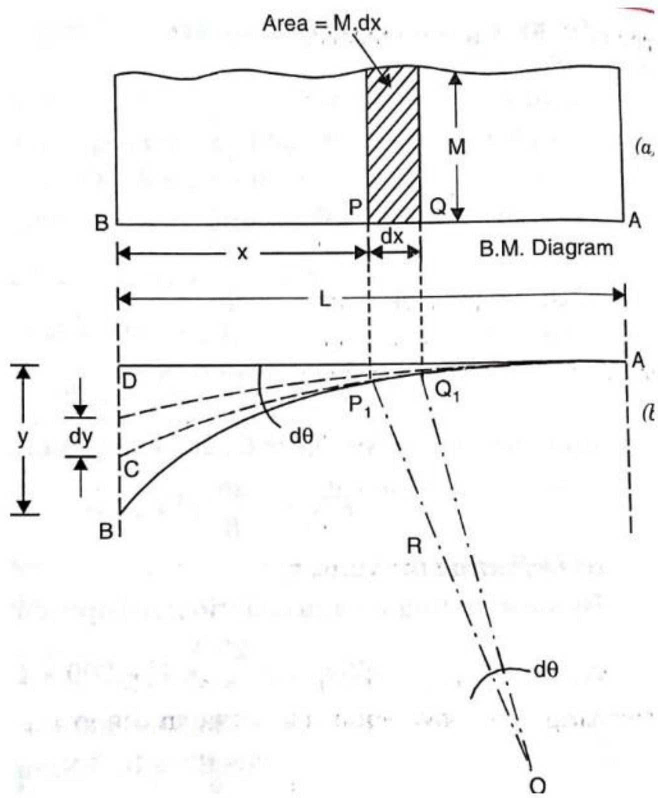
Let, R = Radius of curvature of deflected beam

dθ = Angle included between the tangent P_1 and Q_1

M = Bending moment between P and Q

dx = Length of PQ

θ = The angle in radians, included between the tangents drawn at the extremities of the beam i.e., at A and B facing the reference line.



From geometry of the bend up beam

Section P_1Q_1, We have

$$P_1Q_1 = R.d\theta$$

$$P_1Q_1 = dx$$

$$dx = R.d\theta$$

$$d\theta = \frac{dx}{R}$$

From bending moment equation.

$$\frac{M}{I} = \frac{E}{R}$$

Or $$R = \frac{EI}{M}$$

Substituting R value in dθ equation,

$$d\theta = \frac{M}{EI} X dx \qquad \ldots(i)$$

Since A is point of zero slope at B is obtained by integrating the above equation between the limits 0 and L.

$$\theta = \int_0^L \frac{M}{EI}\, dx$$

$$= \frac{1}{EI}\int_0^L M.dx$$

We Know that M.dx represents the B.M diagram of length dx.

Hence $\int_0^L M.dx$ is the area of B.M. diagram between A and B.

∴ slope θ $= \frac{1}{EI}$X Area of B.M diagram between A and B

In case, slope at A is not zero, then "Total change of slope between B and A equals the area of B.M diagram between B and A divided by the flexural rigidity EI" .

Deflection due to the bending of the portion PQ.

$$dy = x.d\theta$$

Substituting the value of dθ in equation (i) we get,

$$dy = x.\frac{M}{EI}Xdx$$

Since the deflection at A is assumed to be zero, the total deflection at B is obtained by integrating the above equation between the limits 0 and L.

$$y = \int_0^L x\frac{M}{EI}\ dx$$

$$= \frac{1}{EI}\int_0^L x.M.dx$$

But x.M.dx represents the moment of area of the BM diagram of length dx about B. This is equal to the total area of BM diagram multiplied by the distance of the C.G. of the BM diagram area from B.

$$y = \frac{1}{EI}X\ \bar{x}\ X \text{ Area of B.M diagram}$$

$$= \frac{A\bar{x}}{EI}$$

Where, A = Area of BM diagram

X= Distance of C.G of the area from B

In case the point A is not a point of zero slope and deflection

" The deflection of B with respect to the tangent at A equal to B, the first moment area about B of the area of the B.M diagram between B and A".

4.17. MOHR'S THEOREM 1:

The change of slope between any two points is equal to the net area of the B.M diagram between these points divided by EI

$$\text{Slope}\quad (\theta) = \frac{\text{Area of Bending Moment Diagram}}{\text{EI}}$$

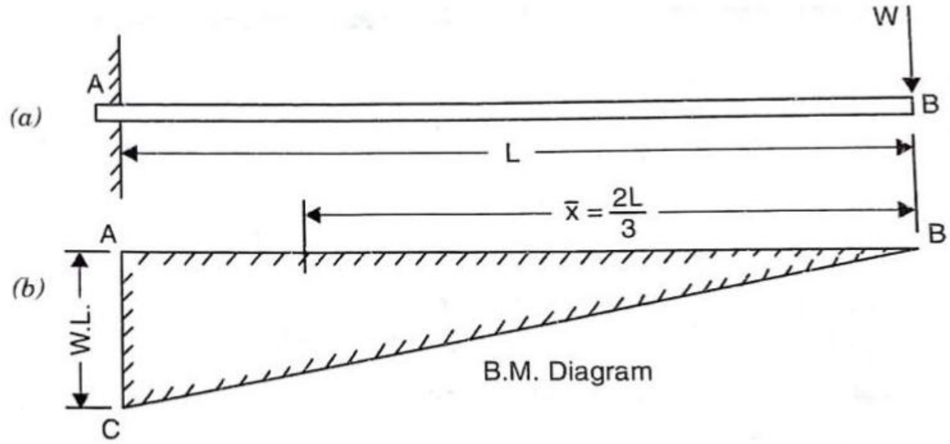
4.18. MOHR'S THEOREM 2:

The total deflection between any two points is equal to the net moment of area of BM diagram between these points divided by EI.

$$\text{Deflection (y)} = \frac{\text{Area of Bending Moment Diagram}X\bar{x}}{\text{EI}}$$

4.19.MAXIMUM SLOPE AND DEFLECTION FOR THE CANTILEVER BEAM WITH A POINT LOAD AT FREE END.

A cantilever beam AB of length L fixed at end A free at end B carrying a point load W at the free end as shown in Fig.



BM at the free end, B= 0

BM at the fixed end, A = -W.L = -WL

Let, y_B = deflection at end B with respect to A

θ_B = Slope at B

According to Mohr's Theorem I,

Slope, $\theta_B = \frac{Area\ of\ BM\ diagram\ between\ A\ and\ B}{EI}$

Area of BM diagram $= \frac{1}{2}.L.WL = \frac{WL^2}{2}$

∴ Slope at free end $\boldsymbol{\theta_B = \frac{WL^2}{2EI}}$

According to Mohr's Theorem II,

Deflection $y_B = \frac{A\,\bar{x}}{EI}$

$\bar{x}$ from B $= \frac{2}{3}L$

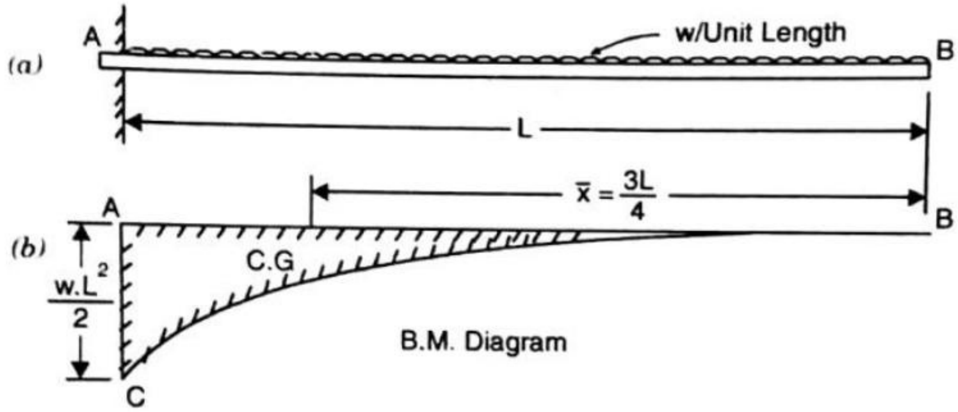
Deflection $y_B = \frac{\frac{WL^2}{2} X \frac{2L}{3}}{EI}$

Deflection at free end

$$\boldsymbol{y_B = \frac{WL^3}{3EI}}$$

4.20. MAXIMUM SLOPE AND DEFLECTION FOR THE CANTILEVER BEAM CARRYING A UNIFORMLY DISTRIBUTED LOAD.

A cantilever beam AB of length L fixed at end A free at end B carrying a uniformly distributed load of w/unit length over the entire length as shown in Fig.



BM at the free end, B= 0

BM at the fixed end, A = -W.L.$\frac{L}{2}$=- $\frac{WL^2}{2}$

Let, y_B = deflection at end B with respect to A

θ_B = Slope at B

According to Mohr's Theorem I,

Slope, $\theta B = \frac{Area\ of\ BM\ diagram\ between\ A\ and\ B}{EI}$

Area of BM diagram $= \frac{1}{3} \times L \times \frac{WL^2}{2} = \frac{WL^3}{6}$

∴ Slope at free end $\boldsymbol{\theta_B = \frac{WL^3}{6EI}}$

According to Mohr's Theorem II,

Deflection $y_B = \frac{A\,\bar{x}}{EI}$

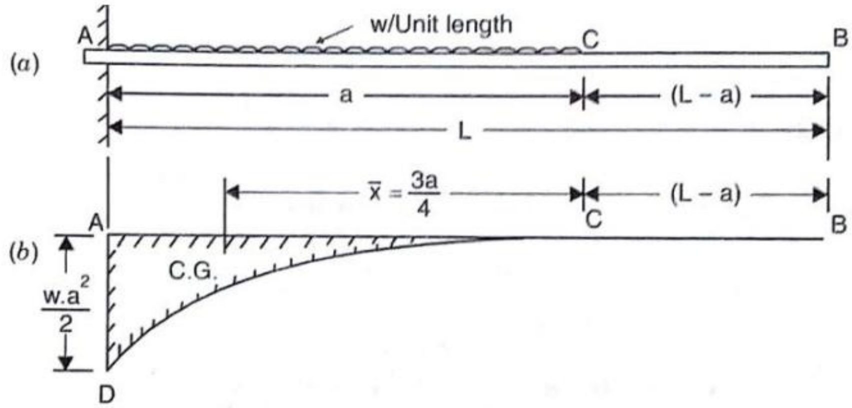
$\bar{x}$ from B $= \frac{3}{4}L$

Deflection $y_B = \frac{\frac{WL^3}{6} X \frac{3L}{4}}{EI}$

$\boldsymbol{y_B = \frac{WL^4}{8EI}}$

4.21. Maximum slope and Deflection for the Cantilever beam carrying a uniformly distributed load upto a length 'a' from the fixed end.

A cantilever beam AB of length L fixed at end A free at end B carrying a uniformly distributed load of w/unit length up to a length of 'a' from the fixed end as shown in Fig.



BM at the free end, B= 0

BM at C = 0

BM at the fixed end, A = -W.a.$\frac{a}{2}$=- $\frac{Wa^2}{2}$

Let, y_B = deflection at end B with respect to A

θ_B = Slope at B

According to Mohr's Theorem I,

Slope, $\theta B = \frac{Area\ of\ BM\ diagram\ between\ A\ and\ B}{EI}$

Area of BM diagram $= \frac{1}{3} \times a \times \frac{Wa^2}{2} = \frac{Wa^3}{6}$

∴ Slope at free end $\boldsymbol{\theta_B = \frac{Wa^3}{6EI}}$

According to Mohr's Theorem II,

Deflection $y_B = \frac{A\,\bar{x}}{EI}$

$\bar{x}$ from B $= (L - a) + \frac{3}{4}a$

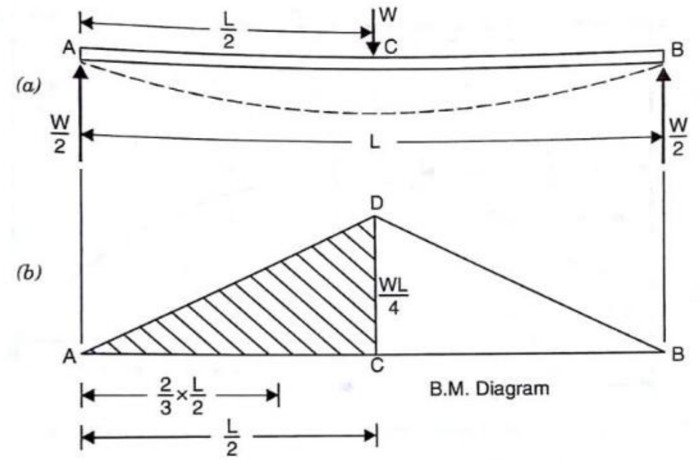
Deflection $y_B = \frac{\frac{Wa^3}{6} \times \left[(L-a)+\frac{3a}{4}\right]}{EI}$

Deflection at free end,

$$\boldsymbol{y_B = \frac{Wa^3}{6EI}\left[(L - a) + \frac{3a}{4}\right]}$$

4.22.MAXIMUM SLOPE AND DEFLECTION FOR THE SIMPLY SUPPORTED BEAM WITH A CENTRAL POINT LOAD.

A Simply supported beam AB of length L carrying a point load W at the centre of the beam (i.e., at a point C) as shown in Fig.



Since the beam is symmetrically loaded,

$R_A = R_B = \frac{Total\ load}{2} = \frac{W}{2}$

BM at the ends A and B= 0 (since A and B are simply supported ends)

BM at Centre, $C = R_A.\frac{L}{2} = \frac{W}{2}.\frac{L}{2} = \frac{WL}{4}$

Let, y_c = deflection at the centre.C

$\theta_A = \theta_B$ = Slope at Supports.A and B.

According to Mohr's Theorem I,

Slope, $\theta_A = \theta_B = \frac{Area\ of\ BM\ diagram\ between\ A\ and\ C}{EI}$

Area of BM diagram $= \frac{1}{2}.\frac{L}{2}.\frac{WL}{4} = \frac{WL^2}{16}$

∴ Slope at Supports $\boldsymbol{\theta_A = \theta_B = \frac{WL^2}{16EI}}$

According to Mohr's Theorem II,

Deflection at centre $y_C = \frac{A\bar{x}}{EI}$

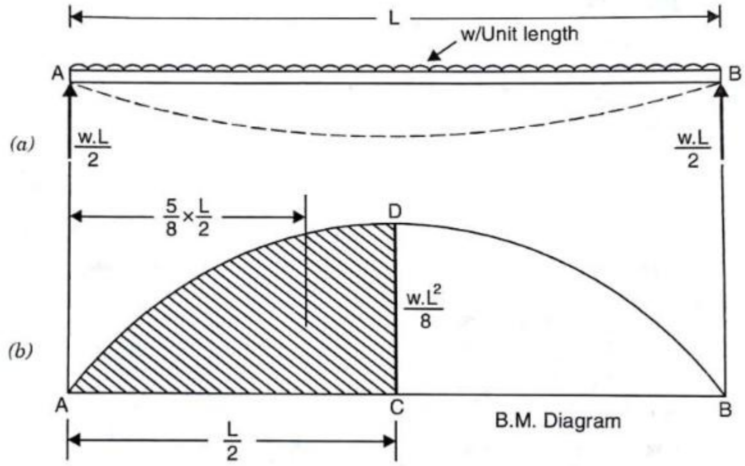
$\bar{x}$ from A $= \frac{2}{3}\frac{L}{2} = \frac{2L}{6}$

Deflection $y_c = \frac{\frac{WL^2}{16} X \frac{2L}{6}}{EI}$

Deflection at centre $\boldsymbol{y_c = \frac{WL^3}{48EI}}$

4.23. MAXIMUM SLOPE AND DEFLECTION FOR THE SIMPLY SUPPORTED BEAM WITH UNIFORMLY DISTRIBUTED LOAD.

A Simply supported beam AB of length L carrying a udl of w/unit length as shown in Fig.



Since the beam is symmetrically loaded,

$$R_A = R_B = \frac{Total\ load}{2} = \frac{wL}{2}$$

BM at the ends A and B= 0 (since A and B are simply supported ends)

BM at Centre, $C = R_A.\frac{L}{2} - \frac{wL}{2}.\frac{L}{4} = \frac{wL}{2}.\frac{L}{2} - \frac{wL^2}{8} = \frac{WL^2}{8}$

Let, y_c = deflection at the centre.C

$\theta_A = \theta_B$ = Slope at Supports.A and B.

According to Mohr's Theorem I,

Slope, $\theta_A = \theta_B = \frac{Area\ of\ BM\ diagram\ between\ A\ and\ C}{EI}$

Area of BM diagram $= \frac{2}{3}.\frac{L}{2}.\frac{WL^2}{8} = \frac{WL^3}{24}$

∴ Slope at Supports $\boldsymbol{\theta_A = \theta_B = \frac{WL^3}{24EI}}$

According to Mohr's Theorem II,

Deflection at centre $y_C = \frac{A\,\bar{x}}{EI}$

$\bar{x}$ from A $= \frac{5}{8}\frac{L}{2} = \frac{5L}{16}$

Deflection $y_c = \frac{\frac{WL^3}{24} X \frac{5L}{16}}{EI}$

Deflection at centre $\boldsymbol{y_c = \frac{5WL^4}{384EI}}$

Example.3.11. A cantilever beam of 4m long carries a point load of 9KN at the free end and an UDL of 8kN/m over a length of 2m from the fixed end. Determine the maximum slope and deflection by area moment method. Take $E = 2.2 \times 10^5$Mpa and $I = 22.5 \times 10^6 mm^4$.

Given Data:

Span, $L = 4m$

Point load at free end $W = 9KN$

Udl, $w = 8kN/m$ over a length of 2m from the fixed end

Young's Modulus, $E = 2.2 X 10^5 Mpa$

$= 2.2 X 10^5 X 10^6 pa$

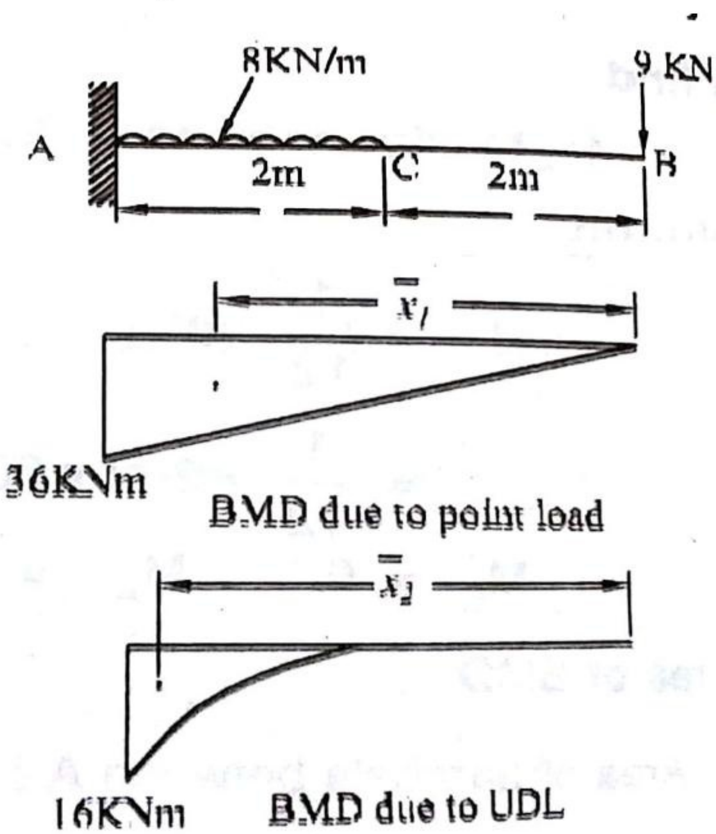
$= 2.2 X 10^5 X 10^6 N/m^2$

$= 2.2 X 10^5 X \frac{10^6}{10^6} = 2.2 X 10^5 N/mm^2$.

Moment of Inertia $I = 22.5 X 10^6 mm^4$.

To Find

The maximum slope and deflection



Solution

Bending moments

Due to point load,

$M_B = 0$ $M_A = -9 X 4 = -36kNm$

Due to UDL

$M_B = M_C = 0$ $M_A = -8 X 2 X \frac{4}{2}$ = -16 kNm

Area of BMD

Area of BMD due to point load,

$$A_1 = \frac{1}{2} X 4 X 36$$

$$=72kNm^2 = 72 X 10^9 Nmm^2.$$

Area of BMD due to UDL,

$$A2 = \frac{1}{3} X 2 X 16$$

$$= 10.67 kNm^2 = 10.67 X 10^9 Nmm^2$$

Centrodial distances from free end

$$\overline{x_1} = \frac{2}{3} X 4 = 2.67m = 2.67 X 10^3 mm$$

$$\overline{x_2} = 2 + \frac{3}{4} X 2 = 3.5m = 3.5 X 10^3 mm$$

Maximum slope

Applying Mohr's Theorem I

Maximum slope at free end,

$$\theta_B = \frac{Area\ of\ BMD}{EI} = \frac{A_1 + A_2}{EI}$$

$$= \frac{(72+10.67)\ X\ 10^9}{(2.2 X 10^5 X 22.5 X 10^6)}$$

= 0.0167 radians

Maximum Deflection

Applying Mohr's Theorem II

Maximum Deflection at free end,

$$y_B = \frac{(Area\ of\ BMD) X\ \bar{x}}{EI}$$

$$= \frac{A_1\bar{x}_1 + A_2\bar{x}_2}{EI}$$

$$= \frac{(72X10^9X2.67X10^3)+(10.67X10^9X3.5X10^3)}{(2.2X10^5X22.5X10^6)}$$

= 46.38mm.

Example.3.12. A Cantilever of 4m span carries a UDL of 20 kN/m run spread over its entire length. In addition to UDL it carries a concentrated load of 30 kN at the free end. Calculate the slope and deflection at the free end by moment area method. Take $E = 2 X 10^5 N/mm^2$ and $I = 8X10^7 mm^4$.

Given Data:

Span, $L = 4m$

Udl, $w = 20kN/m$

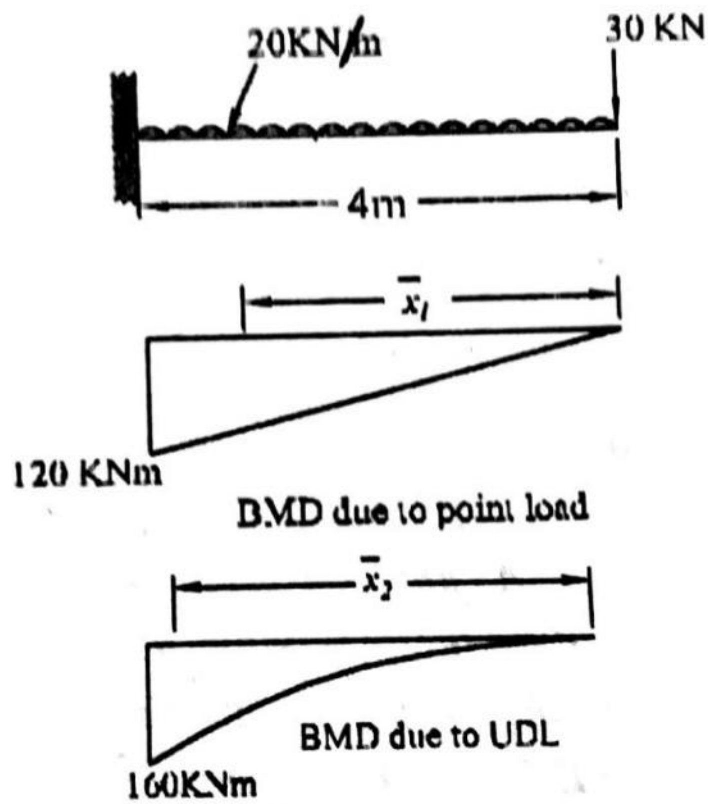
Point load at free end $W = 30KN$

Young's Modulus, $E = 2 X 10^5 N/mm^2$

Moment of Inertia $I = 8 X 10^7 mm^4$.

To Find

The maximum slope and deflection



Solution

Bending moments

Due to point load,

$M_B = 0$ $M_A = -30 X 4 = -120kNm$

Due to UDL

$M_B = 0$ $MA = -20 X 4 X \frac{4}{2} = -160kNm$

Area of BMD

Area of BMD due to point load,

$A_1 = \frac{1}{2} X 4 X 120$

$=240kNm^2 = 240 X 10^9 Nmm^2$.

Area of BMD due to UDL,

$A_2 = \frac{1}{3} X 4 X 160$

$= 213.33 kNm^2 = 213.33 X 10^9 Nmm^2$

Centrodial distances from free end

$\overline{x_1} = \frac{2}{3} X 4 = 2.67m = 2.67 X 10^3 mm$

$\overline{x_2} = \frac{3}{4} X 4 = 3m = 3 X 10^3 mm$

Maximum slope

Applying Mohr's Theorem I

Maximum slope at free end,

$$\theta_B = \frac{Area\ of\ BMD}{EI} = \frac{A_1 + A_2}{EI}$$

$$= \frac{(240 + 213.33)\ X\ 10^9}{(2X10^5X8X10^7)}$$ **= 0.0283 radians**

Maximum Deflection

Applying Mohr's Theorem II

Maximum Deflection at free end,

$$y_B = \frac{(Area\ of\ BMD) X\ \bar{x}}{EI}$$

$$= \frac{A_1\bar{x}_1 + A_2\bar{x}_2}{EI}$$

$$= \frac{(240X10^9X2.67X10^3)+(213.33X10^9X3X10^3)}{(2X10^5X8X10^7)} = \mathbf{80mm}.$$

Example.3.13. A simply supported beam of hollow circular section of external diameter 200mm and internal diameter 150mm has a span of 6m. It is subjected to a central concentrated load of 50kN and a UDL of 5kN/m over the entire span. Determine the maximum slope at supports and maximum deflection at centre. Take E = 2 X 10^5N/mm^2.

Given Data:

External diameter, D = 200mm

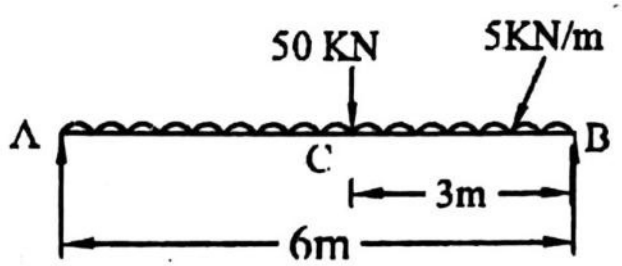
Internal diameter, d = 150mm

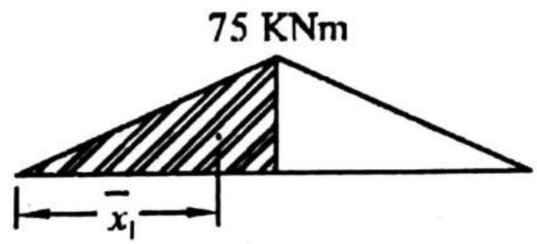
Span, L = 6m

Central point load W = 50kN

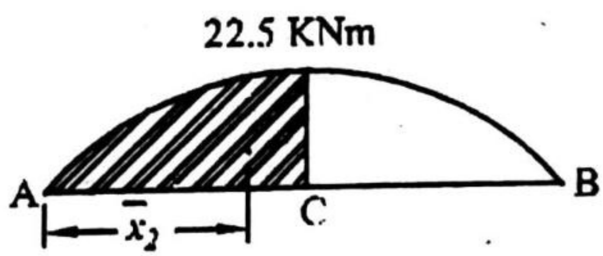
Udl w = 5 KN/m

Young's Modulus, E = 2 X 10^5N/mm^2





BMD due to Point load



B.M.D due to UDL

To Find

The maximum slope and deflection

Solution

Bending moments

Moment of inertia for a hollow circular section,

$$I = \frac{\pi}{64} X (D^4 - d^4)$$

$$= \frac{\pi}{64} X (200^4 - 150^4)$$

$$= 53.69 X 10^6 mm^4$$

Bending Moments

Due to point load, B.M at supports,

$M_A = M_B = 0$ (since A and B are simply supported)

B.M at centre, $M_C = \frac{WL}{4}$

$$= \frac{50X6}{4} = 75 kNm = 75 X 10^6 Nmm$$

Due to UDL, B.M at supports,

$M_A = M_B = 0$ (since A and B are simply supported)

B.M at centre, $M_C = \frac{wL^2}{8}$

$$= \frac{50X6^2}{4} = 22.5 kNm = 22.5 X 10^6 Nmm$$

Area of BMD

Area of BMD due to point load,

$$A_1 = \frac{1}{2} X 3 X 75$$

$$=112.5 kNm^2 = 112.5 X 10^9 Nmm^2.$$

Area of BMD due to UDL,

$$A_2 = \frac{2}{3} X 3 X 22.5$$

$$= 45 kNm^2 = 45 X 10^9 Nmm^2$$

Centrodial distances from free end

$\bar{x_1} = \frac{2}{3} X 3 = 2m \quad = 2 X 10^3 mm$

$\overline{x_2} = \frac{5}{8}$ X 3 = 1.875m= 1.875 X 10^3 mm

Maximum slope

Applying Mohr's Theorem I

Maximum slope at Supports,

$$\theta_A = \theta_B = \frac{Area\ of\ BMD}{EI} = \frac{A_1 + A_2}{EI}$$

$$= \frac{(112.5 + 45)\ X\ 10^9}{(2X10^5 X53.69X10^6)} = \mathbf{0.0147\ radians}$$

Maximum Deflection

Applying Mohr's Theorem II

Maximum Deflection at Centre,

$$y_c = \frac{(Area\ of\ BMD)X\ \bar{x}}{EI}$$

$$= \frac{A_1\bar{x}_1 + A_2\bar{x}_2}{EI}$$

$$= \frac{(112.5X10^9 X2X10^3) + (45X10^9 X1.875X10^3)}{(2X10^5 X53.69X10^6)} = \mathbf{28.81mm}$$

4.24.CONJUGATE BEAM METHOD

Conjugate beam is an imaginary beam of length equal to that of the original beam but for which the load diagram is the $\frac{M}{EI}$ diaagram (i.e., the load at any pint on the conjugate beam is equal to the B.M at that point divided by EI).

1. The slope at any section of the given beam is equal to the shear force at the corresponding section of the conjugate beam.
2. The deflection at any section for the given beam is equal to the bending moment at the corresponding section of the conjugate beam.

Hence before applying the conjugate beam method, conjugate beam is constructed. The load on the conjugate beam at any point is equal to the B.M at that point divided by EI. Hence the loading on the conjugate beam is known. Then the shear force at any point on the conjugate beam gives the slope at the corresponding point of actual beam. And the B.M at any point on the conjugate beam gives the deflection at the corresponding point of the actual beam.

4.25. SLOPE AND DEFLECTION OF A SIMPLY SUPPORTED BEAM WITH A POINT LOAD AT CENTRE

Fig. shows a simply supported beam AB of length L carrying a point load W at the centre C.

Since the beam is symmetrically loaded,

$$R_A = R_B = \frac{Total\ load}{2} = \frac{W}{2}$$

BM at the ends A and B= 0 (since A and B are simply supported ends)

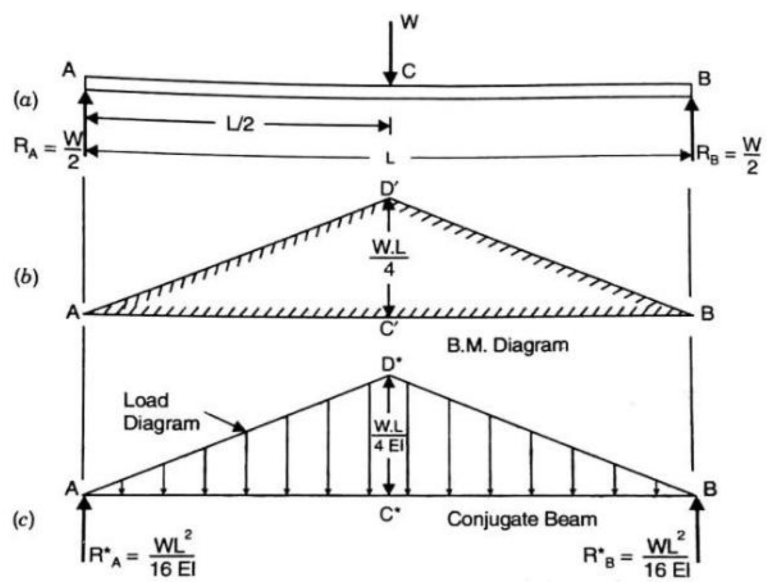
BM at Centre, $C = RA.\frac{L}{2} = \frac{W}{2}.\frac{L}{2} = \frac{WL}{4}$

The B.M. diagram is shown in Fig.

Now the conjugate beam AB can be constructed. The load on the conjugate beam will be obtained by dividing the B.M at that point by EI.

The shape of the loading on the conjugate beam will be same as of B.M diagram.

The ordinate of loading on conjugate beam will be equal to $\frac{M}{EI} = \frac{\frac{WL}{4}}{EI} = \frac{WL}{4EI}$



Let R_A^* = Reaction at A for conjugate beam

R_B^* = Reaction at B for conjugate beam

Total load on the conjugate beam

= Area of the load diagram

$= \frac{1}{2} \times AB \times C^*D^* = \frac{1}{2} \times L \times \frac{WL}{4EI}$

$= \frac{WL^2}{8EI}$

Reaction at each support for the conjugate beam will be half of the total load

$$R_A^* = R_B^* = \frac{1}{2} \times \frac{WL^2}{8EI} = \frac{WL^2}{16EI}$$

According to Conjugate beam method,

Slope at supports, θ_A = Shear force at A for the conjugate beam = R_A^*

$= \frac{\boldsymbol{WL^2}}{\boldsymbol{16EI}} = \theta_B$

And y_C = B.M at C for the conjugate beam

$= R_A^* \times \frac{L}{2}$ - Load corresponding to AC*D* X Distance of C.G. of AC*D* from C

$= \frac{WL^2}{16EI} \times \frac{L}{2} - \left(\frac{1}{2} \times \frac{L}{2} \times \frac{WL}{4EI}\right) \text{X} \left(\frac{1}{3} \times \frac{L}{2}\right)$

$= \frac{WL^3}{32EI} - \frac{WL^3}{96EI}$

$= \frac{\boldsymbol{WL^3}}{\boldsymbol{48EI}}$

4.26. SLOPE AND DEFLECTION OF A SIMPLY SUPPORTED BEAM WITH UNIFORMLY DISTRIBUTED LOAD

Fig. shows a simply supported beam AB of length L carrying a UDL w/m over its entire length

Since the beam is symmetrically loaded,

$$R_A = R_B = \frac{Total\ load}{2} = \frac{wL}{2}$$

BM at the ends A and B= 0 (since A and B are simply supported ends)

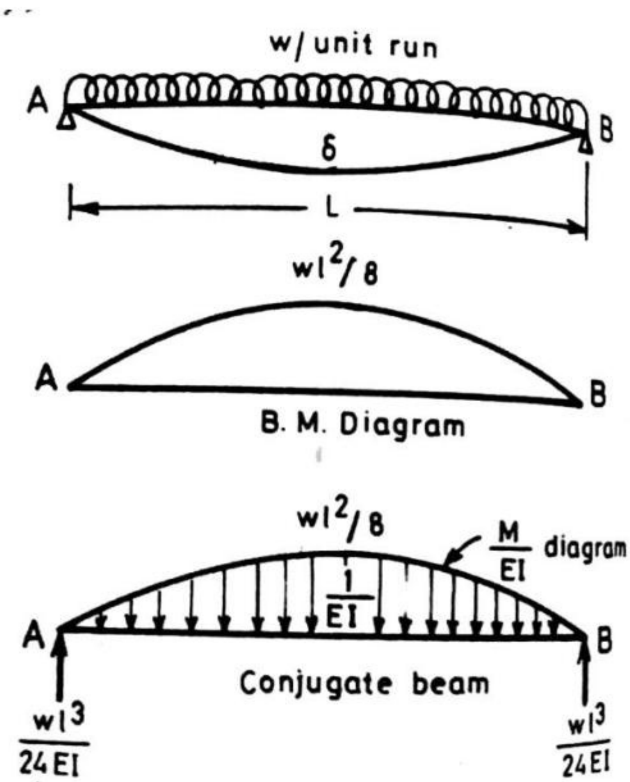
BM at Centre, $C = \frac{WL^2}{8}$

The B.M. diagram is shown in Fig.

Now the conjugate beam AB can be constructed. The load on the conjugate beam will be obtained by dividing the B.M at that point by EI.

The shape of the loading on the conjugate beam will be same as of B.M diagram.

The ordinate of loading on conjugate beam will be equal to $\frac{M}{EI} = \frac{\frac{WL^2}{8}}{EI} = \frac{WL^2}{8EI}$



Let R_A^* = Reaction at A for conjugate beam

R_B^* = Reaction at B for conjugate beam

Total load on the conjugate beam

= Area of the load diagram

$= \frac{2}{3} X\ L\ X\ \frac{WL^2}{8EI}$

$= \frac{WL^3}{12EI}$

Reaction at each support for the conjugate beam will be half of the total load

$R_A^* = R_B^* = \frac{1}{2} \times \frac{WL^3}{12EI}$

$= \frac{WL^3}{24EI}$

According to Conjugate beam method,

Slope at supports, θ_A = Shear force at A for the conjugate beam = R_A^*

$= \frac{\boldsymbol{WL^3}}{\boldsymbol{24EI}} = \theta_B$

And y_C = B.M at C for the conjugate beam

$= R_A^* \times \frac{L}{2}$ - Load corresponding to AC*D* X Distance of C.G. of AC*D* from C

$= \frac{WL^3}{24EI} \times \frac{L}{2} - \left(\frac{2}{3} \times \frac{L}{2} \times \frac{WL^2}{8EI}\right) X \left(\frac{3}{8} \times \frac{L}{2}\right)$

$= \frac{WL^4}{48EI} - \frac{6WL^4}{768EI} = \boldsymbol{\frac{5WL^4}{384EI}}$

Example.3.14. A beam ABCD is simply supported at it's A and D over a span of 30 m. It is made up of three portions AB, BC and CD each 10 metres in length. The moments of inertia of the section of these portions are I, 3I and 2I respectively. Where I = 2 X $10^{10}mm^4$. The beam carries a point load of 150 KN at B and a point load of 300 KN at C. Neglecting the weight of the beam calculate the slopes and deflections at A, B, C and D. Take E = 200 kN/mm^2.

Solution:

Let R_A and R_B be the reactions at the supports. Taking moment about D, We have,

R_D X 30 – 150 X 10 - 300 X 20 = 0

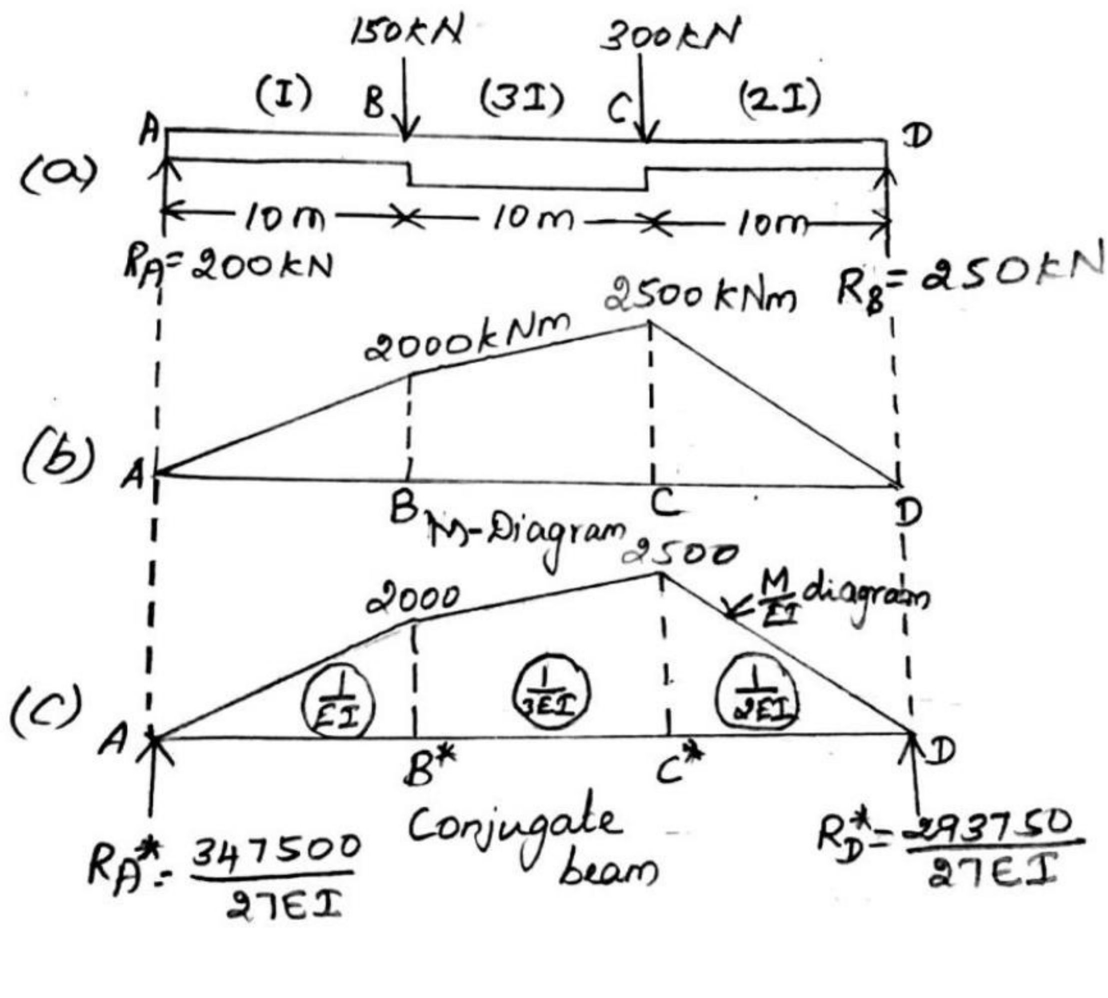
R_D X 30 = 150 X 10 + 300 X 20

R_D = 250 kN

$R_A + R_D$ = 150 +300

R_A + 250 = 450

R_A = 450-250 =200 kN.



B.M at A = 0

B.M at B = 200 X 10 = 2000 kNm

B.M at C = 200 X 20 – 150 X 10 = 2500 kNm.

B.M at D = 0

Fig. (b) shows the B.M diagram for the given beam.

Fig (c) shows the $\frac{M}{EI}$ diagram which is the loading on the conjugate beam. The thickness on the diagram is $\frac{1}{EI}$ for the portion AB, $\frac{1}{3EI}$ for the portion BC and $\frac{1}{2EI}$ for the portion CD.

The properties of the load on the conjugate beam are given below:

Load component	Magnitude	Distance From A	Moment About A
Load on AB* $= \frac{1}{2}$ X 2000 X $\frac{1}{EI}$	$\frac{10000}{EI}$	$\frac{20}{3}$	$\frac{200000}{3EI}$
Load on BC* = 2000 X 10 X 500 X $\frac{1}{3EI}$	$\frac{20000}{3EI}$	15	$\frac{100000}{EI}$
$\frac{1}{2}$ X 10 X 500 X $\frac{1}{3EI}$	$\frac{2500}{3EI}$	$\frac{50}{3}$	$\frac{125000}{9EI}$
Load on CD* $= \frac{1}{2}$ X 10 X 2500 X $\frac{1}{2EI}$	$\frac{6250}{EI}$	$\frac{70}{3}$	$\frac{437500}{3EI}$
Total	$\frac{71250}{3EI}$		$\frac{2937500}{9EI}$

Let R_A* and R_D* be the reactions at A and D for the conjugate beam.

Taking moments about A, we have,

$$R_D^* X 30 = \frac{2937500}{9EI}$$

$$\therefore R_D^* = \frac{2937500}{270EI} = \frac{293750}{27EI}$$

$$R_A^* = \frac{71250}{3EI} - \frac{293750}{27EI} = \frac{347500}{27EI}$$

Now we can easily determine the slopes and deflections at A,B,C,D for the given beam

Slope at A for the given beam = S.F. at A for the conjugate beam

$$= \frac{347500}{27EI} = \frac{347500 \, X \, 10^9}{27 \, X \, 200 \, X \, 10^3 X \, 2 \, X \, 10^{10}}$$

= **0.003218 radians**

Slope at B for the given beam = S.F. at B for the conjugate beam

$$= \frac{347500}{27EI} - \frac{1000}{EI} = \frac{77500}{27EI}$$

$$= \frac{77500\ X\ 10^9}{27\ X\ 200\ X\ 10^3 X\ 2\ X\ 10^{10}}$$ **= 0.0007176 radians**

Slope at C for the given beam = S.F. at C for the conjugate beam

$$= \frac{293750}{27EI} - \frac{6250}{EI} = \frac{125000}{27EI}$$

$$= \frac{125000\ X\ 10^9}{27\ X\ 200\ X\ 10^3 X\ 2\ X\ 10^{10}}$$ **= 0.001157 radians**

Slope at D for the given beam = S.F. at D for the conjugate beam

$$= \frac{293750}{27EI}$$

$$= \frac{293750\ X\ 10^9}{27\ X\ 200\ X\ 10^3 X\ 2\ X\ 10^{10}}$$ **= 0.00272 radians**

Deflection at A = B.M at A for the conjugate beam = 0

(Since A is simply supported)

Deflection at B = B.M at B for conjugate beam

$$= \frac{347500}{27EI} \times 10 - \frac{10000}{EI} \times \frac{10}{2} = \frac{2575000}{27EI}$$

$$= \frac{2575000 \times 10^{12}}{27\ X\ 200\ X\ 10^3 X\ 2\ X\ 10^{10}}$$ = **23.84 mm**

Deflection at C = B.M at C for conjugate beam

$$= \frac{293750}{27EI} \times 10 - \frac{6250}{EI} \times \frac{10}{3}$$

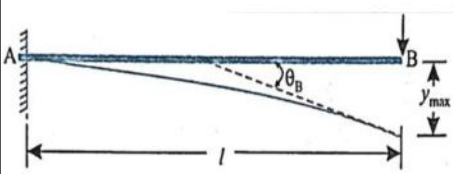
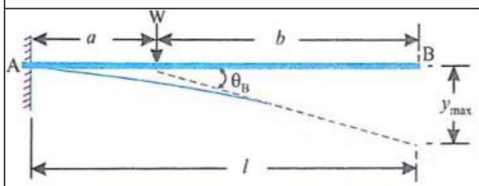
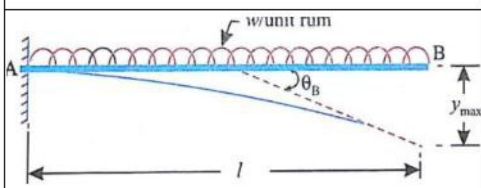
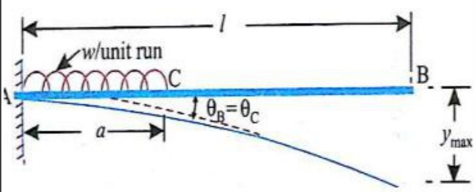
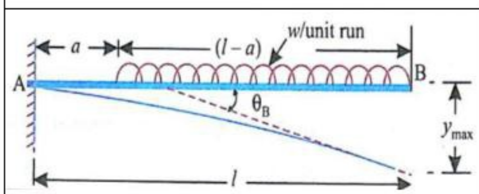
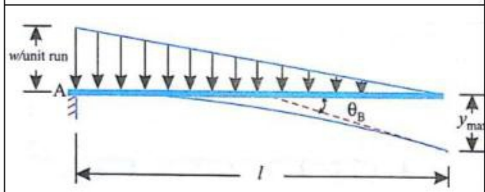
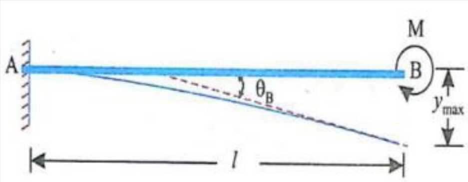
$$= \frac{2375000}{27EI}$$

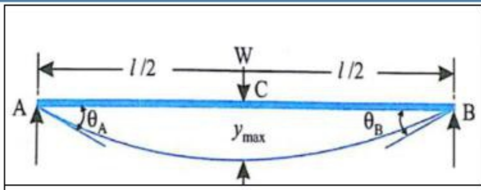
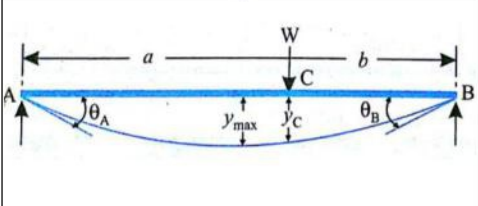
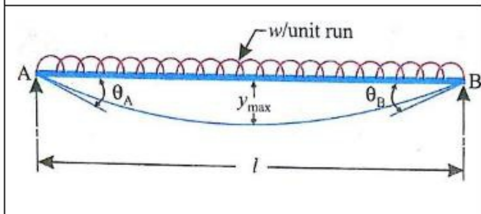
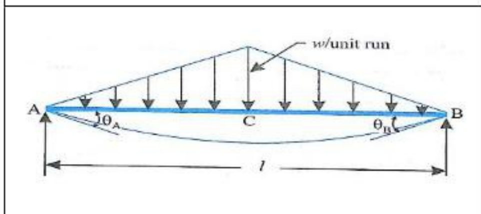
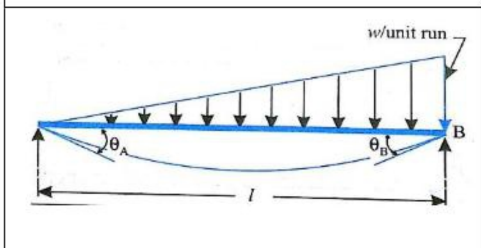
$$= \frac{2375000\ X\ 10^{12}}{27\ X\ 200\ X\ 10^3 X\ 2\ X\ 10^{10}}$$ = **21.99 mm**

Deflection at D = B.M at D for the conjugate beam = 0

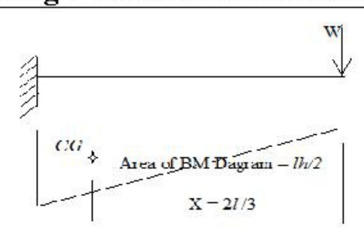
(Since D is simply supported)

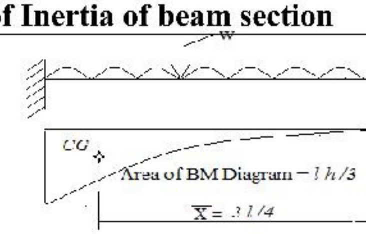
IMPORTANT TERMS

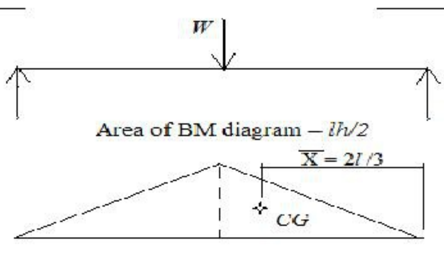
DESCRIPTION	SLOPE	DEFLECTION	MAX. BM
	$\theta_B = \dfrac{Wl^2}{2\,E\,I}$	$y_{B_{Max}} = \dfrac{Wl^3}{3\,E\,I}$	$M_A = Wl$
	$\theta_B = \theta_C = \dfrac{Wa^2}{2\,E\,I}$	$y_C = \dfrac{Wa^3}{3\,E\,I}$ and $y_{B_{Max}} = \dfrac{Wa^3}{3\,E\,I} + \dfrac{Wa^2}{3\,E\,I}(l - a)$	$M_A = Wa$
	$\theta_B = \dfrac{wl^3}{6\,E\,I}$	$y_{B_{Max}} = \dfrac{wl^4}{8\,E\,I}$	$M_A = \dfrac{wl^2}{2}$
	$\theta_B = \theta_C = \dfrac{wa^3}{6\,E\,I}$	$y_C = \dfrac{wa^4}{8\,E\,I}$ and $y_{B_{Max}} = \dfrac{wa^4}{8\,E\,I} + \dfrac{wa^3}{6\,E\,I}(l - a)$	$M_A = \dfrac{wa^2}{2}$
	$\theta_B = \dfrac{wl^3}{6\,E\,I} - \dfrac{w(l-a)^3}{6\,E\,I}$	$y_{B_{Max}} = \dfrac{wl^4}{8\,E\,I} - \left[\dfrac{w(l-a)}{8\,E\,I} + \dfrac{w(l-a)^3}{6\,E\,I}a\right]$	$M_A = w(l - a)\left[\dfrac{l-a}{2} + a\right]$
	$\theta_B = \dfrac{wl^3}{24\,E\,I}$	$y_{B_{Max}} = \dfrac{wl^4}{30\,E\,I}$	$M_A = \dfrac{wl^2}{6}$
	$\theta_B = \dfrac{Ml}{E\,I}$	$y_{B_{Max}} = \dfrac{Ml^2}{2\,E\,I}$	$M_A = M$

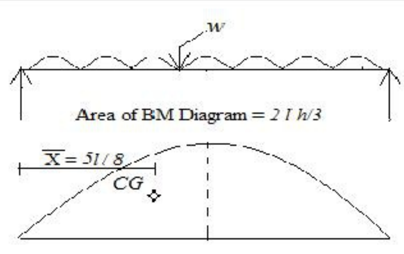
	$\theta_A = \theta_B = \frac{Wl^2}{16\,E\,I}$	$y_{B_{Max}} = \frac{Wl^3}{48\,E\,I}$	$M_C = \frac{Wl}{4}$
	$\theta_A = \frac{Wb(l^2 - b^2)}{6\,E\,I\,l}$ $\theta_B = \frac{Wa(l^2 - a^2)}{6\,E\,I\,l}$	$y_{max} = \frac{Wb(l^2 - b^2)^{3/2}}{9\sqrt{3}\,E\,I\,l}$ and $y_C = \frac{Wa^2a^2}{3\,E\,I\,l}$	$M_C = \frac{Wab}{l}$
	$\theta_A = \theta_B = \frac{wl^3}{24\,E\,I}$	$y_{max} = \frac{5}{384}\frac{wl^4}{E\,I}$	$M_C = \frac{wl^2}{8}$
	$\theta_A = \frac{7}{360}\frac{wl^3}{E\,I}$ $\theta_{B=}$	$y_{max} = \frac{2.5}{384}\frac{wl^4}{E\,I}$	$M_C = \frac{wl^3}{9\sqrt{3}}$
	$\theta_A = \theta_B = \frac{5}{192}\frac{Wl^2}{E\,I}$	$y_{max} = \frac{5}{120}\frac{wl^4}{E\,I}$	$M_C = \frac{wl^2}{12}$
Moment Area Method			
Slope	$\boldsymbol{\theta}$ **= A/EI**	**Y =** $A\bar{X}$ **/EI**	**Deflection**

A = Area of BM diagram; $\bar{X}$ **= Centre of gravity distance;**
E= Young's modulus of beam material ; I = Moment of Inertia of beam section









MECAULAY'S METHOD	
Step1: set the point XX at a distance of x m from free end/right support (OR) Near to fixed end/left support. Step2: Take moment about XX Step3: integrate the moment equation wrt x with adding of constant C_1 at the first part of equation is slope equation Step4: integrate again wrt x with adding another constant C_2 at the first part of equation is deflection equation Step5: put condition x = 0 ; y = 0 ; in first part of deflection eqn to find C_2 value ($C_2 = 0$) and Put another condition x = l ; y = 0 in whole part of same eqn to find C_1 value. Step6: then substitute C_1, C_2 value in slope & deflection Eqn to get real Slope and deflection eqn. Step7: Now substitute required point distance in slope and deflection eqn in first part to find Near right support slope and deflection. Step8: similarly include second part of eqn and substitute another x value to find slope and deflection of every point.	➢ $MI \frac{d^2y}{dx^2}$ = Moment equation ➢ $MI \frac{dy}{dx} = Slope\ equation$ ➢ $MI\ y = deflection\ equation$ ➢ To find maximum deflection put $\frac{dy}{dx} = 0$ to get x value and substitute x value in deflection to find max. deflection ➢ To evaluate integration in whole part ➢ In UDL, unload distance also assume both side UDL load act.
DOUBLE INTEGRATION METHOD	
Same procedure followed as mecaulay's method But some difference are 1. Constant C_1, C_2 are add at the end of equation 2. Equation are not separated only whole eqn used 3. Two condition are applied in whole eqn to find C_1, C_2 4. Ordnary integration followed	

THEORETICAL QUESTIONS

1. Derive an expression for the slope and deflection of a beam subjected to uniform bending moment.
2. Prove that the relation that $M = EI\ \frac{d^2y}{dx^2}$

   where M = Bending moment, E = Young's modulus, I = M.O.I.
3. Find an expression for the slope at the supports of a simply supported beam, carrying a point load at the centre.
4. Prove that the deflection at the centre of a simply supported beam, carrying a point load at the centre, is given by $y_C = \frac{WL^3}{48EI}$

   where W = Point load, L = Length of beam.

5. Find an expression for the slope and deflection of a simply supported beam, carrying a point load W at a distance 'a' from left support and at a distance 'b' from right support where a > b.
6. Prove that the slope and deflection of a simply supported beam of length L and carrying a uniformly distributed load of w per unit length over the entire length are given by

   Slope at supports $= -\frac{WL^2}{24\,EI}$, and Deflection at centre $= \frac{5}{384}\frac{WL^3}{EI}$

   Where W = Total load = w × L.
7. What is Macaulay's method ? Where is it used ? Find an expression for deflection at any section of a simply supported beam with an eccentric point load, using Macaulay's method.
8. What is moment- area method ? Where is it conveniently used ? Find the slope and deflection of a simply supported beam carrying a (i) point load at the centre and (ii) uniformly distributed load over the entire length using moment-area method.
9. What is a cantilever? What are the different methods of finding of slope and deflection of a cantilever ?
10. Derive an expression for the slope and deflection of a cantilever of length L, carrying a point load W at the free end by double integration method.
11. Solve questions 2, by moment area method.
12. Prove that the slope and deflection of a cantilever carrying uniformly distributed load over the whole length are given by,

    $\theta_B = \frac{wL^3}{6EI}$ and $y_B = \frac{wL^4}{8EI}$

    Where w = Uniformly distributed load and
    EI = Flexural rigidity.
13. Find the expression for the slope and deflection of a cantilever of length L which carries a uniformly distributed load over a length 'a' from the fixed end by
    (i) Double integration method and
    (ii) Moment area method.

14. Prove that the slope and deflection of a cantilever length L, which carries a gradually varying load from zero at the free end to w/m run at the fixed end are given by :

$$\theta_B = \frac{wL^3}{24EI} \qquad \text{and} \qquad y_B = \frac{wL^4}{30EI}$$

Where EI = Flexural rigidity.

NUMERICAL PROBLEMS

1. A wooden beam 4 m long, simply supported at its ends, is carrying a point load of 7.25 kN at its centre. The cross-section of the beam is 140 mm wide and 240 mm deep. If E for the beam = 6×10^3 N/mm^2, find the deflection at the centre.
2. A beam 5 m long , simply supported at its ends, carries a point load W at its centre. If the slope at the ends of the beam is not to exceed 1°, find the deflection at the centre of the beam.
3. Determine : (i) slope at the left support, (ii) deflection under the load and (iii) maximum deflection of a simply supported beam of length 10 m, which is carrying a point load of 10 kN at a distance 6 m from the left end.
   Take E = 2×10^5 N/mm^2 and I = 1×10^8 mm^4.
4. A beam of uniform rectangular section 100 mm wide and 240 mm deep is simply supported at its ends. It carries a uniformly distributed load of 9.125 kN/m run over the entire span of 4 m. Find the deflection at the centre if E = 1.1×10^4 N/mm^2.
5. A beam of length 4.8 m and of uniform rectangular section is simply supported at its ends. It carries a uniformly distributed load of 9.375 kN/m run over the entire length. Calculate the width and depth of the beam if permissible bending stress is 7 N/mm^2 and maximum deflection is not to exceed 0.95 cm.
   Take E for beam material = 1.05×10^4 N/mm^2.
6. Solve problem 3, using Macaulay's method.
7. A beam of length 10 m is simply supported at its ends and carries two point loads of 100 kN and 60 kN at a distance of 2 m and 5 m respectively from the left support. Calculate the deflections under each load. Find also the maximum deflection.

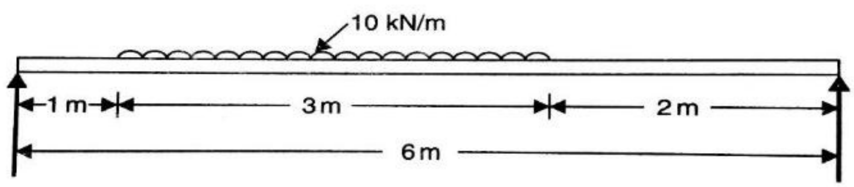
Take I = 18 × 10^8 mm^4 and E = 2 × 10^5 N/mm^2.

8. A beam of length 20 m is simply supported at its ends and carries two point loads of 4 kN and 10 kN at a distance of 8 m and 12 m from left end respectively. Calculate : (i) deflection under each load (ii) maximum deflection.

   Take E = 2 × 10^6 N/mm^2 and I = 1 × 10^9 mm^4.

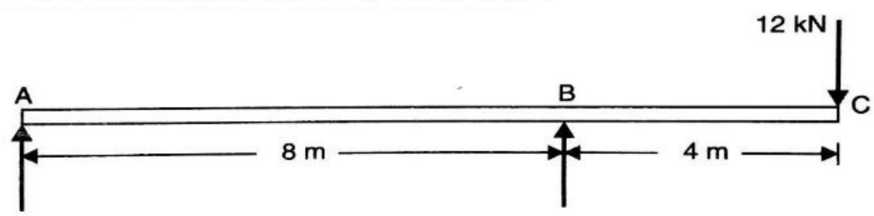
9. A beam of length 6 m is simply supported at its ends. It carries a uniformly distributed load of 10 kN/m as shown in Fig. Determine the deflection of the beam at its mid-point and also the position and the maximum deflection.

   Take $EI = 4.5 \times 10^8$ N/mm^2.



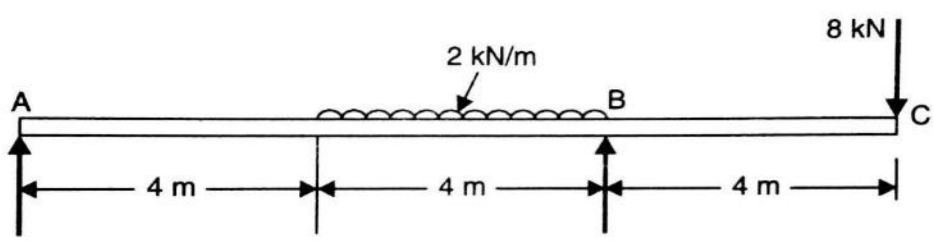
10. A beam ABC of length 12 metre has one support at the left end and other support at a distance of 8 m from the left end. The beam carries a point load of 12 kN at the right end as shown in Fig. Find the slopes over each supports and at the right end. Find also the deflection at the right end.

    Take E = 2 × 10^5 N/mm^2 and I = 5 × 10^8 mm^4.



11. An overhanging beam ABC is loaded as shown in Fig.. Determine the deflection of the beam at point C.

    Take E = 2 × 10^5 N/mm^2 and I = 5 × 10^8 mm^4.



12. A beam of span 8 m and of uniform flexural rigidity EI = 40 MN-m^2, is simply supported at its ends. It carries a uniformly distributed load of 15 kN/m run over the entire span. It is also subjected to a clockwise moment of 160 kNm at a distance of 3 m from the left support. Calculate the slope of the beam at the point of application of the moment.
13. A cantilever of length 2 m carries a point load of 30 kN at the free end. If moment of inertia $I = 10^8$ mm^4 and value of E = 2×10^5 N/mm^2, then find :
    i) slope of the cantilever at the free end and
    ii) deflection at the free end.
14. A cantilever of length 3 m carries a point load of 60 kN at a distance of 2 m from the fixed end. If E = 2×10^5 N/mm^2 and $I = 10^8$ mm^4, find :
    i) slope at the free end and
    ii) deflection at the free end.
15. A cantilever of length 30 m carries a uniformly distributed load of 24 kN/m length over the entire length. If moment of inertia of the beam =10^8 mm^3 and value of E = 2×10^5 N/mm^2, determine the slope and deflection at the free end.
16. A cantilever of length 3 m carries a uniformly distributed load over the entire length. If the slope at the free end is 0.01777 radians, find the deflection at the free end.
17. Determine the slope and deflection at the free end of a cantilever of length 4 m which is carrying a uniformly distributed load of 12 kN/m over a length of 3 m from the fixed end. Take $EI = 2 \times 10^{13}$ N/mm^2.
18. A cantilever of length 3 m carries a uniformly distributed load of 15 kN/m over a length of 2 m from the free end. If $I = 10^8$ mm^4 and E = 2×10^5 N/mm^2, find :
    i) slope at the free end and
    ii) deflection at the free end.
19. A cantilever of length 2 m carries a load of 20 kN at the free end and 30 kN at a distance 1 m from the end. Find the slope and deflection at the free end. Take E = 2.0×10^5 N/mm^2 and $I = 1.5 \times 10^8$ mm^4.

20. Determine the deflection at the free end of a cantilever which is 2 m long and carries a point load of 9 kN at the free end and a uniformly distributed load of 8 kN/m over a length of 1 m from the fixed end.
21. A cantilever of length 2 m carries a uniformly varying load of zero intensity at the free end, and 45 kN/m at the fixed end. If $E = 2 \times 10^5$ N/mm^2 and $I = 10^8$ mm^4, find the slope and deflection of the free end.
22. A cantilever of length 2 m carries a point load of 30 kN at the free end and another load of 30 kN at its centre. If $EI = 10^{13}$ N/mm^2 for the cantilever then determine by moment area method, the slope and deflection at the free end of cantilever.
23. A cantilever of length 'L' carries a U.D.L. of w per unit for a length of $\frac{L}{2}$ from the fixed end.

    Determine the slope and deflection at the free end using area moment method.