Duality in Lattice:

When " $\leq$ " is a partial order relation on a set S, then its converse " $\geq$ " is also a partial order relation on S.

Distributive lattice:

A lattice $(L, \wedge, \vee)$ is said to be distributive lattice if $\wedge$ and $\vee$ satisfies the following conditions $\forall a, b, c \in L$

$D_1: a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$D_2: a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Modular Inequality:

If $(L, \wedge, \vee)$ is a Lattice, then for any $a, b, c \in L$, $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$.

Proof:

Assume $a \leq c$

$\Rightarrow a \vee c = c$ . . . (1)

By, distributive inequality, we have

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

$\Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$ (Using (1))

Therefore, $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$. . . . (2)

Conversely, assume $a \vee (b \wedge c) \leq (a \vee b) \wedge c$

Now, by the definition of LUB and GLB, we have

$a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$

$\Rightarrow a \leq c$

Hence $a \vee (b \wedge c) \leq (a \vee b) \wedge c \Rightarrow a \leq c$ . . . (3)

From (2) and (3), we have $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$.

Hence the proof.

Modular Lattice:

A Lattice $(L, \wedge, \vee)$ is said to be Modular lattice if it satisfies the following condition.

M_1: if $a \leq c$ then $a \vee (b \wedge c) = (a \vee b) \wedge c$

Theorem: 1

Every distributive Lattice is Modular, but not conversely.

Proof:

Let $(L, \wedge, \vee)$ be the given distributive lattice

$D_1: a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ . . . (1)

Now, if $a \leq c$ then $a \vee c = c$ . . . (2)

(1)(1) $\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$= (a \vee b) \wedge c$ (using (2))

If $a \leq c$ then $a \vee (b \wedge c) = (a \vee b) \wedge c$

Therefore every distributive lattice is Modular.

But, converse is not true.

i.e., Every Modular Lattice need not be distributive.

For example, M_5 Lattice is Modular but it is not distributive.

Hence the proof.

Theorem: 2

In any distributive lattice $(L, \wedge, \vee)$ $\forall a, b, c \in L$. Prove that

$\boldsymbol{a \vee b = a \vee c, a \wedge b = a \wedge c \Rightarrow b = c}$

Proof:

Consider $b = b \vee (b \wedge a)$ (Absorption law)

$= b \vee (a \wedge b)$ (Commutative law)

$= b \vee (a \wedge c)$ (Given condition)

$= (b \vee a) \wedge (b \vee c)$ (D1 – Condition)

$= (a \vee b) \wedge (b \vee c)$ (Commutative law)

$= (a \vee c) \wedge (b \vee c)$ (Using given condition)

$= (c \vee a) \wedge (c \vee b)$ (Commutative law)

$= c \vee (a \wedge b)$ (By D1- condition)

$= c \vee (a \wedge c)$ (Given Condition)

$= c \vee (c \wedge a)$ (Commutative law)

$= c$ (Absorption law)

Lattice as a Algebraic system

A Lattice is an algebraic system $(L, \wedge, \vee)$ with two binary operation $\wedge$ and $\vee$ on L which are both commutative, associative and satisfies absorption laws.

SubLattice:

Let $(L, \wedge, \vee)$ be a lattice and let $S \subseteq L$ be a subset of L. Then $(S, \wedge, \vee)$ is a sublattice of $(L, \wedge, \vee)$ iff S is closed under both operation $\wedge$ and $\vee$.

$\forall a, b \in S \Rightarrow a \wedge b \in S$ and $a \vee b \in S$

Lattice Homomorphism:

Let $(L_1, \wedge, \vee)$ and $(L_2, *, \oplus)$ be two given lattices.

A mapping $f: L_1 \rightarrow L_2$ is called Lattice homomorphism if $\forall a, b \in L_1$

$f(a \wedge b) = f(a) * f(b)$

$f(a \vee b) = f(a) \oplus f(b)$

A homomorphism which is also 1 – 1 is called an isomorphism.

Bounded lattice:

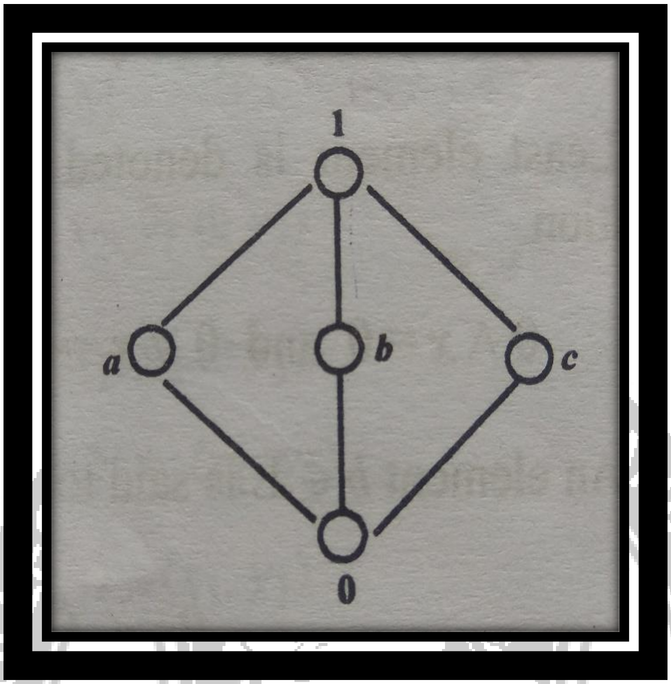
Let $(L, \wedge, \vee)$ be a given Lattice. If it has both “0” element and “1” element then it is said to be bounded Lattice. It is denoted by $(L, \wedge, \vee, 0, 1)$

Complement:

Let $(L, \wedge, \vee, 0, 1)$ be given bounded lattices. Let "a" be any element of L. We say that "b" is complement of a, if $a \wedge b = 0$ and $a \vee b = 1$ and "b" is denoted by the symbol a'. i.e., $(b = a')$. Therefore $a \wedge a' = 0$ and $a \vee a' = 1$.

Note: An element may have no complement or may have more than 1 complement.

Example for a complement.

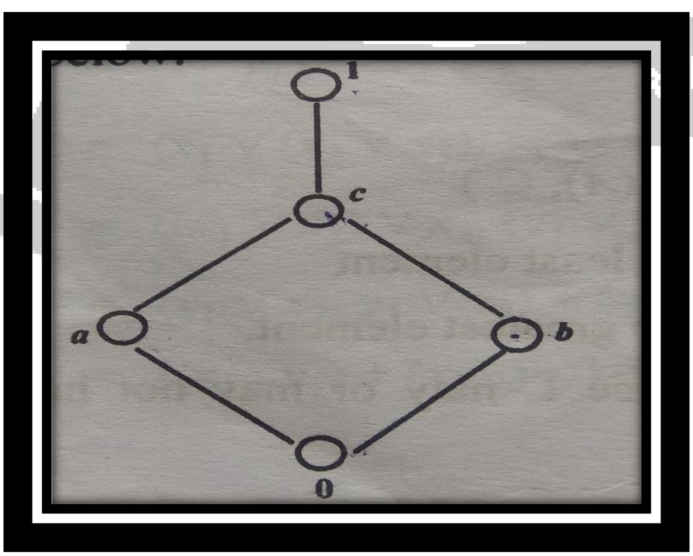


Complement of $a = a'$ is b and c.

Complement of $b = b'$ is a and c.

Complement of $c = c'$ is a and b.

In the example given below:



Complement of does not exist.

Complement of b does not exist.

Complement of c does not exist.

Complemented Lattice:

A bounded lattice $(L, \wedge, \vee, 0, 1)$ is said to be a complemented lattice if every element of L has atleast one complement.

Complete Lattice:

A lattice $(L, \wedge, \vee)$ is said to be complete lattice if every non empty subsets of L has both glb &lub.

1. Prove that in a bounded distributive lattice, the complement of any element is unique.

Proof:

Let L be a bounded distributive lattice.

Let b and c be complements of an element $a \in L$.

To prove $b = c$

Since b and c are complements of a we have

$a \wedge b = 0, a \vee b = 1, a \wedge c = 0, a \vee c = 1$

Now $b = b \wedge 1$

$= b \wedge (a \vee c)$

$= (b \wedge a) \vee (b \wedge c)$

$= (a \wedge b) \vee (b \wedge c)$

$= 0 \vee (b \wedge c)$

$= (a \wedge c) \vee (b \wedge c)$

$= (a \wedge b) \wedge c$

$= 1 \wedge c$

$= c$

Hence the proof.

2. Prove that every distributive lattice is modular.

Proof:

Let $(L, \leq)$ be a distributive lattice.

Let $a, b, c \in L$ such that $a \leq c$

To prove that $a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

Assume that $a \leq c$

To prove that $a \vee (b \wedge c) = (a \vee b) \wedge c$

When $a \leq c \Rightarrow a \vee c = c$

Therefore $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$$= (a \vee b) \wedge c$$

Hence $a \vee (b \wedge c) = (a \vee b) \wedge c$

Hence the proof.

3. Show that in a complemented distributive lattice, $a \leq b \Leftrightarrow a * b' = 0 \Leftrightarrow a' \oplus b = 1 \Leftrightarrow b' \leq a'$ (or) , $a \leq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = 1 \Leftrightarrow b' \leq a'$

Proof:

To prove $(i) \Rightarrow (ii)$

We assume that $a \leq b$

To prove that $a \wedge b' = 0$

We know that $a \leq b \Rightarrow a \wedge b = a$ and $a \vee b = b$

We take $a \vee b = b$

$\Rightarrow (a \vee b) \wedge b' = b \wedge b' = 0$

$\Rightarrow (a \wedge b') \vee (b \wedge b') = 0$

$\Rightarrow (a \wedge b') \vee 0 = 0$

$\Rightarrow (a \wedge b') = 0$

Hence $(i) \Rightarrow (ii)$

To prove $(ii) \Rightarrow (iii)$

We assume that $a \wedge b' = 0$

To prove that $a' \vee b = 1$

Taking complement on both sides

$\Rightarrow (a \wedge b')' = 0'$

$\Rightarrow a' \vee b = 1$

Therefore $a \wedge b' = 0 \Rightarrow a' \vee b = 1$

Hence $(ii) \Rightarrow (iii)$

To prove $(iii) \Rightarrow (iv)$

Assume that $a' \vee b = 1$

To prove that $b' \leq a'$

Now $a' \vee b = 1$

$\Rightarrow (a' \vee b) \wedge b' = 1 \cdot b'$

$\Rightarrow (a' \vee b) \wedge b' = b'$

$\Rightarrow (a' \wedge b') \wedge (b \wedge b') = b'$

$\Rightarrow (a' \wedge b') \vee 0 = b'$

$\Rightarrow (a' \wedge b') = b'$

$\Rightarrow (b' \wedge a') = b'$ by Commutative law

Therefore $a' \vee b = 1 \Rightarrow b' \leq a'$

Hence $(iii) \Rightarrow (iv)$

To prove $(\boldsymbol{iv}) \Rightarrow (\boldsymbol{i})$

Assume that $b' \leq a'$

To prove that $a \leq b$

We have $(b' \wedge a') = b'$

Taking complement on both sides

$\Rightarrow (b' \wedge a')' = (b')'$

$\Rightarrow b \vee a = b$

Therefore $a \vee b = b \Rightarrow a \leq b$

Hence $(iv) \Rightarrow (i)$

Hence $a \leq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = 1 \Leftrightarrow b' \leq a'$

Hence the proof.

4. State and prove DeMorgan's law of lattice.

(OR)

Let $(L, \wedge, \vee, 0, 1)$ is a complemented lattice, then prove that

1. $(a \wedge b)' = a' \vee b'$

2. $(a \vee b)' = a' \wedge b'$

Proof:

1. Claim: $(a \wedge b)' = a' \vee b'$

To prove the above, it is enough to prove that

(i) $(a \wedge b) \wedge (a' \vee b') = 0$

(ii) $(a \wedge b) \vee (a' \vee b') = 1$

(i) Let $(a \wedge b) \wedge (a' \vee b')$

$\Rightarrow \left((a \wedge b) \wedge a'\right) \vee \left((a \wedge b) \wedge b'\right)$ (Distributive law)

$\Rightarrow (a \wedge b \wedge a') \vee (a \wedge b \wedge b')$ (Associative law)

$\Rightarrow (0 \wedge b) \vee (a \wedge 0)$ $(b \wedge b' = 0)$

$\Rightarrow 0 \vee 0$ $(a \wedge 0 = 0)$

Hence $(a \wedge b) \wedge (a' \vee b') = 0$ . . . (1)

(ii) Let $(a \wedge b) \wedge (a' \vee b')$

$\Rightarrow \big(a \vee (a' \vee b')\big) \wedge \big(b \vee (a' \vee b')\big)$ (Distributive law)

$\Rightarrow (a \vee b \vee a') \wedge (a \vee b \vee b')$ (Associative law)

$\Rightarrow (1 \vee b) \wedge (a \vee 1)$ $(b \vee b' = 1)$

$\Rightarrow 1 \wedge 1 = 1$ $(a \wedge 0 = 0)$

Hence $(a \wedge b) \wedge (a' \vee b') = 1$ . . . (2)

From (1) and (2) we have, $(a \wedge b)' = a' \vee b'$

2. Claim: $(\boldsymbol{a} \vee \boldsymbol{b})' = \boldsymbol{a}' \wedge \boldsymbol{b}'$

To prove the above, it is enough to prove that

(i) $(a \vee b) \wedge (a' \wedge b') = 0$

(ii) $(a \vee b) \vee (a' \wedge b') = 1$

(i) Let $(a \vee b) \wedge (a' \wedge b')$

$\Rightarrow \big(a \wedge (a' \wedge b')\big) \vee \big(b \wedge (a' \wedge b')\big)$ (Distributive law)

$\Rightarrow (a \wedge a' \wedge b') \vee (b \wedge b' \wedge a')$ (Associative law)

$\Rightarrow (0 \wedge b') \vee (0 \wedge a')$ $(b \wedge b' = 0)$

$\Rightarrow 0 \vee 0$ $(a \wedge 0 = 0)$

Hence $(a \vee b) \wedge (a' \wedge b') = 0$ . . . (3)

(ii) Let $(a \vee b) \vee (a' \wedge b')$

$\Rightarrow \big((a \vee b) \vee a'\big) \wedge \big((a \vee b) \vee b'\big)$ (Distributive law)

$\Rightarrow (a \vee b \vee a') \wedge (a \vee b \vee b')$ (Associative law)

$\Rightarrow (1 \vee b) \wedge (a \vee 1)$ $(b \vee b' = 0)$

$\Rightarrow 1 \wedge 1 = 1$ (Idempotent law)

Hence $(a \vee b) \vee (a' \wedge b') = 1$ . . . (4)

From (3) and (4) we have, $(a \vee b)' = a' \wedge b'$