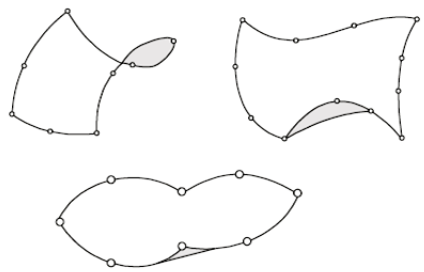
UNIT V ISOPARAMETRIC FORMULATION

PART A

1. What do you mean by uniqueness of mapping?

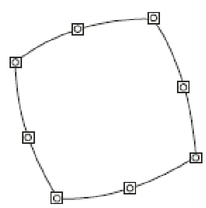
It is absolutely necessary that a point in parent element represents only one point in the isoperimetric element. Some times, due to violent distortion it is possible to obtain undesirable situation of nonuniqueness. Some of such situations are shown in Fig. If this requirement is violated determinant of Jacobiam matrix (to be explained latter) becomes negative. If this happens coordinate transformation fails and hence the program is to be terminated and mapping is corrected.



Non Uniqueness of Mapping

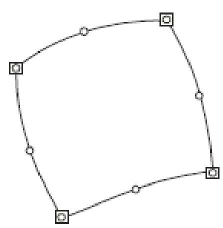
2. What do you mean by iso parametric element?*(April/May 2011)*

If the shape functions defining the boundary and displacements are the same, the element is called as **isoparametric element and** all the eight nodes are used in defining the geometry and displacement.



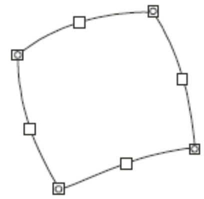
3. What do you mean by super parametric element?

The element in which more number of nodes are used to define geometry compared to the number of nodes used to define displacement are known as **superparametric element.**



4. What do you mean by sub parametric element?

The fig shows subparametric element in which less number of nodes are used to define geometry compared to the number of nodes used for defining the displacements. Such elements can be used advantageously in case of geometry being simple but stress gradient high.



5. What do you mean by iso parametric formulation?*(April/May 2011)*

The principal concept of isoparametric finite element formulation is to express the element coordinates and element displacements in the form of interpolations using the natural coordinate system of the element. These isoparametric elements of simple shapes expressed in natural coordinate system, known as master elements, are the transformed shapes of some arbitrary curves sided actual elements expressed in Cartesian coordinate system.

6. What is a Jacobian matrix of transformation?*(April/May 2011)*

It's the transformation between two different co-ordinate system. This transformation is used to evaluate the integral expression involving 'x' interms of expressions involving ε.

$$\int_{xA}^{XB} f(x)dx = \int_{-1}^{1} f(\varepsilon)d\varepsilon$$

The differential element dx in the global co-ordinate system x is related to differential element dε in natural co-ordinate system ε by

$$dx = dx/\,d\varepsilon\ .\ d\varepsilon$$

$$dx = J\ .\ d\varepsilon$$

Jacobian matrix of transformation $J = dx/\,d\varepsilon = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$

7. Differentiate the serendipity and langrangian elements

Serendipity elements	langrangian elements
In discretized element If nodes lies on corner, then the element are known as serendipity elements.	In discretized element, if nodes are present in both centre of element and corner are known as langrangian elements.

8. Explain Gauss quadrature rule.*(Nov/Dec 2012), (April/May 2011)*

The idea of Gauss Quadrature is to select "n" Gauss points and "n" weight functions such that the integral provides an exact answer for the polynomial f(x) as far as possible, Suppose if it is necessary to evaluate the following integral using end point approximation then

$$I = \int_{-1}^{1} f(x)dx$$

The solution will be

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2) + ... + w_n f(x_n)$$

$w_1, w_2, ..., w_n$ are weighted function, $x_1, x_2, ..., x_n$ are Gauss points

9. What are the differences between implicit and explicit direct integration methods?

<u>Implicit direct integration methods:</u>

(i) Implicit methods attempt to satisfy the differential equation at time ‘t’ after the solution at time “t-Δt”is found

(ii) These methods require the solution of a set of linear equations at each time step.

(iii) Normally larger time steps may be used.

(iv) Implicit methods can be conditionally or unconditionally stable.

<u>Explicit direct integration methods</u>:

(i) These methods do not involve the solution of a set of linear equations at each step.

(ii) Basically these methods use the differential equations at time ‘t’ to predict a solution at time “t+Δt”

(iii) Normally smaller time steps may be used

(iv) All explicit methods are conditionally stable with respect to size of time step.

(v) Explicit methods initially proposed for parabolic PDES and for stiff ODES with widely separated time constants.

10. State the three phases of finite element method.

The three phases of FEM is given by,

(i) Preprocessing

(ii) Analysis

(iii) Post Processing

11. List any three FEA software.*(Nov/Dec 2014)*

The following list represents FEA software as,

(i) ANSYS

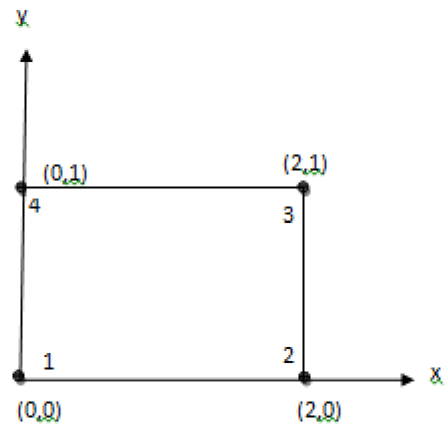
(ii) NASTRAN

(iii) COSMOS

PART-B

1. A four noded rectangular element is shown in Fig. Determine the following

1. jacobian matrix 2. Strain – Displacement matrix 3. Element Stresses.

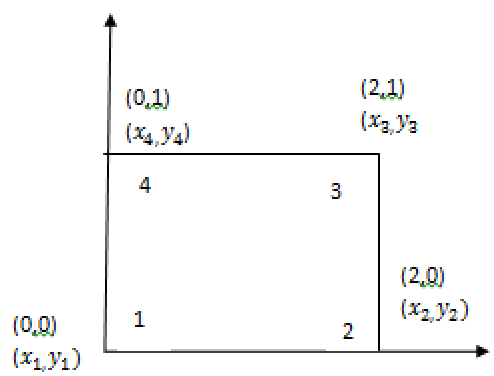


Take $E = 2\times 10^5$ N/mm^2 ; v = 0.25 ; u = $\begin{bmatrix} 0, & 0, & 0, & 0.003, & 0.004, & 0.006, & 0.004, & 0, & 0 \end{bmatrix}^T$ ε = 0 ; η = 0

Assume the plane Stress condition.

Given Data

Cartesian co – ordinates of the points 1,2,3 and 4



$x_1 = 0$; $y_1 = 0$
$x_2 = 2$; $y_2 = 0$
$x_3 = 2$; $y_3 = 1$
$x_4 = 0$; $y_4 = 1$

Young's modulus, $E = 2\times 10^5$ N/mm^2
Poisson's ratio v = 0.25

$$\text{Displacements, } u = \begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

Natural co-ordinates , ε = 0 , η = 0

To find:

1. Jacobian matrix, J
2. Strain – Displacement matrix [B]
3. Element Stress σ.

Formulae used

$$[J] = \begin{bmatrix} J_{11} J_{12} \\ J_{21} J_{22} \end{bmatrix}$$

$$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} - J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} J_{11} \\ -J_{21} J_{11} & J_{22} - J_{12} & \end{bmatrix} \times \frac{1}{4}$$

$$\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

Solution :Jacobian matrix for quadrilateral element is given by,

$$[J] = \begin{bmatrix} J_{11} J_{12} \\ J_{21} J_{22} \end{bmatrix}$$

Where ,

$J_{11} = \frac{1}{4}[-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4]$ ——————— (1)

$J_{12} = \frac{1}{4}[-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4]$ ——————— (2)

$J_{21} = \frac{1}{4}[-(1-\varepsilon)x_1 - (1+\varepsilon)x_2 + (1+\varepsilon)x_3 + (1-\varepsilon)x_4]$ ——————— (3)

$J_{22} = \frac{1}{4}[-(1-\varepsilon)y_1 - (1+\varepsilon)y_2 + (1+\varepsilon)y_3 + (1-\varepsilon)y_4]$ ——————— (4)

Substitute $x_{1,}x_{2,}x_{3,}x_{4,}y_{1,}y_{2,}y_{3,}y_{14,}$ ε and η values in equation (1), (2),(3) and (4)

(1) $J_{11} = \frac{1}{4}[0 + 2 + 2 - 0]$

⟹ **$J_{11} = 1$**

(2) ⟹ $J_{12} = \frac{1}{4}[0 + 0 + 1 - 1]$

$J_{12} = 0$

(3) ⟹ $J_{21} = \frac{1}{4}[0 - 2 + 2 - 0]$

$J_{21} = 0$

(4) ⟹ $J_{22} = \frac{1}{4}[-0 - 0 + 1 + 1]$

$J_{22} = 0.5$

$[J] = \begin{bmatrix} J_{11} J_{12} \\ J_{21} J_{22} \end{bmatrix}$

Jacobian matrix $\quad [J] = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$ ———————— (5)

$[J] = 1\times0.5 - 0$

$[J] = 0.5$

We Know that, Strain – Displacement matrix for quadrilateral element is,

$$[B] = \frac{1}{|J|}\begin{bmatrix} J_{22} - J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} J_{11} \\ -J_{21} J_{11} & J_{22} - J_{12} & \end{bmatrix} \times \frac{1}{4}$$

$$\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\varepsilon) & 0 & -(1+\varepsilon) & 0 & (1+\varepsilon) & 0 & (1-\varepsilon) \end{bmatrix}$$

Substitute $\mathbf{J_{11}, J_{12}, J_{21}, J_{22} |J|, \varepsilon}$ and η values

$$[B] = \frac{1}{0.5}\begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0.5 & 1 \end{bmatrix} \times \frac{1}{4} \begin{bmatrix} -1 & 0 & 1 & 0 & 10 - 1 & 0 \\ -1 & 0 & -1 & 0 & 10 & 1 & 0 \\ 0 & -1 & 0 & 1 & 01 & 0 & -1 \\ 0 & -1 & 0 & -101 & 0 & 1 \end{bmatrix}$$

$$[B] = \frac{1}{0.5\times4}\begin{bmatrix} -0.5 & 0 & 0.5 & 0 & 0.5 & 0 & -0.5 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 1 \\ -1 & -0.5 - 10.5 & 1 & 0.5 & 1 & -0.5 \end{bmatrix}$$

$$= \frac{0.5}{0.5\times4}\begin{bmatrix} -1 & 0 & 1 & 0 & 10 - 1 & 0 \\ 0 & -2 & 0 & -202 & 0 & 2 \\ -2 - 1 - 2 & 1 & 21 & 2 & -1 \end{bmatrix}$$

$$[B] = 0.25\begin{bmatrix} -1 & 0 & 1 & 0 & 10 - 1 & 0 \\ 0 & -2 & 0 & -202 & 0 & 2 \\ -2 - 1 - 2 & 1 & 21 & 2 & -1 \end{bmatrix}$$

We know that,

Element stress, $\boldsymbol{\sigma} = [\mathbf{D}][\boldsymbol{B}]\{\boldsymbol{u}\}$

For plane stress condition,

Stress- strain relationship matrix, $\{D\} = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$

$$= \frac{2\times10^5}{1-(0.25)^2}\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & \frac{1-0.25}{2} \end{bmatrix}$$

$$= 213.33\times 10^3\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

$$= 213.33\times10^3\times0.25 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

$$= 53.333\times10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}$$

Substitute $[D], [B]$ and $\{u\}$

$$\sigma = 53.333\times10^3 \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}\times0.25\begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \\ 0 & -2 & 0 & -2 & 0 & 2 & 0 & 2 \\ -2 & -1 & -2 & 1 & 2 & 1 & 2 & -1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 53.333\times10^3\times0.25\begin{bmatrix} -4 & 2 & 4 & -2 & 4 & 2 & -4 & 2 \\ -1 & -8 & 1 & -8 & 1 & 8 & -1 & 8 \\ -3 & -1.5 & -3 & 1.5 & 3 & 1.5 & 3 & -1.5 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.003 \\ 0.004 \\ 0.006 \\ 0.004 \\ 0 \\ 0 \end{Bmatrix}$$

$$=13.333\times10^3 \begin{Bmatrix} 0 + 0 + (4 \times 0.003) + (-2 \times 0.004) + (4 \times 0.006) + (2 \times 0.004) + 0 + 0 \\ 0 + 0 + (1 + 0.003) + (-8 \times 0.004) + (1 \times 0.006) + (8 \times 0.004) + 0 + 0 \\ 0 + 0 + (-3 \times 0.003) + (1.5 \times 0.004) + (3 \times 0.006) + (1.5 \times 0.004) + 0 + 0 \end{Bmatrix}$$

$$\{\sigma\}= 13.333\times10^3 \begin{Bmatrix} 0.036 \\ 0.009 \\ 0.021 \end{Bmatrix}$$

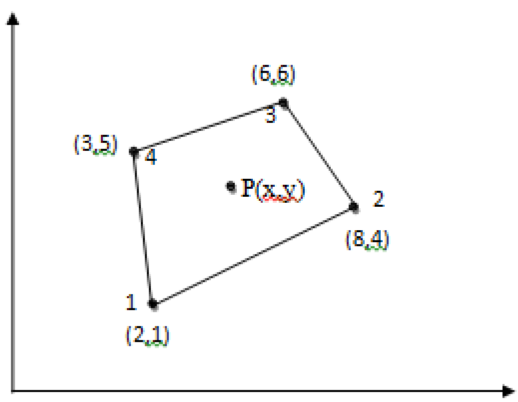
$$\{\sigma\}=\begin{Bmatrix}480\\120\\280\end{Bmatrix}\text{N/m}^2$$

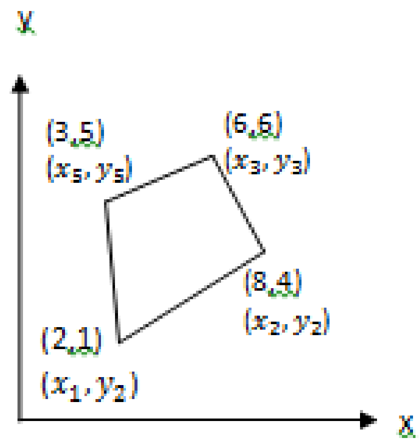
Result :

$$[J] = 0.5$$

$$\{\sigma\}=\begin{Bmatrix}480\\120\\280\end{Bmatrix}\text{N/m}^2$$

2. For the isoparametric quadrilateral element shown in Fig. the Cartesian co-ordinate of point P are (6,4). The loads 10KN and 12KN are acting in x and y direction on the point P. Evaluate the nodal equivalent forces.





Givendata :

Cartesian co- ordinates of point P,

X = 6; y = 4

The Cartesian co-ordinates of point 1,2,3 and 4 are

x_1 = 2; y_1 = 1

x_2 = 8; y_2 = 4

x_3 = 6; y_3 = 6

x_4 = 3; y_4 = 5

Loads ,$F_x = 10KN F_y = 12KN$

To find : Nodal equivalent forces for x and y directions, i,e., $F_{1x}, F_{2x}, F_{3x}, F_{4x}, F_{1y}, F_{2y}, F_{3y}, F_{4y}$

Formulae Used

$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta)$

$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta)$

$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta)$

$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta)$

Element force vector, $\{F\}_e = [N]^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$

solution:

Shape functions for quadrilateral elements are,

$$N_1 = \frac{1}{4}(1-\varepsilon)(1-\eta) \quad \text{———} \quad (1)$$

$$N_2 = \frac{1}{4}(1+\varepsilon)(1-\eta) \quad \text{———} \quad (2)$$

$$N_3 = \frac{1}{4}(1+\varepsilon)(1+\eta) \quad \text{———} \quad (3)$$

$$N_4 = \frac{1}{4}(1-\varepsilon)(1+\eta) \quad \text{———} \quad (4)$$

Cartesian co-ordinates of the point,P(x,y)

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 \quad \text{———} \quad (5)$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 \quad \text{———} \quad (6)$$

Substitute $x, x_1, x_2, x_3, x_4, N_1, N_2, N_3, and\ N_4$ values in equation.

$6 = \frac{1}{4}[(1-\varepsilon)(1-\eta)2 + (1+\varepsilon)(1-\eta)8 + (1+\varepsilon)(1+\eta)6 + (1-\varepsilon)(1+\eta)3]$

$24 = [(1-\eta-\varepsilon+\varepsilon\eta)2 + (1-\eta+\varepsilon-\varepsilon\eta)8 + (1+\eta+\varepsilon+\varepsilon\eta)6 + (1+\eta-\varepsilon-\varepsilon\eta)3]$

$24 = 19 - \eta + 9\varepsilon - 3\varepsilon\eta$

$5 = -\eta + 9\varepsilon - 3\varepsilon\eta$

$$\boxed{9\varepsilon - \eta - 3\varepsilon\eta = 5} \quad \text{———} \quad (7)$$

Substitute $y, y_1, y_2, y_3, y_4, N_1, N_2, N_3, and\ N_4$ values in equation.

$4 = \frac{1}{4}[(1-\varepsilon)(1-\eta)1 + (1+\varepsilon)(1-\eta)4 + (1+\varepsilon)(1+\eta)6 + (1-\varepsilon)(1+\eta)5]$

$16 = [1-\eta-\varepsilon+\varepsilon\eta+4-4\eta+4\varepsilon-4\varepsilon\eta+6+6\eta+6\varepsilon+6\varepsilon\eta+5+5\eta-5\varepsilon-5\varepsilon\eta]$

$16 = [16+6\eta+4\varepsilon-2\varepsilon\eta]$

$$4\varepsilon + 6\eta - 2\varepsilon\eta = 0 \quad \text{———— (8)}$$

Equation (7) multiplied by 2 and equation (8) multiplied by (-3).

$$18\varepsilon - 2\eta - 6\varepsilon\eta = 10 \quad \text{———— (9)}$$

$$-12\varepsilon - 18\eta + 6\varepsilon\eta = 0 \quad \text{———— (10)}$$

$$6\varepsilon - 20\eta = 10$$

$$-20\eta = 10 - 6\varepsilon$$

$$20\eta = 6\varepsilon - 10$$

$$\eta = \frac{6\varepsilon - 10}{20}$$

$$\eta = 0.3\varepsilon - 0.5 \quad \text{———— (11)}$$

Substituting η value in equation (7),

$9\varepsilon - (0.3\varepsilon - 0.5) - 3\varepsilon(0.3\varepsilon - 0.5) = 5$

$$10.2\varepsilon - 0.9\varepsilon^2 - 4.5 = 0$$

$$0.9\varepsilon^2 - 10.2\varepsilon + 4.5 = 0$$

$$\varepsilon = \frac{10.2 \pm \sqrt{(-10.2)^2 - 4(0.9)(4.5)}}{2(0.9)}$$

$$= \frac{10.2 - 9.372}{1.8}$$

$$\varepsilon = 0.46$$

Substitute ε and η values in equation (1),(2),(3) and (4)

⟹ (1) $N_1 = \frac{1}{4}(1 - 0.46)(1 + 0.362)$

$$N_1 = 0.18387$$

(2) ⟹ $N_2 = \frac{1}{4}(1 + 0.46)(1 + 0.362)$

$N_2 = 0.49713$

(3) $\Longrightarrow$ $N_3 = \frac{1}{4}(1 + 0.46)(1 - 0.362)$

$N_3 = 0.23287$

(4) $\Longrightarrow$ $N_4 = \frac{1}{4}(1 - 0.46)(1 - 0.362)$

$N_3 = 0.08613$

We know that,

Element force vector, $\{F\}_e = [N]^T \begin{Bmatrix} F_x \\ F_y \end{Bmatrix}$ ———————— (12)

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} \{F_x\}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} 0.18387 \\ 0.49713 \\ 0.23287 \\ 0.08613 \end{Bmatrix} \{10\}$$

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} 1.8387 \\ 4.9713 \\ 2.3287 \\ 0.8613 \end{Bmatrix} \text{KN}$$

Similarly,

$$\begin{Bmatrix} F_{1y} \\ F_{2y} \\ F_{3y} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix} \{F_y\}$$

$$\begin{Bmatrix} F_{1y} \\ F_{2y} \\ F_{3y} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} 0.18387 \\ 0.49713 \\ 0.23287 \\ 0.08613 \end{Bmatrix} \{12\}$$

$$\begin{Bmatrix} F_{1y} \\ F_{2y} \\ F_{3y} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} 2.20644 \\ 5.96556 \\ 2.79444 \\ 1.03356 \end{Bmatrix} KN$$

Result:

Nodal forces for x directions,

$$\begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{Bmatrix} = \begin{Bmatrix} 1.8387 \\ 4.9713 \\ 2.3287 \\ 0.8613 \end{Bmatrix} KN$$

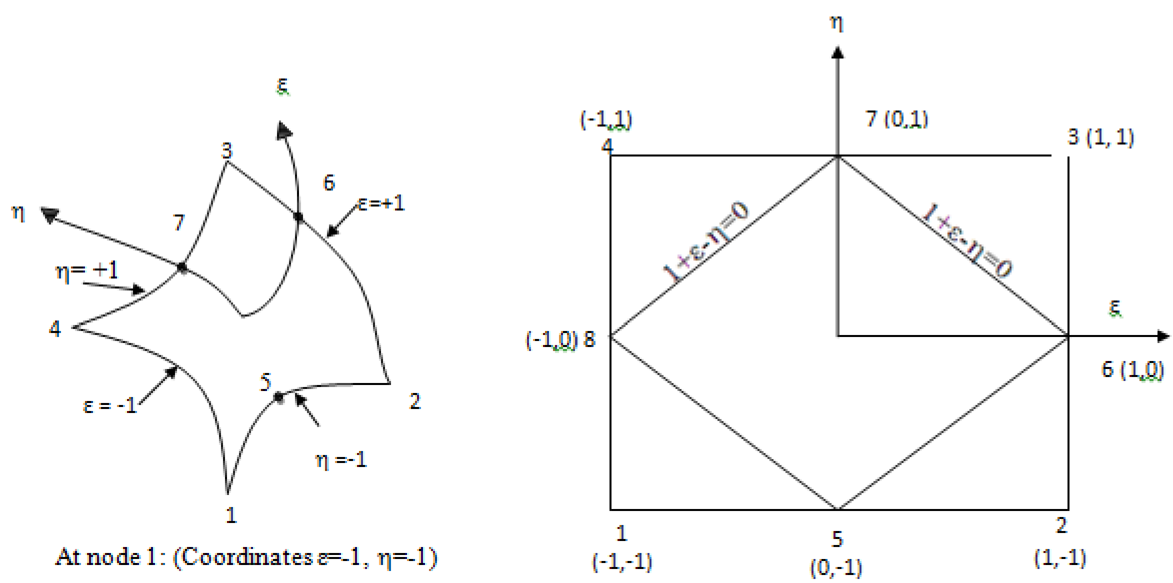
Nodal forces for y directions,

$$\begin{Bmatrix} F_{1y} \\ F_{2y} \\ F_{3y} \\ F_{4y} \end{Bmatrix} = \begin{Bmatrix} 2.20644 \\ 5.96556 \\ 2.79444 \\ 1.03356 \end{Bmatrix} KN$$

4. Derive the shape function for the Eight Noded Rectangular Element

Consider a eight noded rectangular element is shown in fig. It belongs to the serendipity family of elements. It consists of eight nodes, which are located on the boundary.

We know that, shape function $N_1 = 1$ at node 1 and 0 at all other nodes.



At node 1: (Coordinates ε=-1, η=-1)

Shape Function $N_1 = 1$ at node 1

$N_1=0$ at all other nodes

N1 has to be in the form of $N_1 = C(1-\varepsilon)(1-\eta)(1+\varepsilon+\eta)$ ——————— (1)

Where C is constant

Substitute $\varepsilon = -1$ and $\eta = -1$ in equation (1)

$N_1 = C(1+1)(1+1)(-1)$

$1 = -4C$

$C = -\frac{1}{4}$

Substitute C value in equation

$$N_1 = -\frac{1}{4}(1+\varepsilon)(1+\eta)(1+\varepsilon+\eta) \quad \text{——————— (2)}$$

At node 2 :(Coordinates $\varepsilon = 1, \eta = -1$)

Shape Function $N_2 = 1$ at node 2

N2 = 0 at all other nodes

N_2has to be in the form of $N_2 = C(1+\varepsilon)(1-\eta)(1-\varepsilon+\eta)$ ——————— (3)

Substitute $\varepsilon = 1$ and $\eta = -1$ in equation (3)

$N_2 = C(1+1)(1+1)(-1)$

$1 = -4C$

$C = -\frac{1}{4}$

Substitute C value in equation (3)

$$N_2 = -\frac{1}{4}(1+\varepsilon)(1-\eta)(1-\varepsilon+\eta) \quad \text{——————— (4)}$$

At node 3 :(Coordinates $\varepsilon = 1, \eta = 1$)

Shape Function $N_3 = 1$ at node 3

$N_3 = 0$ at all other nodes

N_3has to be in the form of $N_3 = C(1+\varepsilon)(1+\eta)(1-\varepsilon-\eta)$ ——————— (5)

Substitute $\varepsilon = 1$ and $\eta = 1$ in equation (5)

$N_3 = C\ (1+1)\ (1+1)\ (-1)$

$1 = -4C$

$C = -\frac{1}{4}$

Substitute C value in equation (5)

$$N_3 = -\frac{1}{4}(1+\varepsilon)(1+\eta)(1-\varepsilon-\eta) \quad \text{———————— (6)}$$

At node 4 :(Coordinates $\varepsilon = -1, \eta = 1$)

Shape Function $N_4 = 1$ at node 4

$N_4 = 0$ at all other nodes

N_4 has to be in the form of $N_4 = C(1-\varepsilon)(1+\eta)(1+\varepsilon-\eta)$ ———————— (7)

Substitute $\varepsilon = -1$ and $\eta = 1$ in equation (7)

$N_4 = C\ (1+1)\ (1+1)\ (-1)$

$1 = -4C$

$C = -\frac{1}{4}$

Substitute C value in equation (3)

$$N_4 = -\frac{1}{4}(1-\varepsilon)(1+\eta)(1+\varepsilon-\eta) \quad \text{———————— (8)}$$

Now , we define N_5, N_6, N_7 and N_8 at the mid points.

At node 5 :(Coordinates $\varepsilon = -1, \eta = -1$)

Shape Function $N_5 = 1$ at node 5

$N_5 = 0$ at all other nodes

N_5 has to be in the form of $N_5 = C(1-\varepsilon)(1-\eta)(1+\varepsilon)$

$N_5 = C\ (1-\varepsilon^2)(1-\eta)$ ———————— (9)

Substitute $\varepsilon = 0$ and $\eta = -1$ in equation (9)

$N_5 = C\ (1-0)(1+1)$

$1 = 2C$

$C = \frac{1}{2}$

Substitute C value in equation (9)

$$N_5 = \frac{1}{2}(1-\varepsilon^2)(1-\eta) \qquad \text{——————} (10)$$

At node 6 :(Coordinates ε = 1,η= - 1)

Shape Function N_6 = 1 at node 6

N_6 = 0 at all other nodes

N_6 has to be in the form of $N_6 = C\,(1+\varepsilon)(1-\eta)(1+\eta)$

$$N_6 = C\,(1+\varepsilon)(1-\eta^2) \qquad \text{——————} (11)$$

Substitute ε = 1 and η = 0 in equation (11)

$$N_6 = C\,(1+1)\,(1-0)$$

$$1 = 2C$$

$$C = \frac{1}{2}$$

Substitute C value in equation (11)

$$N_6 = \frac{1}{2}(1+\varepsilon)(1-\eta^2) \qquad \text{——————} (12)$$

At node 7 :(Coordinates ε = 1,η= 1)

Shape Function N_7 = 1 at node 7

N_7 = 0 at all other nodes

N_7 has to be in the form of $N_7 = C\,(1+\varepsilon)(1+\eta)(1-\varepsilon)$

$$N_7 = C\,(1-\varepsilon^2)(1+\eta) \qquad \text{——————} (13)$$

Substitute ε = 0 and η = 1 in equation (12)

$$N_7 = C\,(1-0)\,(1+1)$$

$$1 = 2C$$

$$C = \frac{1}{2}$$

Substitute C value in equation (13)

$$N_7 = \frac{1}{2}(1-\varepsilon^2)(1+\eta) \qquad \text{——————} (14)$$

At node 8 :(Coordinates ε = -1,η= 1)

Shape Function N_8 = 1 at node 8

N_8 = 0 at all other nodes

N_8 has to be in the form of $N_8 = C\,(1-\varepsilon)(1+\eta)(1-\eta)$

$$N_8 = C\,(1 - \varepsilon)(1 - \eta^2) \qquad \text{———— (15)}$$

Substitute $\varepsilon = -1$ and $\eta = 0$ in equation (15)

$$N_8 = C\,(1+1)\,(1 - 0)$$

$$1 = 2C$$

$$C = \frac{1}{2}$$

Substitute C value in equation (15)

$$N_8 = \frac{1}{2}(1 - \varepsilon)(1 - \eta^2) \qquad \text{———— (16)}$$

Shape Functions are,

$$N_1 = -\frac{1}{4}(1+\varepsilon)\,(1+\eta)\,(1+\varepsilon+\eta)$$

$$N_2 = -\frac{1}{4}(1+\varepsilon)\,(1-\eta)\,(1-\varepsilon+\eta)$$

$$N_3 = -\frac{1}{4}(1+\varepsilon)\,(1+\eta)\,(1-\varepsilon-\eta)$$

$$N_4 = -\frac{1}{4}(1-\varepsilon)\,(1+\eta)\,(1+\varepsilon-\eta)$$

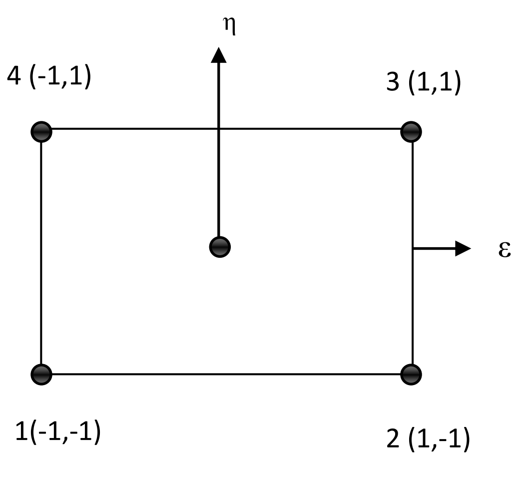
$$N_5 = \frac{1}{2}(1-\varepsilon^2)(1-\eta)$$

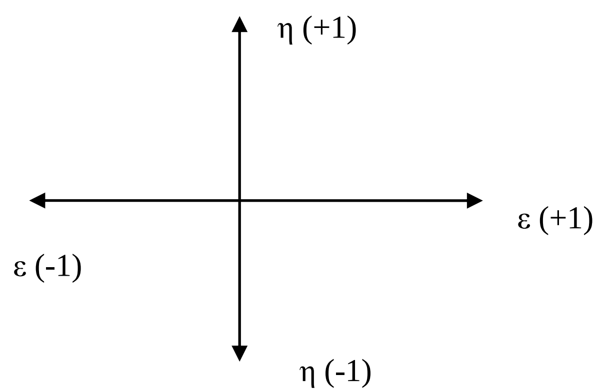
$$N_6 = \frac{1}{2}(1+\varepsilon)(1-\eta^2)$$

$$N_7 = \frac{1}{2}(1-\varepsilon^2)(1+\eta)$$

$$N_8 = \frac{1}{2}(1-\varepsilon)(1-\eta^2)$$

5. Derive the shape function for 4 noded rectangular parent element by using natural co-ordinate system and co-ordinate transformation





Consider a four noded rectangular element as shown in FIG. The parent element is defined in ε and η co-ordinates i.e., natural co-ordinates ε is varying from -1 to 1 and η is also varying -1 to 1.

We know that,

Shape function value is unity at its own node and its value is zero at other nodes.

At node 1: (co-ordinate $\varepsilon = -1$, $\eta = -1$)

Shape function $N_1 = 1$ at node 1.

$N_1 = 0$ at nodes 2, 3 and 4

N_1 has to be in the form of $N_1 = C\,(1 - \varepsilon)\,(1 - \eta)$ ———— (1)

Where, C is constant.

Substitute $\varepsilon = -1$ and $\eta = -1$ in equation (1)

$\Rightarrow$ $N_1 = C\,(1+1)(1+1)$

$\Rightarrow$ $N_1 = 4C$

$\Rightarrow$ $C = \frac{1}{4}$

Substitute C value in equation (1)

$\Rightarrow$ $\boxed{N_1 = \frac{1}{4}(1 - \varepsilon)\,(1 - \eta)}$ ———— (2)

At node 2: (co-ordinate $\varepsilon = 1$, $\eta = -1$)

Shape function $N_2 = 1$ at node 2.

$N_2 = 0$ at nodes 1, 3 and 4

N_1 has to be in the form of $N_2 = C\,(1 + \varepsilon)\,(1 - \eta)$ ———— (3)

Where, C is constant.

Substitute $\varepsilon = 1$ and $\eta = -1$ in equation (3)

$\Rightarrow$ $N_2 = C\,(1+1)\,(1+1)$

$\Rightarrow$ $N_2 = 4C$

$\Rightarrow$ $C = \frac{1}{4}$

Substitute C value in equation (1)

$\Rightarrow$ $\boxed{N_2 = \frac{1}{4}(1 + \varepsilon)\,(1 - \eta)}$ ———— (4)

At node 3: (co-ordinate $\varepsilon = 1$, $\eta = 1$)

Shape function $N_3 = 1$ at node 3.

$N_3 = 0$ at nodes 1, 2 and 4

N_1has to be in the form of $N_3 = C\,(1 + \varepsilon)\,(1 + \eta)$ ———————— (5)

Where, C is constant.

Substitute $\varepsilon = 1$ and $\eta = 1$ in equation (5)

$\Rightarrow$ $N_3 = C\,(1+1)(1+1)$

$\Rightarrow$ $N_3 = 4C$

$\Rightarrow$ $C = \frac{1}{4}$

Substitute C value in equation (1)

$\Rightarrow$ $$N_3 = \frac{1}{4}(1 + \varepsilon)\,(1 + \eta)$$ ———————— (6)

At node 4: (co-ordinate $\varepsilon = -1$, $\eta = 1$)

Shape function $N_4 = 1$ at node 4.

$N_4 = 0$ at nodes 1, 2 and 3

N_1has to be in the form of $N_4 = C\,(1 - \varepsilon)\,(1 + \eta)$ ———————— (7)

Where, C is constant.

Substitute $\varepsilon = -1$ and $\eta = 1$ in equation (1)

$\Rightarrow$ $N_4 = C\,(1+1)\,(1+1)$

$\Rightarrow$ $N_4 = 4C$

$\Rightarrow$ $C = \frac{1}{4}$

Substitute C value in equation (1)

$\Rightarrow$ $$N_4 = \frac{1}{4}(1 - \varepsilon)\,(1 + \eta)$$ ———————— (8)

Consider a point p with co-ordinate (ε, η). If the displacement function $u = \begin{Bmatrix} u \\ v \end{Bmatrix}$ represents the displacements components of a point located at (ε, η) then,

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v = N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

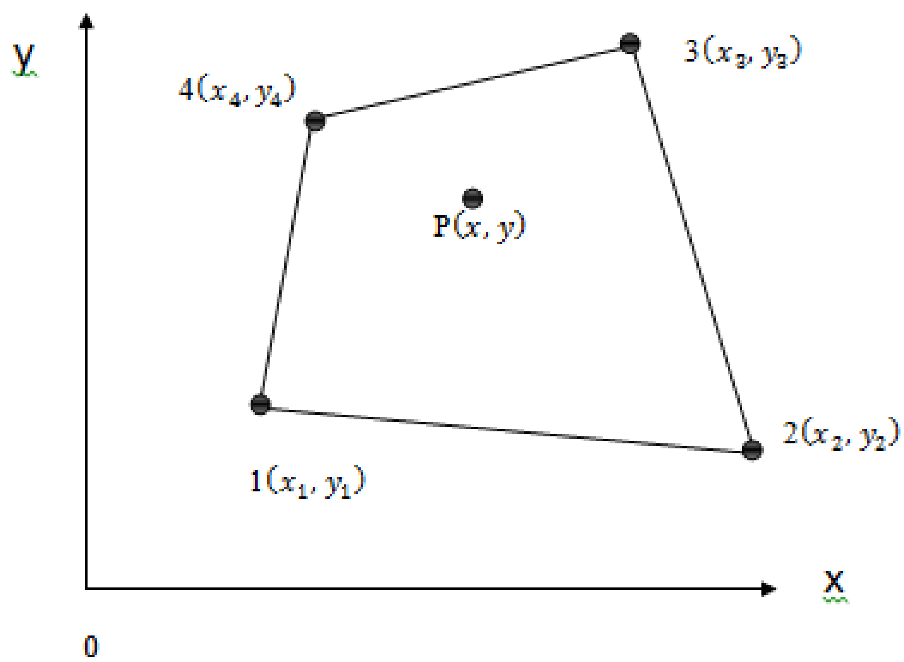
It can be written in matrix form as,

$$u = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad (9)$$

In the isoparametric formulation i,e., for global system, the co-ordinates of the nodal points are (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4). In order to get mapping the co-ordinate of point p is defined as

$$x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$$

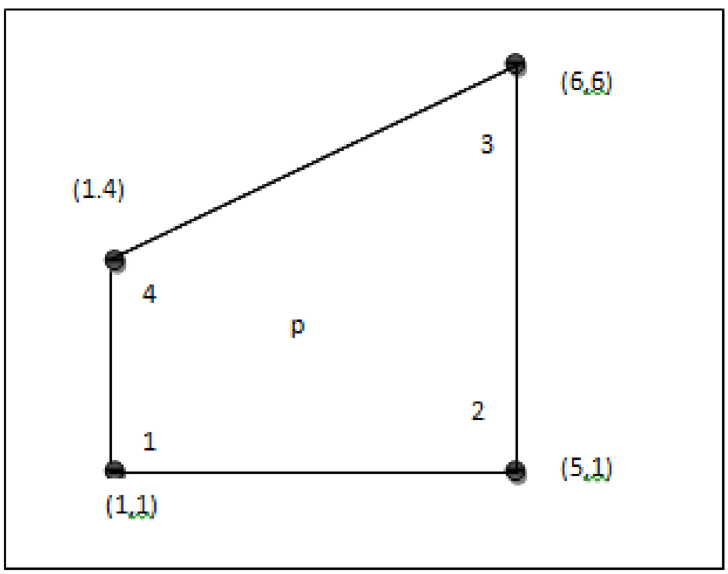
$$y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$$



The above equation can be written in matrix form as,

$$u = \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ x_4 \\ y_4 \end{Bmatrix} \qquad (10)$$

6. For the isoparametric four noded quadrilateral element shown in fig. Determine the Cartesian co-ordinates of point P which has local co-ordinatesε= 0.5 , η =0.5



Given data

Natural co-ordinates of point P

ε= 0.5

η =0.5

Cartesian co-ordinates of the point 1,2,3 and 4 P(x, y)

$x_1 = 1; \quad y_1 = 1$

$x_2 = 5; \quad y_2 = 1$

$x_3 = 6; \quad y_3 = 6$

$x_4 = 1; \quad y_4 = 4$

To find : Cartesian co-ordinates of the point P(x,y)

Formulae used:

Co -ordinate, $x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$

Co-ordinate, $y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$

Solution

Shape function for quadrilateral elements are,

$N_1 = \frac{1}{4}(1 - \varepsilon)(1 - \eta)$
$N_2 = \frac{1}{4}(1 + \varepsilon)(1 - \eta)$
$N_3 = \frac{1}{4}(1 + \varepsilon)(1 + \eta)$
$N_4 = \frac{1}{4}(1 - \varepsilon)(1 + \eta)$

Substitute ε and η values in the above equations,

$N_1 = \frac{1}{4}(1 - 0.5)(1 - 0.5) = 0.0625$

$N_2 = \frac{1}{4}(1 + 0.5)(1 - 0.5) = 0.1875$

$N_3 = \frac{1}{4}(1 + 0.5)(1 + 0.5) = 0.5625$

$N_4 = \frac{1}{4}(1 - 0.5)(1 + 0.5) = 0.1875$

We know that,

Co-ordinate, $x = N_1 x_1 + N_2 x_2 + N_3 x_3 + N_4 x_4$

$= 0.0625 \times 1 + 0.1875 \times 5 + 0.5625 \times 6 + 0.1875 \times 1$

$x = 4.5625$

Similarly,

Co-ordinate, $y = N_1 y_1 + N_2 y_2 + N_3 y_3 + N_4 y_4$

$= 0.0625\times1+0.1875\times1+0.5625\times6+0.1875\times4$

$y = 4.375$

7. Evaluate the integral I = $\int_{-1}^{1}\left(e^x + x^2 + \frac{1}{x+7}\right)dx$ using Gaussian integration with one, ,two , three integration points and compare with exact solution

<u>Given:</u>

$$I = \int_{-1}^{1}\left(e^x + x^2 + \frac{1}{x+7}\right)dx$$

To Find:

Evaluate the integral by using Gaussian.

<u>Formulae used:</u>

$$I = \int_{-1}^{1}\left(e^x + x^2 + \frac{1}{x+7}\right)dx$$

$f(x_1), w_1 f(x_1)$,

$w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$

Solution

1. point Gauss quadrature

$x_1 = 0$; $w_1 = 2$

$$f(x) = e^x + x^2 + \frac{1}{x+7}$$

$$f(x_1) = e^0 + 0 + \frac{1}{0+7}$$

$f(x_1) = 1.1428$

$w_1 f(x_1) = 2 \times 1.1428$

$= 2.29$

2. point Gauss quadrature

$x_1 = \sqrt{\frac{1}{3}}$=0.5773;

$x_2 = -\sqrt{\frac{1}{3}}$= -0.5773;

$w_1 = w_2 = 1$

$f(x) = e^x + x^2 + \frac{1}{x+7}$

$f(x_1) = e^{0.5773} + 0.5773^2 + \frac{1}{0.5773+7}$

$f(x_1)$ = 1.7812 + 0.33327 + 0.13197

$f(x_1)$ = 2.246

$w_1 f(x_1)$= 1 ×2.246

= 2.246

$f(x_2) = e^{-0.5773} + (-0.5773)^2 + \frac{1}{-0.5773+7}$

= 0.5614 + 0.3332+0.15569

$f(x_2)$ = 1.050

$w_2 f(x_2)$= 1 ×1.050

= 1.050

$w_1 f(x_1) + w_2 f(x_2)$ = 2.246 + 1.050

= 3.29

3. point Gauss quadrature

$x_1 = \sqrt{\frac{3}{5}}$=0.7745;

x_2 = 0:

$x_1 = -\sqrt{\frac{3}{5}}$= - 0.7745;

$w_1 = \frac{5}{9}$ = 0.5555;

$w_2 = \frac{8}{9}$ = 0.8888

$w_2 = \frac{5}{9}$ = 0.5555

$f(x) = e^x + x^2 + \frac{1}{x+7}$

$f(x_1) = e^{0.7745} + (0.7745^2) + \frac{1}{0.7745+7}$

$f(x_1)$ = 2.1697 + 0.6 + 0.1286

$f(x_1)$ = 2.898

$w_1 f(x_1)$= 0.55555×2.898

= 1.610

$f(x_2) = 1 + \frac{1}{7}$

$f(x_2)$ = 1.050

$w_2 f(x_2)$= 0.888×1.143

= 1.0159

$w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$ = 1.160 + 1.0159 +0.6786

= 2.8545

Exact Solution $I = \int_{-1}^{1}\left(e^x + x^2 + \frac{1}{x+7}\right)dx$

$= [e^x]_{-1}^{1} + \left[\frac{x^3}{3}\right]_{-1}^{1} + [\ln(x+7)]_{-1}^{1}$

$=(e^{+1} - e^{-1}) + \left[\frac{1}{3} - \frac{-1}{3}\right] + [\ln(1+7) - \ln(-1+7)]$

$=[2.7183 - 0.3678] + \frac{2}{3} + [\ln(8) - \ln(6)]$

$= 2.3505 + 0.6666 + [2.0794 - 1.7917]$

= 3.0171 + 0.2877 = 3.3048