BILINEAR TRANSFORMATION

A bilinear transformation is also called a linear fractional transformation because $\frac{az+b}{cz+d}$ is a fraction formed by the linear fuctions $az - b$ and $cz + d$.

Theorem: 1 Under a bilinear transformation no two points in z plane go to the same point in w plane.

Proof:

Suppose z_1 and z_2 go to the same point in the w plane under the transformation $w = \frac{az+b}{cz+d}$.

Then $\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}$

$$\Rightarrow (az_1 + b)(cz_2 + d) = (az_2 + b)(cz_1 + d)$$

$$i.e., \quad (az_1 + b)(cz_2 + d) - (az_2 + b)(cz_1 + d) = 0$$

$$\Rightarrow acz_1 z_2 + adz_1 + bcz_2 + bd - acz_1 z_2 - adz_2 - bcz_1 - bd = 0$$

$$\Rightarrow (ad - bc)(z_1 - z_2) = 0$$

$$\text{or} \quad z_1 = z_2 \qquad [\because ad - bc \neq 0]$$

This implies that no two distinct points in the z plane go to the same point in w plane. So, each point in the z plane go to a unique point in the w plane.

Theorem: 2 The bilinear transformation which transforms z_1, z_2, z_3, into w_1, w_2, w_3 is $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

Proof:

If the required transformation $w = \frac{az+b}{cz+d}$.

$$\Rightarrow w - w_1 = \frac{az+b}{cz+d} - \frac{az_1+b}{cz_1+d} = \frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}$$

$$\Rightarrow (cz + d)(cz_1 + d)(w - w_1) = (ad - bc)(z - z_1)$$

$$\Rightarrow (cz_2 + d)(cz_3 + d)(w_2 - w_3) = (ad - bc)(z_2 - z_3)$$

$$\Rightarrow (cz + d)(cz_3 + d)(w - w_3) = (ad - bc)(z - z_3)$$

$$\Rightarrow (cz_2 + d)(cz_1 + d)(w_2 - w_1) = (ad - bc)(z_2 - z_1)$$

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{\left[\frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}\right]\left[\frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}\right]}{\left[\frac{(ad-bc)(z-z_3)}{(cz+d)(cz_3+d)}\right]\left[\frac{(ad-bc)(z_2-z_1)}{(cz_2+d)(cz_1+d)}\right]}$$

$$= \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Now, $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \qquad \dots (1)$

Let : $A = \frac{w_2 - w_3}{w_2 - w_1}, B = \frac{z_2 - z_3}{z_2 - z_1}$

$$(1) \Rightarrow \frac{w - w_1}{w - w_3} A = \frac{z - z_1}{z - z_3} B$$

$$\frac{wA - w_1 A}{w - w_3} = \frac{zB - z_1 B}{z - z_3}$$

$$\Rightarrow wAz - wAz_3 - w_1 Az + w_1 Az_3 = wBz - wz_1 B - w_3 zB + w_3 z_1 B$$

$$\Rightarrow w[(A - B)z + (Bz_1 - Az_3)] = (Aw_1 - Bw_3)z + (Bw_3 z_1 - Aw_1 z_3)$$

$$\Rightarrow w = \frac{(Aw_1 - Bw_3)z + (Bw_3 z_1 - Aw_1 z_3)}{(A - B)z + (Bz_1 - Az_3)}$$

$\frac{az+b}{cz+d}$, Hence $a = Aw_1 - Bw_3, b = Bw_3 z_1 - Aw_1 z_3, c = A - B, d = Bz_1 - Az_3$

Cross ratio

Definition:

Given four point z_1, z_2, z_3, z_4 in this order, the ratio $\frac{(z-z_1)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross ratio of the points.

Note: (1) $w = \frac{az+b}{cz+d}$ can be expressed as $cwz + dw - (az + b) = 0$

It is linear both in w *and* z that is why, it is called bilinear.

Note: (2) This transformation is conformal only when $\frac{dw}{dz} \neq 0$

$$i.e., \frac{ad - bc}{(cz + d)^2} \neq 0$$

$$i.e., ad - bc \neq 0$$

If $ad - bc \neq 0$, every point in the z plane is a critical point.

Note: (3) Now, the inverse of the transformation $w = \frac{az+b}{cz+d}$ is $z = \frac{-dw+b}{cw-a}$ which is also a bilinear transformation except $w = \frac{a}{c}$.

Note: (4) Each point in the plane except $z = \frac{-d}{c}$ corresponds to a unique point in the w plane.

The point $z = \frac{-d}{c}$ corresponds to the point at infinity in the w plane.

Note: (5) The cross ratio of four points

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$ is invariant under bilinear transformation.

Note: (6) If one of the points is the point at infinity the quotient of those difference which involve this points is replaced by 1.

Suppose $z_1 = \infty$, then we replace $\frac{z-z_1}{z_2-z_1}$ by 1 (or)Omit the factors involving ∞

Example: Find the fixed points of $w = \frac{2zi+5}{z-4i}$.

Solution:

The fixed points are given by replacing $w\ by\ z$

$$z = \frac{2zi+5}{z-4i}$$

$$z^2 - 4iz = 2zi + 5\ ;\ z^2 - 6iz - 5 = 0$$

$$z = \frac{6i \pm \sqrt{-36+20}}{2} \qquad \therefore z = 5i, i$$

Example: Find the invariant points of $w = \frac{1+z}{1-z}$

Solution:

The invariant points are given by replacing $w\ by\ z$

$$z = \frac{1+z}{1-z}$$

$$\Rightarrow z - z^2 = 1 + z$$

$$\Rightarrow z^2 = -1$$

$$\Rightarrow z = \pm i$$

Example: Obtain the invariant points of the transformation $w = 2 - \frac{2}{z}$.

Solution:

The invariant points are given by

$$z = 2 - \frac{2}{z}\ ; \qquad z = \frac{2z-2}{z}$$

$$z^2 = 2z - 2; \qquad z^2 - 2z + 2 = 0$$

$$z = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

Example: Find the fixed point of the transformation $w = \frac{6z-9}{z}$.

Solution:

The fixed points are given by replacing $w = z$

i.e., $w = \frac{6z-9}{z} \Rightarrow z = \frac{6z-9}{z}$

$$\Rightarrow z^2 = 6z - 9$$

$$\Rightarrow z^2 - 6z + 9 = 0$$

$$\Rightarrow (z-3)^2 = 0$$

$$\Rightarrow z = 3,3$$

The fixed points are $3, 3$.

Example: Find the bilinear transformation that maps the points $z = 0, -1, i$ into the points $w = i, 0, \infty$ respectively.

Solution:

Given $z_1 = 0,\ \ z_2 = -1,\ \ z_3 = i,$

$w_1 = i,\ \ w_2 = 0,\ \ w_3 = \infty,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[omit the factors involving w_3, since $w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{w-i}{0-i} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\Rightarrow \frac{w-i}{-i} = \frac{z}{(z-i)}(1+i)$$

$$\Rightarrow w - i = \frac{z}{(z-i)}(-i+1)$$

$$\Rightarrow w = \frac{z}{(z-i)}(-i+1) + i = \frac{-iz+z+iz+1}{(z-i)} = \frac{z+1}{z-i}$$

Aliter: Given $z_1 = 0,\ \ z_2 = -1,\ \ z_3 = i,$

$w_1 = i,\ \ w_2 = 0,\ \ w_3 = \infty,$

Let the required transformation be

$$w = \frac{az+b}{cz+d} \ldots (1),\ ad - bc \neq 0$$

$$i = \frac{b}{d}$$

$$w_1 = \frac{az_1+b}{cz_1+d}$$

$$i = \frac{b}{d}$$

$$b = di$$

$$w_2 = \frac{az_2+b}{cz_2+d}$$

$$0 = \frac{-a+b}{-c+d}$$

$$\Rightarrow -a + b = 0$$

$$\Rightarrow a = b$$

$$w_3 = \frac{az_3+b}{cz_3+d}$$

$$\frac{1}{0} = \frac{ai+b}{ci+d}$$

$$\Rightarrow ci + d = 0$$

$$\Rightarrow d = -ci$$

$$\therefore a = b = di = c$$

$$\therefore (1) \Rightarrow w = \frac{az+a}{az+\frac{a}{i}} = \frac{z+1}{z+\frac{1}{i}} = \frac{z+1}{z-i}$$

Example: Find the bilinear transformation that maps the points $\infty, i, 0$ onto $0, i, \infty$ respectively.

Solution:

Given $z_1 = \infty,\ \ z_2 = i,\ \ z_3 = 0, w_1 = 0,\ \ w_2 = i,\ \ w_3 = \infty,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[omit the factors involving z_1, and w_3, since $z_1 = \infty, w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z_2-z_3)}{z-z_3}$$

$$\Rightarrow \frac{w-0}{i-0} = \frac{i-0}{z-0}$$

$$\Rightarrow w = \frac{-1}{z}$$

Example: Find the bilinear transformation which maps the points $1, i, -1$ onto the points $0, 1, \infty$, show that the transformation maps the interior of the unit circle of the $z-$ plane onto the upper half of the $w-$ plane

Solution:

Given $z_1 = 1,\ z_2 = i,\ z_3 = -1$

$w_1 = 0,\ w_2 = 1,\ w_3 = \infty,$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[Omit the factors involving w_3, since , $w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{w-0}{1-0} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$\because \left[\left(\frac{i+1}{i-1}\right)\left(\frac{i+1}{i+1}\right)\right] = \left[\frac{i^2+i+i+1}{i^2-i^2}\right]$ $=\left[\frac{2i}{-2}\right] = -i$

$$\Rightarrow w = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$= \frac{z-1}{z+1}[-i]$$

$$\Rightarrow w = \frac{(-i)z+i}{(1)z+1}\left[\because w = \frac{az+b}{cz+d}, ad-bc \neq 0 \text{ Form}\right]$$

To find z:

$$\Rightarrow wz + w = -iz + i$$

$$\Rightarrow wz + iz = -w + i$$

$$\Rightarrow z[w+i] = -w + i$$

$$\Rightarrow z = \frac{(w-i)}{w+i}$$

To prove: $|z| < 1$ maps $v > 0$

$$\Rightarrow |z| < 1$$

$$\Rightarrow \left|\frac{-(w-i)}{w+i}\right| < 1$$

$$\Rightarrow \left[\frac{w-i}{w+i}\right] < 1$$
$$\Rightarrow |w-i| < |w+i|$$
$$\Rightarrow |u+iv-i| < |u+iv+i|$$
$$\Rightarrow |u+i(v-1)| < |u+i(v+i)|$$
$$\Rightarrow u^2+(v-1)^2 < u^2+(v+1)^2$$
$$\Rightarrow (v-1)^2 < (v+1)^2$$
$$\Rightarrow \ v^2-2v+1 < v^2+2v+1$$
$$\Rightarrow \ -4v < 0$$
$$\Rightarrow v > 0$$

Example: Find the bilinear transformation which maps $z = 1, i, -1$ respectively onto $w = i, 0, -i$. Hence find the fixed points. [A.U, May 2001] [A.U April 2016 R-15 U.D]

Solution:

Given $z_1 = 1,\ \ z_2 = i,\ \ z_3 = -1,$

$w_1 = i,\ \ w_2 = 0,\ \ w_3 = -i,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$Let \quad A = \frac{w_2-w_3}{w_2-w_1} = \frac{0+i}{0-i} = -1$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{i+1}{i-1} = -i$$

$$\Rightarrow a = Aw_1 - Bw_3 = (-1)(i) - (-i)(-i) = -i+1$$
$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (-i)(-i)(1) - (-1)(i)(-1) = -1-i$$
$$\Rightarrow c = A - B = (-1) - (-i) = -1+i$$
$$\Rightarrow d = Bz_1 - Az_3 = (-i)(1) - (-1)(-1) = -i-1$$

We know that, $w = \frac{az+b}{cz+d}, ad-bc \neq 0$

$$\therefore w = \frac{(-i+1)z+(-1-i)}{(-1+i)z+(-i-1)} = \frac{iz+1}{(-i)z+1}$$

Example: Find the bilinear transformation which maps $z = 0$ onto $w = -i$ and has -1 and 1 as the invariant points. Also show that under this transformation the upper half of the z plane maps onto the interior of the unit circle in the w plane.

Solution:

Given $z_1 = 0,\ \ z_2 = -1,\ \ z_3 = 1,$

$w_1 = -i,\ \ w_2 = -1,\ \ w_3 = 1,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$Let \quad A = \frac{w_2-w_3}{w_2-w_1} = \frac{-1-1}{-1+i} = \frac{-2}{-1+i} = 1+i$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{-1-1}{-1-0} = 2$$

$$\Rightarrow a = Aw_1 - Bw_3 = (1+i)(-i) - 2(1) = -i+1-2 = -i-1$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (2)(1)(0) - (1+i)(-i)(1) = i-1$$

$$\Rightarrow c = A - B = (1+i) - 2 = i-1$$

$$\Rightarrow d = Bz_1 - Az_3 = (2)(0) - (1+i)(1) = -(1+i)$$

We know that, $w = \frac{az+b}{cz+d}, ad-bc \neq 0$

$$\therefore w = \frac{(-i+1)z+(i-1)}{(i-1)z+(-1-i)} = \frac{z+(-i)}{(-i)z+1}$$

We know that, $z = \frac{-dw+b}{cw-a} = \frac{-w-i}{-iw-1} = \frac{w+i}{1+wi}$

$$z = \frac{u+iv+i}{1(u+iv)i}$$

$$= \frac{u+iv+i}{1+iu-v} = \frac{u+iv+i}{(1-v)+iu}$$

$$= \left[\frac{u+iv+i}{(1-v)+iu}\right]\left[\frac{1-v-iu}{(1-v)-iu}\right]$$

$$= \frac{u-uv-iu^2+iv-iv^2+uv+i-iv+u}{(1-v)^2+u^2}$$

$$x+iy = \frac{2u+i[-u^2-v^2+1]}{(1-v)^2+u^2}$$

$$\Rightarrow \quad y = \frac{1-u^2-v^2}{(1-v)^2+u^2}$$

Upper half of the z −plane

$$\Rightarrow y \geq 0$$

$$\Rightarrow \frac{1-u^2-v^2}{(1-v)^2+u^2} \geq 0$$

$$\Rightarrow 1-u^2-v^2 \geq 0$$

$$\Rightarrow 1 \geq u^2+v^2$$

$$\Rightarrow u^2+v^2 \leq 1$$

Therefore the upper half of the z −plane maps onto the interior of the unit circles in the w-plane.