UNIT–II FOURIER SERIES

PROBLEMS BASED ON CHANGE OF INTERVAL

CHANGE OF INTERVAL

In most of the Engineering applications, we require an expansion of a given function over an interval $2l$ other than 2π.

Suppose f(x) is a function defined in the interval $c < x < c+2l$. The Fourier expansion for f(x) in the interval $c<x<c+2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}$$

$$\text{where } a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos(n\pi x / l)\,dx \quad \&$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin(n\pi x / l)\,dx$$

Even and Odd Function

If f(x) is an even function and is defined in the interval (c, c+2 *l*), then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where $$a_0 = \frac{2}{l} \int_0^l f(x)dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos (n\pi x / l) \, dx$$

If f(x) is an odd function and is defined in the interval (c, c+2 *l*), then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin (n\pi x / l) \, dx$$

Half Range Series

Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin (n\pi x / l) \, dx$$

Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

ROHINI COLLEGE OF ENGINEERING AND TECHNOLOGY
PALKULAM, KANYAKUMARI
OBSERVE OPTIMIZE OUTSPREAD

where $a_0 = \frac{2}{l} \int_0^l f(x)dx$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos (n\pi x / l) \, dx$$

Example 14

Find the Fourier series expansion for the function

$$f(x) = (c/\ell)x \quad \text{in } 0<x<\ell$$
$$= (c/\ell)(2\ell - x) \quad \text{in } \ell<x<2\ell$$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right)$

Now, $a_0 = \frac{1}{l} \int_0^{2l} f(x)dx$

$$= \frac{1}{\ell} \left[(c/\ell) \int_0^{\ell} x \, dx + (c/\ell) \int_{\ell}^{2\ell} (2\ell - x) \, dx \right]$$

$$= \frac{1}{\ell} \left[(c/\ell) (x^2 / 2) \Big|_0^{\ell} + (c/\ell) (2\ell x - x^2/2) \Big|_{\ell}^{2\ell} \right]$$

$$= \frac{c}{\ell^2} \ell^2 = c$$

$$a_n = \frac{1}{\ell} \int_0^{2\ell} f(x) \cos (n\pi x / \ell) \, dx$$

$$= \frac{1}{\ell} \left[\int_0^{\ell} (c/\ell)x \cos \frac{n\pi x}{\ell} dx + \int_{\ell}^{2\ell} (c/\ell)(2\ell - x) \cos \frac{n\pi x}{\ell} dx \right]$$

$$= \frac{c}{\ell^2} \left\{ \int_0^{\ell} x \, d \left(\frac{\sin(n\pi x/\ell)}{n\pi/\ell} \right) + \int_{\ell}^{2\ell} (2\ell - x) \, d \left(\frac{\sin(n\pi x/\ell)}{n\pi/\ell} \right) \right\}$$

$$= \frac{c}{\ell^2}\left[\left\{ (x)\left\{\frac{\sin\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right\} - (1)\left\{\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right\}\right\}_0^{\ell}\right.$$

$$\left. + \left\{(2\ell - x)\left\{\frac{\sin\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right\} - (-1)\left\{\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right\}\right\}_{\ell}^{2\ell}\right]$$

$$= \frac{c}{\ell^2}\left\{\left\{\frac{\ell^2 \cos n\pi}{n^2\pi^2} - \frac{\ell^2}{n^2\pi^2} + -\frac{\ell^2 \cos 2n\pi}{n^2\pi^2} + \frac{\ell^2 \cos n\pi}{n^2\pi^2}\right\}\right\}$$

$$= \frac{c}{\ell^2}\,\frac{\ell^2}{n^2\pi^2}\{2\cos n\pi - 2\}$$

$$= \frac{2c}{n^2\pi^2}\{(-1)^n - 1\}$$

$$b_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\,.\sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{1}{\ell}\int_0^{\ell}(c/\ell)x\,\sin\frac{n\pi x}{\ell}\,dx + \int_{\ell}^{2\ell}(c/\ell)(2\ell - x)\sin\frac{n\pi x}{\ell}\,dx$$

$$= \frac{c}{\ell^2}\int_0^{\ell} x\, d\left(-\frac{\cos(n\pi x/\ell)}{n\pi/\ell}\right) + \int_{\ell}^{2\ell}(2\ell - x)\,d\left(-\frac{\cos(n\pi x/\ell)}{n\pi/\ell}\right)$$

ROHINI COLLEGE OF ENGINEERING AND TECHNOLOGY
PALKULAM, KANYAKUMARI
OBSERVE OPTIMIZE OUTSPREAD

$$= \frac{c}{\ell^2}\left[\left\{(x)\left\{-\frac{\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right\} - (1)\ -\frac{\sin\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right\}_0^{\ell}\right.$$

$$\left. + \left\{(2\ell - x)\ -\frac{\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right\} - (-1)\left\{-\frac{\sin\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right\}_{\ell}^{2\ell}\right]$$

$$= \frac{c}{\ell^2}\left\{-\frac{\ell^2\cos n\pi}{n\pi} + \frac{\ell^2\cos n\pi}{n\pi}\right\}$$

$$= 0.$$

Therefore, $f(x) = \frac{c}{2} + \frac{2c}{\pi^2}\sum_{n=1}^{\infty}\frac{\{(-1)^n - 1\}}{n^2}\cos(n\pi x/\ell)$

Example 15

Find the Fourier series of periodicity 3 for $f(x) = 2x - x^2$, in $0 < x < 3$.

Here $2\ell = 3$.

$\therefore\ \ell = 3/2$.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3} + b_n\sin\frac{2n\pi x}{3}\right)$

where $a_o = (2/3)\int_0^3 (2x - x^2)\,dx$

$$= (2/3)\left\{2(x^2/2) - (x^3/3)\,dx\right\}_0^3$$

$= 0.$

$$a_n = (2/3)\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= (2/3)\int_0^3 (2x - x^2)\, d\left(\frac{\sin(2n\pi x/3)}{(2n\pi/3)}\right)$$

$$= (2/3)\left[(2x - x^2)\left(\frac{\sin(2n\pi x/3)}{(2n\pi/3)}\right) - (2-2x)\left(-\frac{\cos(2n\pi x/3)}{(4n^2\pi^2/9)}\right) + (-2)\ -\ \frac{\sin(2n\pi x/3)}{(8n^3\pi^3/27)}\right]_0^3$$

$$= (2/3)\left\{-(9/n^2\pi^2) - (9/2n^2\pi^2)\right\} \qquad = -\,9/n^2\pi^2$$

$$b_n = (2/3)\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\,dx$$

$$= (2/3)\int_0^3 (2x - x^2)\, d\left(-\frac{\cos(2n\pi x/3)}{(2n\pi/3)}\right)$$

$$= (2/3)\left[(2x - x^2)\left(-\frac{\cos(2n\pi x/3)}{(2n\pi/3)}\right) - (2-2x)\left(-\frac{\sin(2n\pi x/3)}{(4n^2\pi^2/9)}\right) + (-2)\ \frac{\cos(2n\pi x/3)}{(8n^3\pi^3/27)}\right]_0^3$$

$$= (2/3)\left\{(9/2n\pi) - (27/4n^3\pi^3) + (27/4n^3\pi^3)\right\}$$

$$= 3/n\pi$$

Therefore, $$f(x) = \sum_{n=1}^{\infty}\left[-(9/n^2\pi^2)\cos\frac{2n\pi x}{3} + (3/n\pi)\sin\frac{2n\pi x}{3}\right]$$

Exercises

1. Obtain the Fourier series for $f(x) = \pi x$ in $0 \le x \le 2$.
2. Find the Fourier series to represent x^2 in the interval $(-l, l)$.
3. Find a Fourier series in $(-2, 2)$, if
   $f(x) = 0, -2 < x < 0$

$= 1, 0 < x < 2.$

4.Obtain the Fourier series forf(x)

$= 1 - x$ in $0 \leq x \leq l$

$= 0$ in $l \leq x \leq 2l$. Hence deduce that

$1 - (1/3) + (1/5) - (1/7) + \ldots = \pi/4$ &

$(1/1^2) + (1/3^2) + (1/5^2) + \ldots = (\pi^2/8)$

5.If $f(x) = \pi x, \quad 0 \leq x \leq 1$

$= \pi(2-x), 1 \leq x \leq 2,$

Show that in the interval (0,2),

$$f(x) = (\pi/2) - (4/\pi)\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right)$$

6.Obtain the Fourier series for

$f(x) = x$ in $0 < x < 1$

$= 0$ in $1 < x < 2$ 7.Obtain

the Fourier series for

$f(x) = (cx/l)$ in $0 < x < l$

$= (c/l)(2l - x)$ in $l < x < 2l$.

8.Obtain the Fourier series

forf(x) $= (l + x), -l \leq x \leq 0.$

$= (l - x), 0 \leq x \leq l.$

Deduce that $\displaystyle\sum_{1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

9.Obtain half-range sine series for the

functionf(x) $= cx$ in $0 < x \leq (l/2)$

$= c(l - x)$ in $(l/2) < x < l$

10.Express $f(x) = x$ as a half – range sine series in $0 < x < 2$

11.Obtain the half-range sine series for e^x in $0 < x < 1$.

12.Find the half –range cosine series for the function $f(x) = (x-2)^2$ in the interval$0 < x < 2$.

Deduce that $\sum_{1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

ROHINI COLLEGE OF ENGINEERING AND TECHNOLOGY
PALKULAM, KANYAKUMARI
OBSERVE OPTIMIZE OUTSPREAD