

## UNIT-II FOURIER SERIES

### PROBLEMS BASED ON CHANGE OF INTERVAL

#### CHANGE OF INTERVAL

In most of the Engineering applications, we require an expansion of a given function over an interval  $2l$  other than  $2\pi$ .

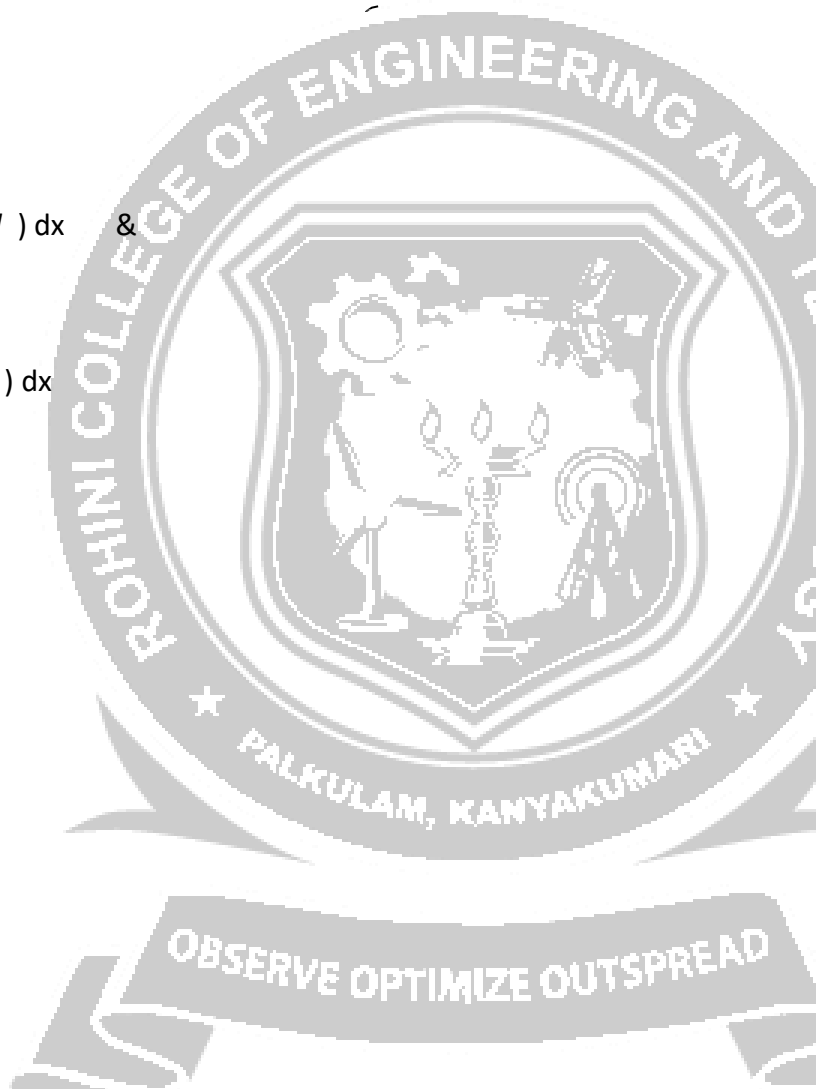
Suppose  $f(x)$  is a function defined in the interval  $c < x < c+2l$ . The Fourier expansion for  $f(x)$  in the interval  $c < x < c+2l$  is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}$$

$$\text{where } a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) \cos \left( \frac{n\pi x}{l} \right) dx \quad \&$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) \sin \left( \frac{n\pi x}{l} \right) dx$$



## Even and Odd Function

If  $f(x)$  is an even function and is defined in the interval  $(c, c+2l)$ , then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$\text{where } a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \left( \frac{n\pi x}{l} \right) dx$$

If  $f(x)$  is an odd function and is defined in the interval  $(c, c+2l)$ , then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \left( \frac{n\pi x}{l} \right) dx$$

## Half Range Series

### Sine Series

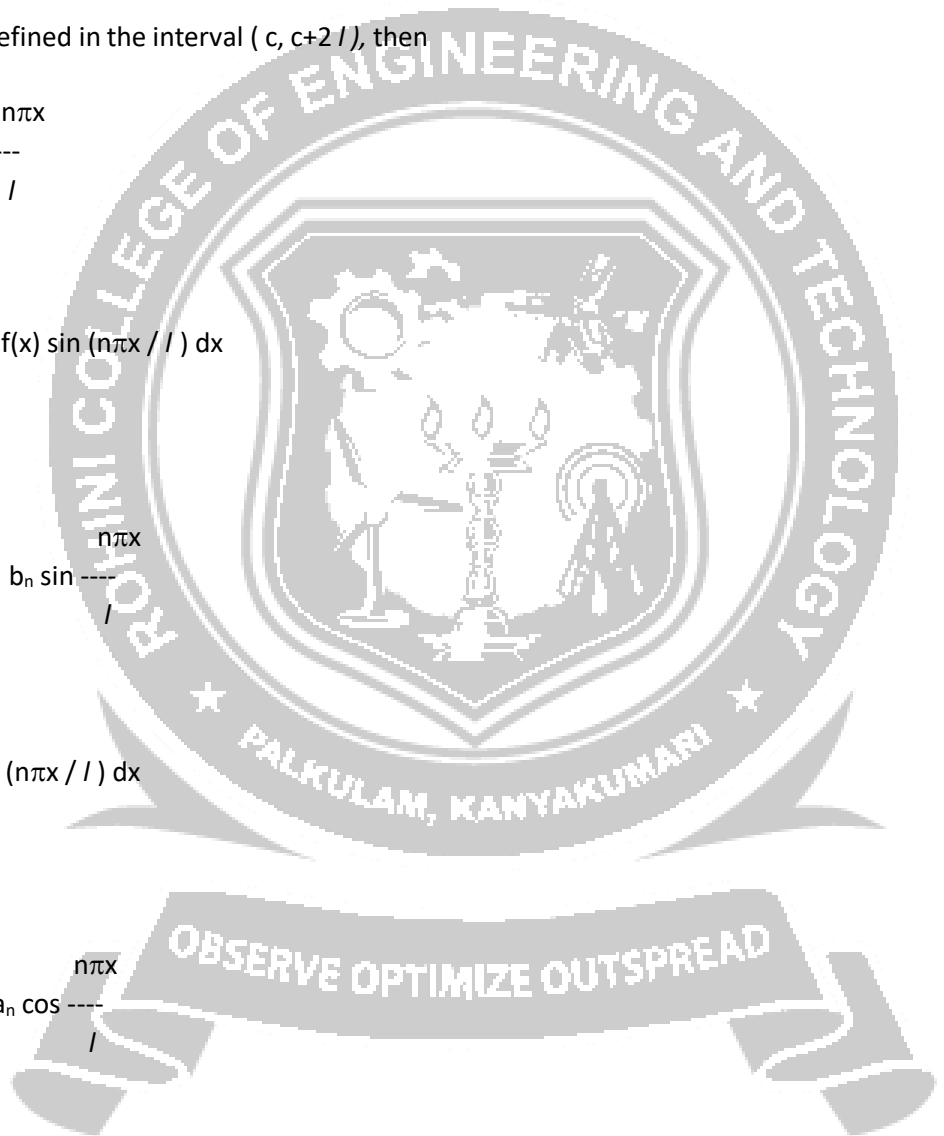
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \left( \frac{n\pi x}{l} \right) dx$$

### Cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$



$$\text{where } a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos(n\pi x / l) dx$$

### Example 14

Find the Fourier series expansion for the function

$$f(x) = (c/l)x \quad \text{in } 0 < x < l \\ = (c/l)(2l-x) \quad \text{in } l < x < 2l$$

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}$$

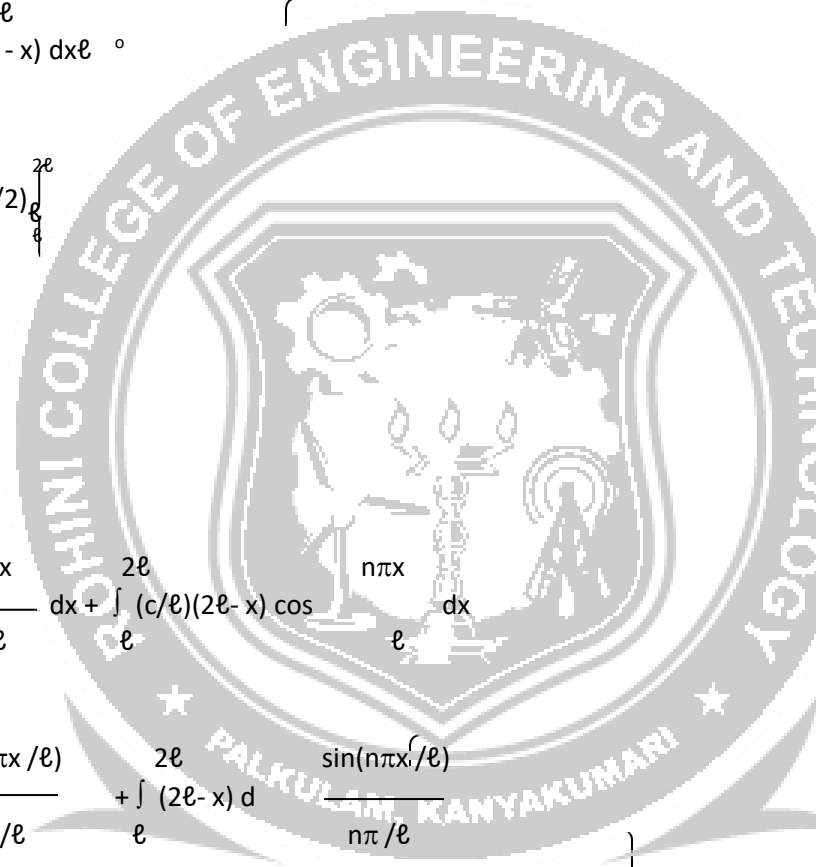
$$\text{Now, } a_0 = \frac{1}{l} \int_0^{2l} f(x) dx \\ = \frac{1}{l} \left[ (c/l) \int_0^l x dx + (c/l) \int_l^{2l} (2l-x) dx \right]$$

$$= \frac{1}{l} \left[ (c/l) \left( \frac{x^2}{2} \right) \Big|_0^l + (c/l) \left( 2lx - \frac{x^2}{2} \right) \Big|_l^{2l} \right] \\ = \frac{c}{l^2} l^2 = c$$

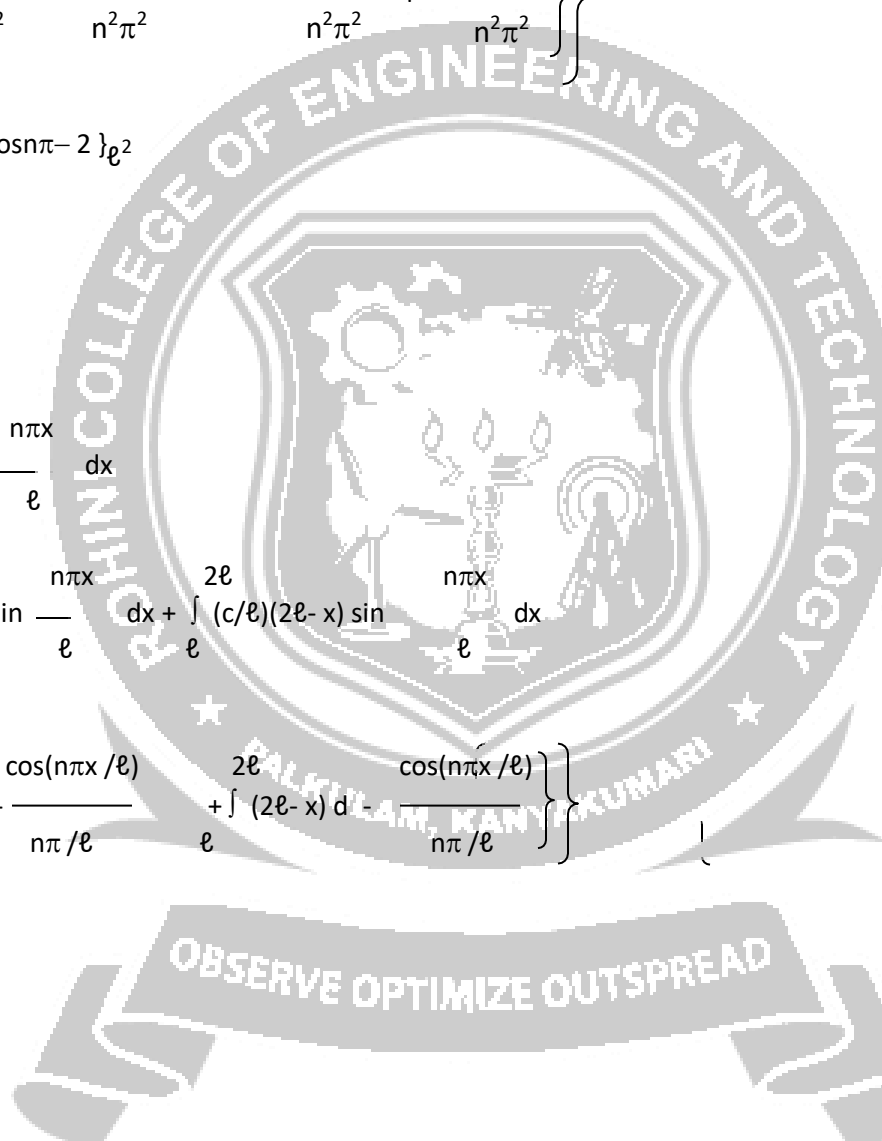
$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos(n\pi x / l) dx$$

$$= \frac{1}{l} \left[ \int_0^l (c/l)x \cos \frac{n\pi x}{l} dx + \int_l^{2l} (c/l)(2l-x) \cos \frac{n\pi x}{l} dx \right]$$

$$= \frac{c}{l^2} \left[ \int_0^l x d \frac{\sin(n\pi x / l)}{n\pi / l} + \int_l^{2l} (2l-x) d \frac{\sin(n\pi x / l)}{n\pi / l} \right]$$



$$\begin{aligned}
&= \frac{c}{e^2} \left\{ (x) \left\{ \frac{\sin \frac{n\pi x}{e}}{\frac{n\pi}{e}} - (-1) \frac{-\cos \frac{n\pi x}{e}}{\frac{n^2\pi^2}{2e^2}} \right\} \right\} \\
&\quad + (2e-x) \left\{ \frac{\sin \frac{n\pi x}{e}}{\frac{n\pi}{e}} - (-1) \frac{-\cos \frac{n\pi x}{e}}{\frac{n^2\pi^2}{e^2}} \right\} \\
&= \frac{c}{e^2} \left\{ \frac{e^2 \cos n\pi}{n^2\pi^2} - \frac{e^2}{n^2\pi^2} + \frac{e^2 \cos 2n\pi}{n^2\pi^2} + \frac{e^2 \cos n\pi}{n^2\pi^2} \right\} \\
&= \frac{c}{e^2} \frac{e^2}{n^2\pi^2} \{ 2 \cos n\pi - 2 \} \\
&= \frac{2c}{n^2\pi^2} \{ (-1)^n - 1 \} \\
b_n &= \frac{1}{e} \int_0^{2e} f(x) \cdot \sin \frac{n\pi x}{e} dx \\
&= \frac{1}{e} \int_0^e (c/e)x \sin \frac{n\pi x}{e} dx + \int_e^{2e} (c/e)(2e-x) \sin \frac{n\pi x}{e} dx \\
&= \frac{c}{e^2} \left\{ \int_0^e x dx - \frac{\cos(n\pi x/e)}{n\pi/e} + \int_e^{2e} (2e-x) dx - \frac{\cos(n\pi x/e)}{n\pi/e} \right\}
\end{aligned}$$



$$= \frac{c}{\ell^2} \left\{ \left\{ (x) \right\} \left\{ \frac{\cos \frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}} \right\} - (-1)^n \left\{ \frac{\sin \frac{n\pi x}{\ell}}{\frac{n^2\pi}{2\ell^2}} \right\} \right\}_0^\ell$$

$$+ (2\ell - x) \left\{ \frac{\cos \frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}} \right\} - (-1)^n \left\{ \frac{\sin \frac{n\pi x}{\ell}}{\frac{n^2\pi}{2\ell^2}} \right\}_\ell^{2\ell}$$

$$= \frac{c}{\ell^2} - \frac{\ell^2 \cos n\pi}{n\pi} + \frac{\ell^2 \cos n\pi}{n\pi}$$

$$= 0.$$

Therefore,  $f(x) = \frac{c}{2} + \frac{2c}{\pi^2} \sum_{n=1}^{\infty} \frac{\{(-1)^n - 1\}}{n^2} \cos(n\pi x / \ell)$

### Example 15

Find the Fourier series of periodicity 3 for  $f(x) = 2x - x^2$ , in  $0 < x < 3$ .

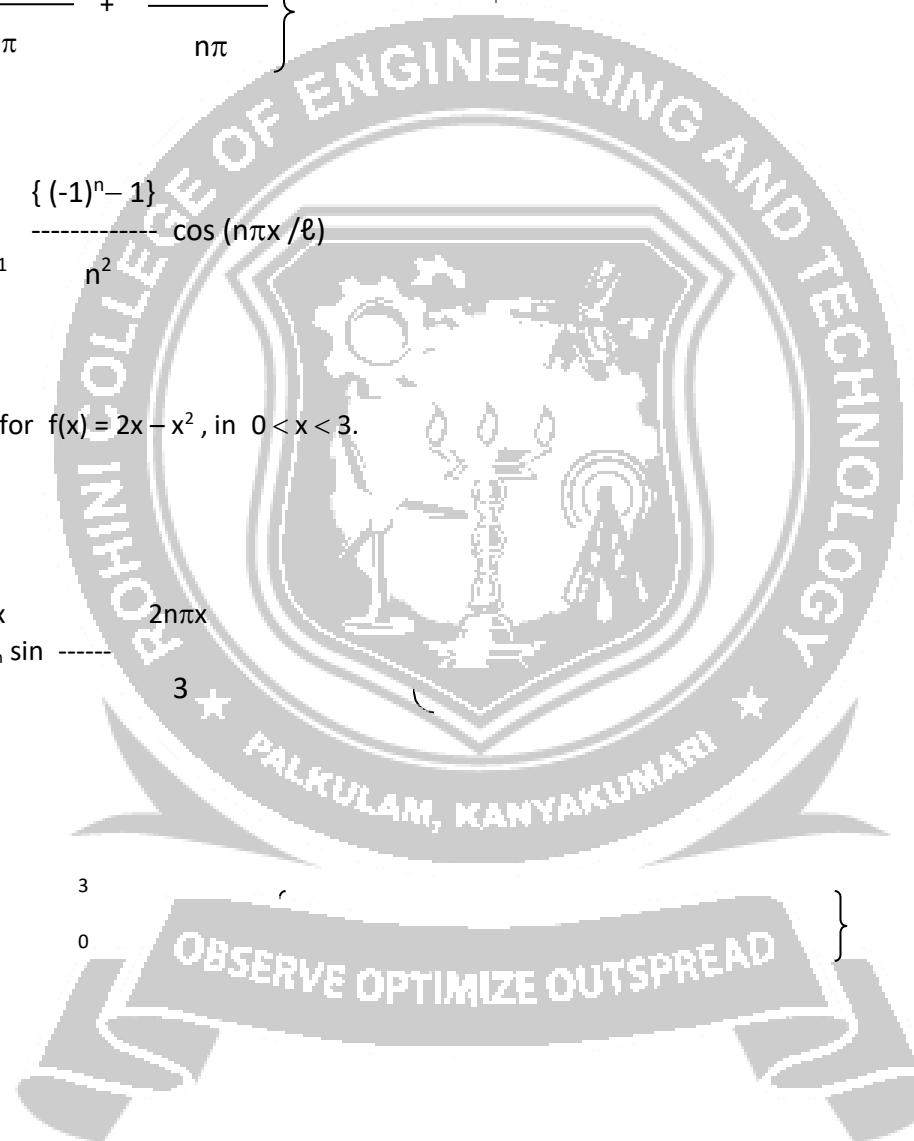
$$\text{Here } 2\ell = 3.$$

$$\therefore \ell = 3/2.$$

Let  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi x}{3} + b_n \sin \frac{2n\pi x}{3}$

where  $a_0 = (2/3) \int_0^3 (2x - x^2) dx$

$$= (2/3) \left[ 2(x^2/2) - (x^3/3) \right]_0^3$$



= 0.

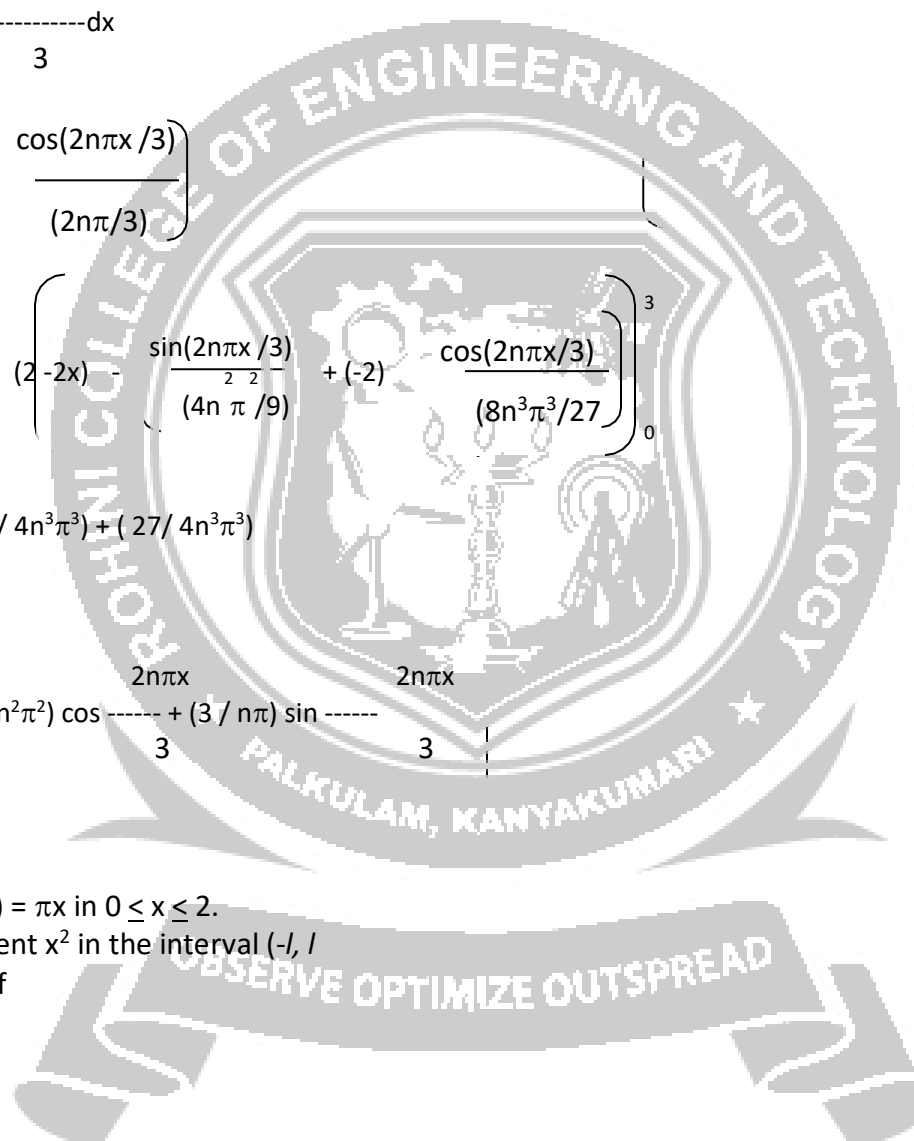
$$\begin{aligned}
 a_n &= \frac{2}{3} \int_0^3 (2x - x^2) \cos \frac{2n\pi x}{3} dx \\
 &= \frac{2}{3} \int_0^3 (2x - x^2) d \left( \frac{\sin(2n\pi x / 3)}{(2n\pi/3)} \right) \\
 &= \frac{2}{3} \left( 2x - x^2 \right) \left( \frac{\sin(2n\pi x / 3)}{(2n\pi/3)} \right) - \left( 2 - 2x \right) \left( \frac{\cos(2n\pi x / 3)}{(4n^2\pi^2/9)} \right) + (-2) \left( \frac{\sin(2n\pi x / 3)}{(8n^3\pi^3/27)} \right) \Bigg|_0^3 \\
 &= \frac{2}{3} \left( -\frac{9}{n^2\pi^2} - \frac{9}{2n^2\pi^2} \right) = -\frac{9}{n^2}\pi^2
 \end{aligned}$$

$$\begin{aligned}
 b_n &= \frac{2}{3} \int_0^3 (2x - x^2) \sin \frac{2n\pi x}{3} dx \\
 &= \frac{2}{3} \int_0^3 (2x - x^2) d \left( -\frac{\cos(2n\pi x / 3)}{(2n\pi/3)} \right) \\
 &= \frac{2}{3} \left( 2x - x^2 \right) \left( -\frac{\cos(2n\pi x / 3)}{(2n\pi/3)} \right) - \left( 2 - 2x \right) \left( -\frac{\sin(2n\pi x / 3)}{(4n^2\pi^2/9)} \right) + (-2) \left( -\frac{\cos(2n\pi x / 3)}{(8n^3\pi^3/27)} \right) \Bigg|_0^3 \\
 &= \frac{2}{3} \left( \frac{9}{2n\pi} - \frac{27}{4n^3\pi^3} + \frac{27}{4n^3\pi^3} \right) \\
 &= \frac{3}{n\pi}
 \end{aligned}$$

Therefore, 
$$f(x) = \sum_{n=1}^{\infty} \left( -\frac{9}{n^2\pi^2} \cos \frac{2n\pi x}{3} + \frac{3}{n\pi} \sin \frac{2n\pi x}{3} \right)$$

### Exercises

1. Obtain the Fourier series for  $f(x) = \pi x$  in  $0 \leq x \leq 2$ .
2. Find the Fourier series to represent  $x^2$  in the interval  $(-l, l)$
3. Find a Fourier series in  $(-2, 2)$ , if  $f(x) = 0, -2 < x < 0$



$$= 1, 0 < x < 2.$$

4. Obtain the Fourier series for  $f(x)$

$$= 1-x \text{ in } 0 \leq x \leq l$$

$$= 0 \text{ in } l \leq x \leq 2l. \quad \text{Hence deduce that}$$

$$1 - (1/3) + (1/5) - (1/7) + \dots = \pi/4 \quad \&$$

$$(1/1^2) + (1/3^2) + (1/5^2) + \dots = (\pi^2/8)$$

5. If  $f(x) = \pi x, \quad 0 \leq x \leq 1$

$$= \pi(2-x), \quad 1 \leq x \leq 2,$$

Show that in the interval  $(0,2)$ ,

$$f(x) = (\pi/2) - (4/\pi) \left[ \frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots \right]$$

6. Obtain the Fourier series for

$$f(x) = x \text{ in } 0 < x < 1$$

$$= 0 \text{ in } 1 < x < 2$$

Obtain the Fourier series for

$$f(x) = (cx/l) \text{ in } 0 < x < l$$

$$= (c/l)(2l-x) \text{ in } l < x < 2l.$$

8. Obtain the Fourier series

$$\text{for } f(x) = (l+x), \quad -l \leq x \leq 0.$$

$$= (l-x), \quad 0 \leq x \leq l.$$

Deduce that 
$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

9. Obtain half-range sine series for the

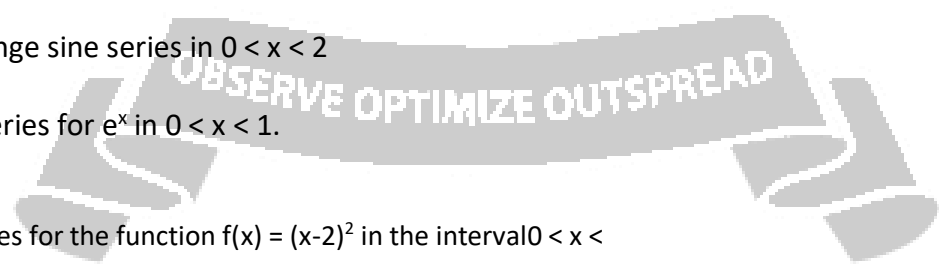
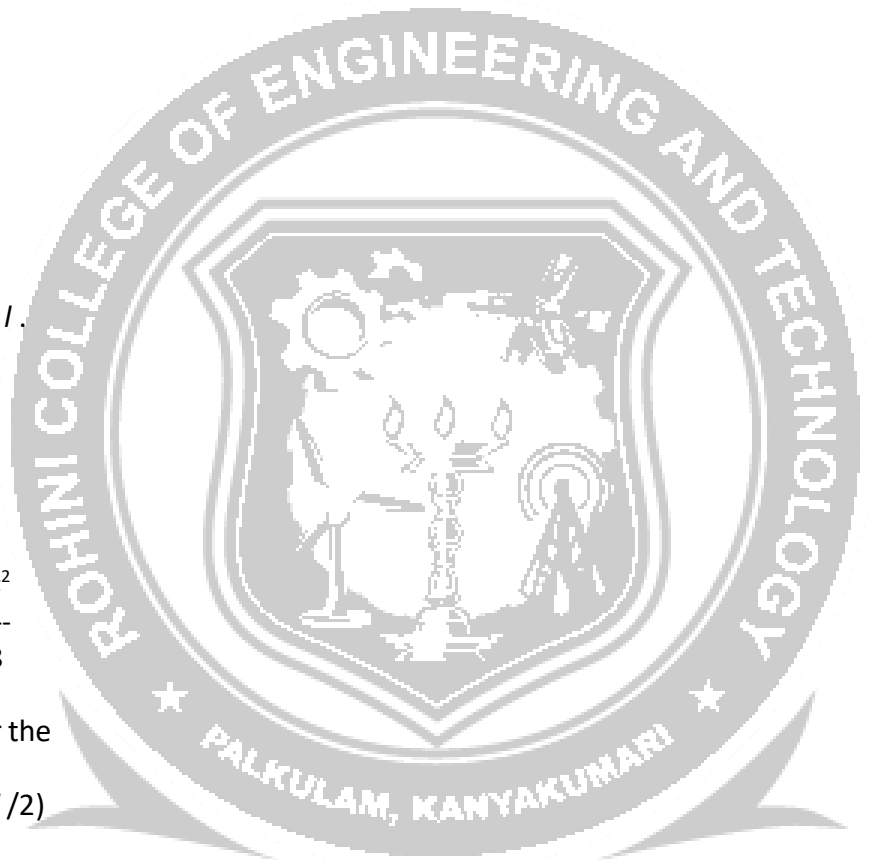
$$\text{function } f(x) = cx \text{ in } 0 < x \leq (l/2)$$

$$= c(l-x) \text{ in } (l/2) < x < l$$

10. Express  $f(x) = x$  as a half-range sine series in  $0 < x < 2$

11. Obtain the half-range sine series for  $e^x$  in  $0 < x < 1$ .

12. Find the half-range cosine series for the function  $f(x) = (x-2)^2$  in the interval  $0 < x < 2$ .



Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

