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DEPARTMENT OF MATHEMATICS

NAME OF THE SUBJECT: TRANSFORMS & PARTIAL DIFFERENTIAL EQUATION

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UNIT – IV : FOURIER TRANSFORMS

UNIT – IV FOURIER TRANSFORMS

	IMPORTANT FORMULAE
1.	**Fourier transform pair:** *i)* The Fourier Transform of *f(x)* is $F\left[f(x)\right]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$ *ii)* The Inverse Fourier Transform of *F(s)* is $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$ Here *F(s) & f(x)* are called Fourier transform pair.
2.	**Fourier Cosine transform pair:** *i)* The Fourier Cosine Transform of *f(x)* is $F_c\left[f(x)\right]=F_c(s)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\cos sx\,dx$ *ii)* The Inverse Fourier Cosine Transform of $F_c(s)$ is $f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} F_c(s)\cos sx\,ds$ Here $F_c(s)$ *& f(x)* are called Fourier cosine transform pair.
3.	**Fourier Sine transform pair:** *i)* The Fourier Sine Transform of *f(x)* is $F_s\left[f(x)\right]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin sx\,dx$ *ii)* The Inverse Fourier Sine Transform of $F_s(s)$ is $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F_s(s)\sin sx\,ds$ Here $F_s(s)$ *& f(x)* are called Fourier sine transform pair.
4.	Parsevals Identity for Fourier transform: $\int_{-\infty}^{\infty}\left\|F(s)\right\|^2 ds=\int_{-\infty}^{\infty}\left\|f(x)\right\|^2 dx$
5.	Parsevals Identity for Fourier Cosine transform: $\int_{0}^{\infty}\left\|F_c(s)\right\|^2 ds=\int_{0}^{\infty}\left\|f(x)\right\|^2 dx$
6.	Parsevals Identity for Fourier Sine transform: $\int_{0}^{\infty}\left\|F_s(s)\right\|^2 ds=\int_{0}^{\infty}\left\|f(x)\right\|^2 dx$
7.	*1)* $\int_{0}^{\infty} e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}$ *2)* $\int_{0}^{\infty} e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}$ 3) $\int_{0}^{\infty} e^{-u^2}du=\frac{\sqrt{\pi}}{2}$

	4) $F\left[xf(x)\right] = (-i)\frac{d}{ds}\left\{F\left[f(x)\right]\right\} = (-i)\frac{d}{ds}\left[F(s)\right]$ 5) $F_s\left[xf(x)\right] = -\frac{d}{ds}\left\{F_c\left[f(x)\right]\right\} = -\frac{d}{ds}\left[F_c(s)\right]$ 6) $F_c\left[xf(x)\right] = \frac{d}{ds}\left\{F_s\left[f(x)\right]\right\} = \frac{d}{ds}\left[F_s(s)\right]$ 7) If *f(x)* and *g(x)* are any two functions and $F_c(s)$ & $G_c(s)$ are there Fourier cosine transforms then $\int_0^\infty f(x)g(x)dx = \int_0^\infty F_c(s)G_c(s)ds$ holds. 8) If *f(x)* and *g(x)* are any two functions and $F_s(s)$ & $G_s(s)$ are there Fourier sine transforms then $\int_0^\infty f(x)g(x)dx = \int_0^\infty F_s(s)G_s(s)ds$ holds.
	PART -A
1.	State Fourier integral theorem. Solution : If $f(x)$ is piecewise continuous, differentiable and absolutely integrable in $(-\infty, \infty)$ then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{is(x-t)}\, dt\,ds$
2.	If F(s) is the Fourier transform of f (x), then show that $F\{f(x-a)\} = e^{ias}F(s)$ Solution : Given $F\left[f(x)\right] = F(s)$ The Fourier Transform of *f(x)* is $F\left[f(x)\right] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$ $F\left[f(x-a)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{isx}dx$ Let $x - a = t \Rightarrow dx = dt$ $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{is(t+a)}dt$ $= e^{isa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt$ $\boxed{F\left[f(x-a)\right] = e^{ias}F\left[f(x)\right]}$
3.	State Convolution theorem in Fourier Transform. Solution : The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms . i.e. $F\left[f(x)*g(x)\right] = F\left[f(x)\right]F\left[g(x)\right] = F(s).G(s)$
4.	If $F\left\{f(x)\right\} = F(s)$, then find $F\left\{e^{iax}f(x)\right\}$. Solution :

	$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$ $F[e^{iax}f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}f(x)e^{isx}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx+iax}dx$ $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}dx$ $F[e^{iax}f(x)] = F(s+a)$
5.	State and prove the change of scale property of Fourier Transform. Statement: If $F[f(x)] = F(s)$ then $F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$ Solution : Given $F[f(x)] = F(s)$ The Fourier Transform of *f(x)* is $F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$ $F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$, If a > 0 $\quad Put\ ax = t \Rightarrow adx = dt \Rightarrow dx = \frac{dt}{a} \quad when\ x = -\infty \Rightarrow t = -\infty\ \ and\ x = \infty \Rightarrow t = \infty$ $F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}\frac{dt}{a}$ $F[f(ax)] = \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}dt = \frac{1}{a}F\left(\frac{s}{a}\right)$. --(1) If a < 0 $\quad Put\ ax = t,\ adx = dt,\ dx = \frac{dt}{a}$ $when\ x = -\infty \Rightarrow t = \infty\ \ and\ x = \infty \Rightarrow t = -\infty$ $\Rightarrow F[f(ax)] = \frac{-1}{\sqrt{2\pi}}\int_{\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}\frac{dt}{a} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}\frac{dt}{a} = \frac{1}{a}F\left(\frac{s}{a}\right)$. ---(2) From (1) & (2) we get $F(f(ax)) = \frac{1}{a}F\left(\frac{s}{a}\right),\ a \neq 0$
6.	Find the Fourier Sine transform of $\frac{1}{x}$.
	Solution : The Fourier Sine Transform of *f(x)* is $F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin sx\, dx$

$$F_s\left(\frac{1}{x}\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin sx}{x}dx$$

$$F_s\left(\frac{1}{x}\right)=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{t}dt=\sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right)=\sqrt{\frac{\pi}{2}} \qquad \because \int_0^\infty \frac{\sin mx}{x}dx=\frac{\pi}{2}$$

PART-B

1. **Find the Fourier transforms of** $f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$ **and hence evaluate** $\int_0^\infty \frac{\sin x}{x}dx$. **Using Parseval's identity, prove that** $\int_0^\infty \frac{\sin^2 t}{t^2}dt=\frac{\pi}{2}$.

Solution: Given $f(x)=\begin{cases}1, & -a<x<a\\ 0, & \text{otherwise}\end{cases}$

The Fourier transform f(x) is

$$F\left[f(x)\right]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\ .$$

$$F(s)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0e^{isx}dx+\int_{-a}^{a} 1e^{isx}dx+\int_{a}^{\infty} 0\ e^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}\cos sx\,dx+i\int_{-a}^{a}\sin sx\,dx\right] \quad \because \sin sx \text{ is an even fn.} \therefore \int_{-a}^{a}\sin sx\,dx=0$$

$$=\frac{1}{\sqrt{2\pi}}2\int_0^a \cos sx\ dx$$

$$=\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[\frac{\sin sx}{s}\right]_0^a \qquad =\sqrt{\frac{2}{\pi}}\left[\frac{\sin as}{s}-0\right]$$

$$\boxed{F(s)=\sqrt{\frac{2}{\pi}}\left(\frac{\sin as}{s}\right)}$$

Deduction: 1

By inverse Fourier transform of *F(s)*.

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\frac{\sin as}{s}\right)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)\left(\cos sx-i\sin sx\right)ds$$

$$=\frac{\sqrt{2}}{\sqrt{2}\sqrt{\pi}\sqrt{\pi}}\left[\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)\left(\cos sx\right)ds-i\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)\left(\sin sx\right)ds\right]$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin as}{s}\right)\cos sx\, ds \qquad \because \left(\frac{\sin as}{s}\right)\sin sx \textit{ is an odd function}$$

$$\int_0^\infty \left(\frac{\sin as}{s}\right)\cos sx\, ds = \frac{\pi}{2} f(x)$$

Put x=0

$$\int_0^\infty \left(\frac{\sin as}{s}\right)\cos 0\, ds = \frac{\pi}{2} f(0)$$

$$\int_0^\infty \left(\frac{\sin as}{s}\right) ds = \frac{\pi}{2}(1) \qquad \because f(x) = 1 \Rightarrow f(0) = 1$$

Put *a=1* and *s=x* we get

$$\boxed{\therefore \int_0^\infty \left(\frac{\sin x}{x}\right) dx = \frac{\pi}{2}}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2 dx = \int_{-\infty}^{\infty} [f(x)]^2 ds$$

$$\int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}}\frac{\sin sa}{s}\right)^2 ds = \int_{-a}^{a} 1^2 dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin sa}{s}\right)^2 ds = [x]_{-a}^{a}$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \left(\frac{\sin sa}{s}\right)^2 ds = [a-(-a)]$$

$$\frac{2}{\pi}2\int_{0}^{\infty} \left(\frac{\sin sa}{s}\right)^2 ds = 2a$$

$$\int_{0}^{\infty} \left(\frac{\sin sa}{s}\right)^2 ds = \frac{2\pi a}{4}$$

Put *a =1* & *s=t* we get,

$$\boxed{\int_0^\infty \frac{\sin^2 t}{t^2} dt = \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}}.$$

2\. **Find the Fourier transform of** $f(x) = \begin{cases} x; & if\ |x| < a \\ 0; & if\ |x| > a \end{cases}$.

Solution: Given $f(x) = \begin{cases} x, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform f(x) is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\ .$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0\, e^{isx}dx + \int_{-a}^{a} x\, e^{isx}dx + \int_{a}^{\infty} 0\, e^{isx}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} x(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} x\cos sx\,dx + i\int_{-a}^{a} x\sin sx\,dx\right] \quad \because x\cos sx \text{ is an } odd \text{ fn} \therefore \int_{-a}^{a} x\cos sx\,dx = 0$$

$$= i\,\frac{1}{\sqrt{2\pi}}\,2\int_{0}^{a} x\sin sx\,dx \qquad \because x\sin x \text{ is an even function} \therefore \int_{-a}^{a} x\sin sx\,dx = 2\int_{0}^{a} x\sin sx\,dx$$

$$= i\,\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[(x)\left(\frac{-\cos sx}{s}\right) - (1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^a$$

$$= i\sqrt{\frac{2}{\pi}}\left[-\frac{x\cos sx}{s} + \frac{\sin sx}{s^2}\right]_0^a$$

$$= i\sqrt{\frac{2}{\pi}}\left[\left(-\frac{a\cos sa}{s} + \frac{\sin sa}{s^2}\right) - (0)\right]$$

$$\boxed{F(s) = i\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - as\cos sa}{s^2}\right]}$$

3\. **Find the Fourier transform of** $f(x) = \begin{cases} a - |x|; & if\ |x| < a \\ 0; & if\ |x| > a > 0 \end{cases}$ **is** $\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s^2}\right)$. **Hence deduce that (i)** $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$. **(ii)** $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}$.

Solution: Given $f(x) = \begin{cases} a - |x|, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform f(x) is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\ .$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0\, e^{isx}dx + \int_{-a}^{a}(a - |x|)e^{isx}dx + \int_{a}^{\infty} 0\, e^{isx}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(a - |x|)(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}(a - |x|)\cos sx\,dx + i\int_{-a}^{a}(a - |x|)\sin sx\,dx\right]$$

$$\because (a - |x|)\sin sx \text{ is an } odd \text{ fn.} \therefore \int_{-a}^{a}(a - |x|)\sin sx\,dx = 0$$

$$= \frac{1}{\sqrt{2\pi}}\,2\int_0^a (a - x)\cos sx\ \ dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[(a-x)\left(\frac{\sin sx}{s}\right) - (-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos sx}{s^2}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{1}{s^2}(\cos sa - \cos 0)\right]$$

$$\boxed{F(s) = \sqrt{\frac{2}{\pi}}\left(\frac{1-\cos sa}{s^2}\right)}$$

$$\boxed{F(s) = 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2\left(\frac{as}{2}\right)}{s^2}\right)} \qquad \because \sin^2\theta = \frac{1-\cos 2\theta}{2} \Rightarrow 1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right) \quad \text{here}\ \theta = \frac{as}{2}$$

Deduction: 1

By inverse Fourier transform of *F(s)*.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2\left(\frac{as}{2}\right)}{s^2}\right)\right]e^{-isx}\,ds$$

$$= \frac{2}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2(\cos sx - i\sin sx)\,ds$$

$$= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2(\cos sx)\,ds - i\int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2(\sin sx)\,ds\right]$$

$$f(x) = \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2(\cos sx)\,ds \qquad \because \left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2(\sin sx)\ \text{is an odd function}$$

$$\int_0^{\infty}\left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2\cos sx\,ds = \frac{\pi}{4}f(x)$$

Put x=0

$$\int_0^\infty \left(\frac{\sin\frac{as}{2}}{s}\right)^2 (\cos 0)\, ds = \frac{\pi}{4} f(0)$$

$$\int_0^\infty \left(\frac{\sin\frac{as}{2}}{s}\right)^2 ds = \frac{\pi a}{4} \qquad \because f(x) = a - |x| \Rightarrow f(0) = a$$

Put a=1 and s=t get

$$\int_0^\infty \left(\frac{\sin\frac{s}{2}}{s}\right)^2 ds = \frac{\pi}{4} \qquad put \frac{s}{2} = t \Rightarrow \frac{ds}{2} = dt$$

$$\int_0^\infty \left(\frac{\sin t}{2t}\right)^2 2dt = \frac{\pi}{4}$$

$$\therefore \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2 dx = \int_{-\infty}^{\infty} [f(x)]^2 ds$$

$$\int_{-\infty}^{\infty} \left(2\sqrt{\frac{2}{\pi}}\left(\frac{\sin^2\left(\frac{as}{2}\right)}{s^2}\right)\right)^2 ds = \int_{-a}^{a} (a - |x|)^2 dx$$

$$\frac{8}{\pi} 2\int_0^\infty \left(\frac{\sin\frac{as}{2}}{s}\right)^4 ds = 2\int_0^a (a-x)^2 dx \quad \because (a-|x|)^2 \text{ and } \frac{\sin^2\left(\frac{as}{2}\right)}{s^2} \text{ are even functions}$$

$$\frac{8}{\pi}\int_0^\infty \left(\frac{\sin\frac{as}{2}}{s}\right)^4 ds = \int_0^a (a-x)^2 dx$$

$$\frac{8}{\pi}\int_0^\infty \left(\frac{\sin\frac{as}{2}}{s}\right)^4 ds = \left(\frac{(a-x)^3}{-3}\right)_0^a$$

$$\frac{8}{\pi}\int_0^\infty \left(\frac{\sin\frac{as}{2}}{s}\right)^4 ds = (0) - \left(-\frac{a^3}{3}\right)$$

$$\int_0^\infty \left(\frac{\sin\frac{as}{2}}{s}\right)^4 ds = \frac{a^3\pi}{3\times 8}$$

Put *a =1* & *s=t* we get,

$$\int_0^\infty \left(\frac{\sin\frac{s}{2}}{s}\right)^4 ds = \frac{\pi}{24} \qquad\qquad put\ \frac{s}{2} = t \Rightarrow \frac{ds}{2} = dt$$

$$\int_0^\infty \left(\frac{\sin t}{2t}\right)^4 2dt = \frac{\pi}{24}$$

$$\boxed{\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt = \frac{\pi}{3}}$$

4. **Find the Fourier transform of** $f(x) = \begin{cases} 1-|x|, & |x| < 1 \\ 0, & |x| \ge 1 \end{cases}$ **and hence find the value of**

(i) $\int_0^\infty \frac{\sin^2 t}{t^2}\, dt$. **(ii)** $\int_0^\infty \frac{\sin^4 t}{t^4}\, dt$.

Slougion:

Hint in the previous problem *a=1*.

5. **Find the Fourier transform of** $f(x) = \begin{cases} a^2 - x^2, & |x| \le a \\ 0, & |x| \ge a \end{cases}$ **and hence evaluate**

(i) $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \frac{\pi}{4}$ **(ii)** $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}$

Solution: Given $f(x) = \begin{cases} a^2 - x^2, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform of f(x) is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\ .$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0\, e^{isx}dx + \int_{-a}^{a}\left(a^2 - x^2\right)e^{isx}dx + \int_{a}^{\infty} 0\ e^{isx}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left(a^2 - x^2\right)(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}\left(a^2-x^2\right)\cos sx\,dx + i\int_{-a}^{a}\left(a^2-x^2\right)\sin sx\,dx\right]$$

$$\because \left(a^2-x^2\right)\sin sx \text{ is an } odd \text{ fn.} \therefore \int_{-a}^{a}\left(a^2-x^2\right)\sin sx\,dx = 0$$

$$= \frac{1}{\sqrt{2\pi}}\,2\int_0^a\left(a^2-x^2\right)\cos sx\,dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[\left(a^2-x^2\right)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$= -2\sqrt{\frac{2}{\pi}}\left[\frac{x\cos sx}{s^2} - \frac{\sin sx}{s^3}\right]_0^a$$

$$= -2\sqrt{\frac{2}{\pi}}\left[\left(\frac{a\cos sa}{s^2} - \frac{\sin sa}{s^3}\right) - (0)\right]$$

$$= -2\sqrt{\frac{2}{\pi}}\left[\frac{as\cos sa - \sin sa}{s^3}\right]$$

$$\boxed{F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - as\cos sa}{s^3}\right]}$$

Deduction: 1

By inverse Fourier transform of *F(s)*.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin sa - as\cos sa}{s^3}\right)\right]e^{-isx}ds$$

$$= \frac{2}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left(\frac{\sin sa - as\cos sa}{s^3}\right)\left(\cos sx - i\sin sx\right)ds$$

$$= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin sa - as\cos sa}{s^3}\right)\left(\cos sx\right)ds - i\int_{-\infty}^{\infty}\left(\frac{\sin sa - as\cos sa}{s^3}\right)\left(\sin sx\right)ds\right]$$

$$f(x) = \frac{4}{\pi}\int_0^{\infty}\left(\frac{\sin sa - as\cos sa}{s^3}\right)\cos sx\,ds \qquad \because \left(\frac{\sin sa - as\cos sa}{s^3}\right)\left(\sin sx\right) \text{is an odd function}$$

$$\int_0^{\infty}\left(\frac{\sin sa - as\cos sa}{s^3}\right)\cos sx\,ds = \frac{\pi}{4} f(x)$$

Put x=0

$$\int_0^{\infty}\left(\frac{\sin sa - as\cos sa}{s^3}\right)\left(\cos 0\right)ds = \frac{\pi}{4} f(0)$$

$$\int_0^{\infty}\left(\frac{\sin sa - as\cos sa}{s^3}\right)ds = \frac{\pi a^2}{4} \qquad \because f(x) = a^2 - x^2 \Rightarrow f(0) = a^2$$

Put *a*=1 and s=t get

$$\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right) dt = \frac{\pi}{4}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2\, ds = \int_{-\infty}^{\infty} [f(x)]^2\, dx$$

$$\int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin sa - as\cos sa}{s^3}\right)\right]^2 ds = \int_{-a}^{a} \left(a^2 - x^2\right)^2 dx$$

$$\frac{8}{\pi}2\int_0^\infty \left(\frac{\sin sa - as\cos sa}{s^3}\right)^2 ds = 2\int_0^a \left(a^4 - 2a^2x^2 + x^4\right) dx$$

$$\because \left(a^2 - x^2\right)^2 \text{ and } \left(\frac{\sin sa - as\cos sa}{s^3}\right)^2 \text{ are even functions}$$

$$\frac{8}{\pi}\int_0^\infty \left(\frac{\sin sa - as\cos sa}{s^3}\right)^2 ds = \left[a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5}\right]_0^a$$

$$\frac{8}{\pi}\int_0^\infty \left(\frac{\sin sa - as\cos sa}{s^3}\right)^2 ds = \left[a^5 - \frac{2a^5}{3} + \frac{a^5}{5}\right]$$

$$\frac{8}{\pi}\int_0^\infty \left(\frac{\sin sa - as\cos sa}{s^3}\right)^2 ds = \left[\frac{15a^5 - 10a^5 + 3a^5}{15}\right]$$

$$\int_0^\infty \left(\frac{\sin sa - as\cos sa}{s^3}\right)^2 ds = \left(\frac{8a^5}{15}\right) \times \frac{\pi}{8}$$

Put $a = 1$ & $s = t$ we get,

$$\boxed{\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt = \frac{\pi}{15}}$$

6. **Find the Fourier transform of** $f(x) = \begin{cases} 1 - x^2\ ; & if\ |x| < 1 \\ 0\ ; & if\ |x| \geq 1 \end{cases}$.

Hence show that $(i)\int_0^\infty \left(\frac{\sin s - s\cos s}{s^2}\right)\cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$ and $(ii)\int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6} dx = \frac{\pi}{15}$

Solution: Given $f(x) = \begin{cases} 1 - x^2, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform f(x) is

$$F\left[f(x)\right] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\ .$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1} 0\, e^{isx}dx + \int_{-1}^{1}\left(1 - x^2\right)e^{isx}dx + \int_{1}^{\infty} 0\ e^{isx}dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\left(1 - x^2\right)(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1}\left(1-x^2\right)\cos sx\,dx + i\int_{-1}^{1}\left(1-x^2\right)\sin sx\,dx\right]$$

$\because \left(1-x^2\right)\sin sx$ is an *odd* fn. $\therefore \int_{-1}^{1}\left(1-x^2\right)\sin sx\,dx = 0$

$$= \frac{1}{\sqrt{2\pi}}\,2\int_{0}^{1}\left(1-x^2\right)\cos sx\,dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[\left(1-x^2\right)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^1$$

$$= -2\sqrt{\frac{2}{\pi}}\left[\frac{x\cos sx}{s^2} - \frac{\sin sx}{s^3}\right]_0^1$$

$$= -2\sqrt{\frac{2}{\pi}}\left[\left(\frac{\cos s}{s^2} - \frac{\sin s}{s^3}\right) - (0)\right]$$

$$= -2\sqrt{\frac{2}{\pi}}\left(\frac{s\cos s - \sin s}{s^3}\right)$$

$$\boxed{F(s) = 2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)}$$

Deduction: 1

By inverse Fourier transform of *F(s)*.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)\right]e^{-isx}ds$$

$$= \frac{2}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\left(\cos sx - i\sin sx\right)ds$$

$$= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\left(\cos sx\right)ds - i\int_{-\infty}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\left(\sin sx\right)ds\right]$$

$$f(x) = \frac{4}{\pi}\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,ds \qquad \because \left(\frac{\sin s - s\cos s}{s^3}\right)(\sin sx) \text{ is an odd function}$$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos sx\,ds = \frac{\pi}{4}f(x)$$

Put $x = \frac{1}{2}$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos\left(\frac{s}{2}\right)ds = \frac{\pi}{4}f\left(\frac{1}{2}\right)$$

$$\int_{0}^{\infty}\left(\frac{\sin s - s\cos s}{s^3}\right)\cos\left(\frac{s}{2}\right)ds = \frac{\pi}{4}\times\frac{3}{4} \qquad \because f(x) = 1-x^2 \Rightarrow f\left(\frac{1}{2}\right) = 1-\frac{1}{4} = \frac{3}{4}$$

	$$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)\cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$$ (ii) By Parseval's identity, $$\int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} [f(x)]^2 dx$$ $$\int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}}\left(\frac{\sin s - s\cos s}{s^3}\right)\right]^2 ds = \int_{-1}^{1} \left(1 - x^2\right)^2 dx$$ $$\frac{8}{\pi} 2\int_0^\infty \left(\frac{\sin sa - as\cos sa}{s^3}\right)^2 ds = 2\int_0^1 \left(1 - 2x^2 + x^4\right) dx$$ $$\because \left(1 - x^2\right)^2 \text{ and } \left(\frac{\sin s - s\cos s}{s^3}\right)^2 \text{ are an even functions}$$ $$\frac{8}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_0^1$$ $$\frac{8}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \left[1 - \frac{2}{3} + \frac{1}{5}\right] - 0$$ $$\frac{8}{\pi}\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \left[\frac{15 - 10 + 3}{15}\right]$$ $$\int_0^\infty \left(\frac{\sin s - s\cos s}{s^3}\right)^2 ds = \left(\frac{8}{15}\right) \times \frac{\pi}{8}$$ Put $s=t$ we get, $$\boxed{\int_0^\infty \frac{\left(\sin t - t\cos t\right)^2}{t^6} dx = \frac{\pi}{15}}$$
7.	**Find the Fourier cosine and sine transform of** $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$ **.** **Solution:** Given $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2 - x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$ The Fourier Cosine transform of f(x) is $$F_c\left[f(x)\right] = F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx .$$ $$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos sx\, dx + \int_1^2 (2 - x)\cos sx\, dx + \int_2^\infty 0\cos sx\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[(x)\left(\frac{\sin sx}{s}\right)-(1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1+\left[(2-x)\left(\frac{\sin sx}{s}\right)-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[\frac{x\sin sx}{s}+\frac{\cos sx}{s^2}\right]_0^1+\left[(2-x)\left(\frac{\sin sx}{s}\right)-\frac{\cos sx}{s^2}\right]_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[\frac{\sin s}{s}+\frac{\cos s}{s^2}-\left(0+\frac{1}{s^2}\right)\right]+\left[0-\frac{\cos 2s}{s^2}-\left(\frac{\sin s}{s}-\frac{\cos s}{s^2}\right)\right]\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\cancel{\frac{\sin s}{s}}+\frac{\cos s}{s^2}-\frac{1}{s^2}-\frac{\cos 2s}{s^2}-\cancel{\frac{\sin s}{s}}+\frac{\cos s}{s^2}\right]$$

$$\boxed{F_c(s)=\sqrt{\frac{2}{\pi}}\left[\frac{2\cos s-\cos 2s-1}{s^2}\right]}$$

The Fourier sine transform of f(x) is

$$F_s\left[f(x)\right]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx\,.$$

$$= \sqrt{\frac{2}{\pi}}\left[\int_0^1 x\sin sx\,dx+\int_1^2 (2-x)\sin sx\,dx+\cancel{\int_2^\infty 0\sin sx\,dx}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[(x)\left(\frac{-\cos sx}{s}\right)-(1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^1+\left[(2-x)\left(\frac{-\cos sx}{s}\right)-(-1)\left(\frac{-\sin sx}{s^2}\right)\right]_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[\frac{-x\cos sx}{s}+\frac{\sin sx}{s^2}\right]_0^1+\left[-(2-x)\left(\frac{\cos sx}{s}\right)-\frac{\sin sx}{s^2}\right]_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\left[-\frac{\cos s}{s}+\frac{\sin s}{s^2}-(0)\right]+\left[0-\frac{\sin 2s}{s^2}-\left(-\frac{\cos s}{s}-\frac{\sin s}{s^2}\right)\right]\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-\cancel{\frac{\cos s}{s}}+\frac{\sin s}{s^2}-\frac{\sin 2s}{s^2}+\cancel{\frac{\cos s}{s}}+\frac{\sin s}{s^2}\right]$$

$$\boxed{F_s(s)=\sqrt{\frac{2}{\pi}}\left[\frac{2\sin s-\sin 2s}{s^2}\right]}$$

8. **Find Fourier transform of $e^{-a|x|}$ and hence deduce that**

(a) $\int_0^\infty \frac{\cos xt}{a^2+t^2}\,dt=\frac{\pi}{2a}e^{-a|x|}$ **(b)** $F\left[xe^{-a|x|}\right]=i\sqrt{\frac{2}{\pi}}\frac{2as}{\left(s^2+a^2\right)^2}$ **.**

The Fourier transform of f(x) is

$$F\left[f(x)\right]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\,.$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\left(\cos sx+i\sin sx\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} e^{-a|x|}\cos sx\,dx + i\int_{-\infty}^{\infty} e^{-a|x|}\sin sx\,dx\right]$$

$\because e^{-a|x|}\sin sx$ is an *odd* fn. $\therefore \int_{-\infty}^{\infty} e^{-a|x|}\sin sx\,dx = 0$

$$= \frac{1}{\sqrt{2\pi}}\,2\int_0^{\infty} e^{-ax}\cos sx\,dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\int_0^{\infty} e^{-ax}\cos sx\,dx$$

$$F(s) = F\left[e^{-a|x|}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right] \qquad \because \int_0^{\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2} \text{ here } a = a; b = s$$

Deduction (a):

By inverse Fourier transform of *F(s)* is

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]e^{-isx}\,ds$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left[\frac{a}{a^2+s^2}\right](\cos sx - i\sin sx)\,ds$$

$$= \frac{a}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{a^2+s^2}\right](\cos sx)\,ds - ia\int_{-\infty}^{\infty}\left[\frac{1}{a^2+s^2}\right](\sin sx)\,ds$$

$$f(x) = \frac{2a}{\pi}\int_0^{\infty}\left[\frac{1}{a^2+s^2}\right]\cos sx\,ds \qquad \because \left[\frac{1}{a^2+s^2}\right](\sin sx) \text{ is an odd function}$$

$$\int_0^{\infty}\left[\frac{1}{a^2+s^2}\right]\cos sx\,ds = \frac{\pi}{2a}f(x)$$

$$\boxed{\int_0^{\infty}\frac{\cos sx}{a^2+s^2}\,ds = \frac{\pi}{2a}e^{-a|x|}}$$

Put $s=t$

$$\boxed{\int_0^{\infty}\frac{\cos tx}{t^2+a^2}\,dt = \frac{\pi}{2a}e^{-a|x|}}$$

Deduction (b):

By Property

$$F\left[x f(x)\right] = -i\frac{d}{ds}\left[F(s)\right]$$

$$F\left[xe^{-a|x|}\right] = -i\frac{d}{ds}F\left(e^{-a|x|}\right)$$

$$= -i\frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\frac{a}{a^2+s^2}\right]$$

	$= -ia\sqrt{\frac{2}{\pi}}\left[\frac{-1}{(a^2+s^2)^2}(0+2s)\right] = i\sqrt{\frac{2}{\pi}}\frac{2as}{(a^2+s^2)^2}$ $F\left[xe^{-a\lvert x\rvert}\right] = i\sqrt{\frac{2}{\pi}}\left[\frac{2as}{(s^2+a^2)^2}\right]$
9.	**Find the Fourier sine and cosine transform of $e^{-ax}, a > 0$ and deduce that** **i)** $\int_0^\infty \frac{s}{s^2+a^2}\sin sx\,dx = \frac{\pi}{2}e^{-ax}$. ***ii)*** $\int_0^\infty \frac{1}{s^2+a^2}\cos sx\,dx = \frac{\pi}{2a}e^{-ax}$ **Solution:** The Fourier sine transform of f(x) is $F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$. $= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$ $F_s(s) = F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]$ $\quad \because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$ here $a = a; b = s$ The Fourier cosine transform of f(x) is $F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$. $= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$ $F_c(s) = F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]$ $\quad \because \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$ here $a = a; b = s$ The inverse Fourier sine transform of $F_s(s)$ is $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$ $= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\sin sx\,dx$ $= \frac{2}{\pi}\int_0^\infty \left[\frac{s}{a^2+s^2}\right]\sin sx\,dx$ $\int_0^\infty \left[\frac{s}{a^2+s^2}\right]\sin sx\,dx = \frac{\pi}{2}f(x)$ $\int_0^\infty \left[\frac{s}{a^2+s^2}\right]\sin sx\,dx = \frac{\pi}{2}e^{-ax}$

The inverse Fourier Cosine transform of $F_c(s)$ is

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right)\cos sx\,dx$$

$$=\frac{2a}{\pi}\int_0^\infty\left(\frac{1}{a^2+s^2}\right)\cos sx\,dx$$

$$\int_0^\infty\left(\frac{a}{a^2+s^2}\right)\cos sx\,dx=\frac{\pi}{2}f(x)$$

$$\int_0^\infty\left(\frac{a}{a^2+s^2}\right)\cos sx\,dx=\frac{\pi}{2a}e^{-ax}$$

10. **Find the Fourier sine and cosine transform of $e^{-ax}, a>0$ and hence find $F_c\left(xe^{-ax}\right)$ and $F_s\left(xe^{-ax}\right)$.**

Solution:

The Fourier sine transform f(x) is

$$F_s[f(x)]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx.$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\boxed{F_s(s)=F_s\left(e^{-ax}\right)=\sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)}\qquad \because\int_0^\infty e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}\quad \text{here } a=a;b=s$$

The Fourier cosine transform f(x) is

$$F_c[f(x)]=F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx.$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\boxed{F_c(s)=F_c\left(e^{-ax}\right)=\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right)}\qquad \because\int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}\quad \text{here } a=a;b=s$$

We know that

i) $F_s[xf(x)]=-\frac{d}{ds}\{F_c[f(x)]\}=-\frac{d}{ds}[F_c(s)]$

$$F_s\left(xe^{-ax}\right)=-\frac{d}{ds}\left\{F_c\left(e^{-ax}\right)\right\}=-\frac{d}{ds}\left(\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right)\right)$$

$$=-a\sqrt{\frac{2}{\pi}}\frac{d}{ds}\left(\frac{1}{a^2+s^2}\right)$$

$$=-a\sqrt{\frac{2}{\pi}}\left(\frac{-1}{\left(a^2+s^2\right)^2}(0+2s)\right)$$

	$F_s\left[xe^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{2as}{\left(a^2+s^2\right)^2}\right]$ ii) $F_c\left[xf(x)\right] = \frac{d}{ds}\left\{F_s\left[f(x)\right]\right\} = \frac{d}{ds}\left[F_s(s)\right]$ $F_s\left[xe^{-ax}\right] = \frac{d}{ds}\left\{F_c\left[e^{-ax}\right]\right\} = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)\right]$ $= \sqrt{\frac{2}{\pi}}\left[\frac{\left(a^2+s^2\right)(1)-s(0+2s)}{\left(a^2+s^2\right)^2}\right]$ $= \sqrt{\frac{2}{\pi}}\left[\frac{a^2+s^2-2s^2}{\left(a^2+s^2\right)^2}\right]$ $F_s\left[xe^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{\left(a^2+s^2\right)^2}\right]$
11.	**Find the Fourier sine transform of** $\frac{e^{-ax}}{x}, a>0$ **and hence find** $F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$. **Solution:** The Fourier sine transform of f(x) is $F_s\left[f(x)\right] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$ $F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx$ Taking diff. on both sides w.r.to s $\frac{d}{ds}\left[F_s\left[\frac{e^{-ax}}{x}\right]\right] = \frac{d}{ds}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx\right]$ $= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial s}\left(\sin sx\right)dx$ $= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\left(\cos sx\right)x\,dx$ $= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$ $\frac{d}{ds}\left[F_s\left[\frac{e^{-ax}}{x}\right]\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]$ Integrating on on both sides w.r.to s $F_s\left[\frac{e^{-ax}}{x}\right] = \sqrt{\frac{2}{\pi}}\int\left[\frac{a}{a^2+s^2}\right]ds$

	$F_s\left(\frac{e^{-ax}}{x}\right) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)$ $\quad \because \int \frac{a}{x^2+a^2}dx = \tan^{-1}\left(\frac{x}{a}\right)$ Similarly $F_s\left(\frac{e^{-bx}}{x}\right) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{b}\right)$ Deduction: $F_s\left(\frac{e^{-ax}-e^{-bx}}{x}\right) = F_s\left(\frac{e^{-ax}}{x} - \frac{e^{-bx}}{x}\right)$ $= F_s\left(\frac{e^{-ax}}{x}\right) - F_s\left(\frac{e^{-bx}}{x}\right)$ $= \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right) - \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{b}\right)$ $F_s\left(\frac{e^{-ax}-e^{-bx}}{x}\right) = \sqrt{\frac{2}{\pi}}\left(\tan^{-1}\left(\frac{s}{a}\right) - \tan^{-1}\left(\frac{s}{b}\right)\right)$
12.	**Find the Fourier cosine transform of** $\frac{e^{-ax}}{x}, a > 0$ **and hence find** $F_c\left(\frac{e^{-ax}-e^{-bx}}{x}\right)$ **Solution:** The Fourier cosine transform f(x) is $F_c\left[f(x)\right] = F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$ $F_c\left(\frac{e^{-ax}}{x}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx$ Taking diff. on both sides w.r.to *s* $\frac{d}{ds}\left(F_c\left(\frac{e^{-ax}}{x}\right)\right) = \frac{d}{ds}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\cos sx\,dx\right)$ $= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial s}(\cos sx)\,dx$ $= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}(-\sin sx)\,x\,dx$ $= -\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$ $\frac{d}{ds}\left(F_c\left(\frac{e^{-ax}}{x}\right)\right) = -\sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)$ Integrating on on both sides w.r.to *s* $F_c\left(\frac{e^{-ax}}{x}\right) = -\sqrt{\frac{2}{\pi}}\int\left(\frac{s}{a^2+s^2}\right)ds$

	$= -\sqrt{\frac{2}{\pi}}\int \left(\frac{s}{a^2+s^2}\right) ds$ $= -\sqrt{\frac{2}{\pi}}\frac{1}{2}\int \left(\frac{2s}{a^2+s^2}\right) ds$ $= -\sqrt{\frac{2}{\pi}}\frac{1}{2}\log\left(s^2+a^2\right) \qquad \because \int \frac{f'(x)}{f(x)}dx = \log\left[f(x)\right]$ $= -\frac{1}{\sqrt{2\pi}}\log\left(s^2+a^2\right)$ $= \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+a^2}\right)$ $F_c\left(\frac{e^{-ax}}{x}\right) = \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+a^2}\right)$ Similarly $F_c\left(\frac{e^{-bx}}{x}\right) = \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+b^2}\right)$ **Deduction:** $F_c\left(\frac{e^{-ax}-e^{-bx}}{x}\right) = F_c\left(\frac{e^{-ax}}{x} - \frac{e^{-bx}}{x}\right)$ $= F_c\left(\frac{e^{-ax}}{x}\right) - F_c\left(\frac{e^{-bx}}{x}\right)$ $= \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+a^2}\right) - \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+b^2}\right)$ $= \frac{1}{\sqrt{2\pi}}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$ $\boxed{F_s\left(\frac{e^{-ax}-e^{-bx}}{x}\right) = \frac{1}{\sqrt{2\pi}}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)}$
13.	**Using Parseval'sidentity evaluate the following integrals.** **1)** $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$ **2)** $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}dx$, where $a > 0$. **Solution:** **Assume** $f(x) = e^{-ax}$ The Fourier sine transform f(x) is $F_s\left[f(x)\right] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$.

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$\boxed{F_s(s) = F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2} \text{ here } a = a; b = s$$

The Fourier cosine transform f(x) is

$$F_c\left[f(x)\right] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos x \, dx$$

$$\boxed{F_c(s) = F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2} \text{ here } a = a; b = s$$

(i) The Parseval's identity for Fourier cosine transform is

$$\int_0^\infty \left|F_c(s)\right|^2 ds = \int_0^\infty \left|f(x)\right|^2 dx$$

$$\int_0^\infty \left[\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right)\right]^2 ds = \int_0^\infty \left(e^{-ax}\right)^2 dx$$

$$\frac{2a^2}{\pi}\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2} ds = \int_0^\infty e^{-2ax} dx$$

$$\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2} ds = \frac{\pi}{2a^2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2} ds = \frac{-\pi}{4a^3}\left[e^{-\infty} - e^{-0}\right]$$

$$\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2} ds = \frac{-\pi}{4a^3}\left[0-1\right] \qquad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2} ds = \frac{\pi}{4a^3}$$

Put s=x we get

$$\boxed{\int_0^\infty \frac{1}{\left(a^2+x^2\right)^2} dx = \frac{\pi}{4a^3}}$$

(ii) The Parseval's identity for Fourier sine transform is

$$\int_0^\infty \left|F_s(s)\right|^2 ds = \int_0^\infty \left|f(x)\right|^2 dx$$

$$\int_0^\infty \left[\sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)\right]^2 ds = \int_0^\infty \left(e^{-ax}\right)^2 dx$$

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$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}ds = \int_0^\infty e^{-2ax}dx$$

$$\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}ds = \frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}ds = \frac{-\pi}{4a}\left[e^{-\infty} - e^{-0}\right]$$

$$\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}ds = \frac{-\pi}{4a}\left[0-1\right] \qquad \because e^{-\infty} = 0;\ e^{-0} = 1$$

$$\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}ds = \frac{\pi}{4a}$$

Put s=x we get

$$\boxed{\int_0^\infty \frac{x^2}{\left(a^2+x^2\right)^2}dx = \frac{\pi}{4a}}$$

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<tr><td>14.</td><td>

Evaluate (a) $\int_0^\infty \frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx$ **(b)** $\int_0^\infty \frac{1}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx$ **using Fourier transforms.**

Solution:

(a) Assume $f(x) = e^{-ax}\ ;g(x) = e^{-bx}$

The Fourier sine transform f(x) is

$$F_s\left[f(x)\right] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx\,.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\boxed{F_s(s) = F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2} \quad \text{here } a = a; b = s$$

Similarly

$$\boxed{G_s(s) = F_s\left[e^{-bx}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{b^2+s^2}\right]}$$

We know that

$$\int_0^\infty F_s(s)G_s(s)ds = \int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{s}{b^2+s^2}\right]ds = \int_0^\infty e^{-ax}e^{-bx}dx$$

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$$\frac{2}{\pi}\int_0^\infty \left[\frac{s^2}{(a^2+s^2)(b^2+s^2)}\right] ds = \int_0^\infty e^{-ax} e^{-bx} dx$$

$$\int_0^\infty \left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] ds = \frac{\pi}{2}\int_0^\infty e^{-(a+b)x} dx$$

$$= \frac{\pi}{2}\int_0^\infty e^{-(a+b)x} dx$$

$$= \frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \frac{-\pi}{2(a+b)}\left[e^{-\infty} - e^{-0}\right]_0^\infty$$

$$= \frac{-\pi}{2(a+b)}[0-1] \qquad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^\infty \left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] ds = \frac{\pi}{2(a+b)}$$

Put s=x we get

$$\boxed{\int_0^\infty \left[\frac{x^2}{(x^2+a^2)(x^2+b^2)}\right] dx = \frac{\pi}{2(a+b)}}$$

(b) Assume $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

The Fourier cosine transform f(x) is

$$F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\boxed{F_c(s) = F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2} \text{ here } a = a; b = s$$

Similarly

$$\boxed{G_c(s) = F_c\left[e^{-bx}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right]}$$

We know that

$$\int_0^\infty F_c(s)G_c(s)ds = \int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right] ds = \int_0^\infty e^{-ax}e^{-bx}dx$$

$$\frac{2ab}{\pi}\int_0^\infty \left[\frac{1}{(a^2+s^2)(b^2+s^2)}\right] ds = \int_0^\infty e^{-ax-bx}dx$$

$$\int_0^\infty \left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] ds = \frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}dx$$

$$= \frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}dx$$

$$= \frac{\pi}{2ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \frac{-\pi}{2ab(a+b)}\left[e^{-\infty} - e^{-0}\right]_0^\infty$$

$$= \frac{-\pi}{2ab(a+b)}[0-1] \qquad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^\infty \left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] ds = \frac{\pi}{2ab(a+b)}$$

Put s=x we get

$$\boxed{\int_0^\infty \left[\frac{1}{(x^2+a^2)(x^2+b^2)}\right] dx = \frac{\pi}{2ab(a+b)}}$$

Evaluate (a) $\int_0^\infty \frac{x^2}{(x^2+9)(x^2+16)}dx$ **, (b)** $\int_0^\infty \frac{1}{(x^2+1)(x^2+4)}dx$ **using Fourier transforms.**

Solution:

***(a)* Assume** $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

The Fourier sine transform f(x) is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx .$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\boxed{F_s(s) = F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2} \text{ here } a = a; b = s$$

Similarly

$$\boxed{G_s(s) = G_s\left[e^{-bx}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{b^2+s^2}\right]}$$

We know that

$$\int_0^\infty F_s(s)G_s(s)ds = \int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{s}{a^2+s^2}\right)\sqrt{\frac{2}{\pi}}\left(\frac{s}{b^2+s^2}\right) ds = \int_0^\infty e^{-ax} e^{-bx} dx$$

$$\frac{2}{\pi}\int_0^\infty \left(\frac{s^2}{(a^2+s^2)(b^2+s^2)}\right) ds = \int_0^\infty e^{-ax-bx} dx$$

$$\int_0^\infty \left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right) ds = \frac{\pi}{2}\int_0^\infty e^{-(a+b)x} dx$$

$$= \frac{\pi}{2}\int_0^\infty e^{-(a+b)x} dx$$

$$= \frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \frac{-\pi}{2(a+b)}\left[e^{-\infty} - e^{-0}\right]_0^\infty$$

$$= \frac{-\pi}{2(a+b)}[0-1] \qquad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\boxed{\int_0^\infty \left(\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right) ds = \frac{\pi}{2(a+b)}} - - - - - - -(1)$$

Put a=3 & b=4 and s=x we get

$$(1) \Rightarrow \int_0^\infty \frac{x^2}{(x^2+9)(x^2+16)} dx = \frac{\pi}{2(3+4)}$$

$$\boxed{\int_0^\infty \frac{x^2}{(x^2+9)(x^2+16)} dx = \frac{\pi}{14}}$$

(b) Assume $f(x) = e^{-ax}$; $g(x) = e^{-bx}$

The Fourier cosine transform f(x) is

$$F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\, dx.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\, dx$$

$$\boxed{F_c(s) = F_c\left(e^{-ax}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right)} \qquad \because \int_0^\infty e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2} \text{ here } a = a; b = s$$

Similarly

$$\boxed{G_c(s) = F_c\left(e^{-bx}\right) = \sqrt{\frac{2}{\pi}}\left(\frac{b}{b^2+s^2}\right)}$$

We know that

$$\int_0^\infty F_c(s)G_c(s)\,ds = \int_0^\infty f(x)g(x)\,dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+s^2}\right)\sqrt{\frac{2}{\pi}}\left(\frac{b}{b^2+s^2}\right)ds = \int_0^\infty e^{-ax}e^{-bx}\,dx$$

$$\frac{2ab}{\pi}\int_0^\infty\left(\frac{1}{(a^2+s^2)(b^2+s^2)}\right)ds = \int_0^\infty e^{-ax-bx}\,dx$$

$$\int_0^\infty\left(\frac{1}{(s^2+a^2)(s^2+b^2)}\right)ds = \frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}\,dx$$

$$= \frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}\,dx$$

$$= \frac{\pi}{2ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \frac{-\pi}{2ab(a+b)}\left[e^{-\infty}-e^{-0}\right]_0^\infty$$

$$= \frac{-\pi}{2ab(a+b)}[0-1] \qquad \because e^{-\infty}=0;\ e^{-0}=1$$

$$\boxed{\int_0^\infty\left(\frac{1}{(s^2+a^2)(s^2+b^2)}\right)ds = \frac{\pi}{2ab(a+b)}}$$

Put a=1 & b=2 s=x we get

$$(1) \Rightarrow \int_0^\infty \frac{1}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{2(1)(2)(1+2)}$$

$$\int_0^\infty \frac{1}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{12}$$

Self reciprocal:

If a transformation of a function *f(x)* is equal to *f(s)* then the function *f(x)* is called self reciprocal.

14.

Find the Fourier transform of $e^{-a^2x^2}$ Hence prove that $e^{-\frac{x^2}{2}}$ is self reciprocal with respect to Fourier Transforms.

Solution:

The Fourier transform f(x) is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2+isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2 - isx\right)} dx$$

$(A - B)^2 = A^2 - 2AB + B^2$

$2AB = isx$

Here $A = ax\ , B = \frac{is}{2a}$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(ax)^2 - isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{\left(\frac{is}{2a}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\left(\frac{is}{2a}\right)^2} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

Let $u = ax - \frac{is}{2a} \Rightarrow du = a\,dx \Rightarrow dx = \frac{du}{a}$; $u : -\infty \ to\ \infty$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{i^2 s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a}$$

$$= \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du \qquad \because i^2 = -1$$

$$= \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} 2\int_{0}^{\infty} e^{-u^2} du \qquad \because e^{-u^2} \text{ is an even function}$$

$$= \frac{1}{a\sqrt{2}\sqrt{\pi}} e^{\frac{-s^2}{4a^2}} 2\frac{\sqrt{\pi}}{2} \qquad \because \int_{0}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}} \qquad ------(1)$$

Deduction:

To prove $e^{\frac{-x^2}{2}}$ is self reciprocal

It is enough to prove that $F\left[e^{-\frac{x^2}{2}}\right]$ is $e^{-\frac{s^2}{2}}$

Put $a = \frac{1}{\sqrt{2}}$ in (1)

$$F\left[e^{-\left(\frac{1}{2}\right)x^2}\right] = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}} e^{\frac{-s^2}{4\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$F\left[e^{-\frac{x^2}{2}}\right] = e^{\frac{\frac{-s^2}{4}}{2}}$$

$$F\left[e^{-\frac{x^2}{2}}\right] = e^{\frac{-s^2}{2}}$$

$\therefore e^{-\frac{x^2}{2}}$ is self reciprocal.

15. **Find the Fourier transform of $e^{-\frac{x^2}{2}}$.**

(or) Show that $e^{-\frac{x^2}{2}}$ is self reciprocal with respect to Fourier Transforms.

Solution:

Let $f(x) = e^{-\frac{x^2}{2}}$

Assume $f(x) = e^{-a^2x^2}$ where $a = \frac{1}{\sqrt{2}}$

The Fourier transform f(x) is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2+isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(a^2x^2-isx\right)}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[(ax)^2-isx+\left(\frac{is}{2a}\right)^2-\left(\frac{is}{2a}\right)^2\right]}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} e^{\left(\frac{is}{2a}\right)^2}dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\left(\frac{is}{2a}\right)^2}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2}dx$$

$(A-B)^2 = A^2 - 2AB + B^2$ $2AB = isx$ Here $A = ax$, $B = \frac{is}{2a}$

Let $u = ax - \frac{is}{2a} \Rightarrow du = adx \Rightarrow dx = \frac{du}{a}$; $u: -\infty \; to \; \infty$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{i^2s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-u^2}\frac{du}{a}$$

$$= \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-u^2}du \quad \because i^2 = -1$$

$$= \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}}\, 2\int_{0}^{\infty} e^{-u^2}du \quad \because e^{-u^2} \text{ is an even function}$$

$$= \frac{1}{a\sqrt{2}\sqrt{\pi}} e^{\frac{-s^2}{4a^2}}\, 2\frac{\sqrt{\pi}}{2} \qquad \because \int_{0}^{\infty} e^{-u^2}du = \frac{\sqrt{\pi}}{2}$$

$$\boxed{F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}}} \qquad ------(1)$$

Deduction:

To prove $e^{\frac{-x^2}{2}}$ is self reciprocal

It is enough to prove that $F\left[e^{-\frac{x^2}{2}}\right]$ is $e^{-\frac{s^2}{2}}$

Put $a = \frac{1}{\sqrt{2}}$ in (1)

$$F\left[e^{-\left(\frac{1}{\sqrt{2}}\right)^2 x^2}\right] = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}} e^{\frac{-s^2}{4\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$F\left[e^{-\frac{x^2}{2}}\right] = e^{\frac{\frac{-s^2}{4}}{2}}$$

$$\boxed{F\left[e^{-\frac{x^2}{2}}\right] = e^{\frac{-s^2}{2}}}$$

$\therefore e^{\frac{-x^2}{2}}$ is self reciprocal.

16. **Find the Fourier cosine transform of** $e^{-a^2x^2}$ **Hence find** $F_s\left[xe^{-a^2x^2}\right]$.

Solution:

Let $f(x) = e^{-a^2x^2}$

The Fourier cosine transform f(x) is

$$F_c\left[f(x)\right] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx\,dx \qquad \because \int_0^\infty f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-a^2x^2}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{1}{2} \int_{-\infty}^{\infty} e^{-a^2x^2}\cos sx\,dx$$

$$F_c\left[f(x)\right] = \text{R.P.of } \frac{1}{2}\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}dx \qquad \because \cos sx = \text{R.P.of } e^{isx}$$

$$F_c\left[f(x)\right] = \text{R.P.of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}dx$$

$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx}dx$$

$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2+isx}dx$$

$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2-isx\right)}dx$$

$(a-b)^2 = a^2 - 2ab + b^2$

$-2ab = isx$

Here $a = ax$

$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(ax)^2 - isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]} dx$

$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{\left(\frac{is}{2a}\right)^2} dx$

$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} e^{\left(\frac{is}{2a}\right)^2} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$

Let $u = ax - \frac{is}{2a} \Rightarrow du = adx \Rightarrow dx = \frac{du}{a}\ ;\ u : -\infty \text{ to } \infty$

$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} e^{\frac{i^2s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a}$

$= \text{R.P.of } \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du \quad \because i^2 = -1$

$= \text{R.P.of } \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}}\, 2\int_{0}^{\infty} e^{-u^2} du \quad \because e^{-u^2}$ is an even function

$= \text{R.P.of } \frac{1}{a\sqrt{2}\sqrt{\pi}} e^{\frac{-s^2}{4a^2}}\, 2\frac{\sqrt{\pi}}{2} \qquad \because \int_{0}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$

$F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}}$ ------(1)

Deduction:

$F_s\left[xf(x)\right] = -\frac{d}{ds}\left\{F_c\left[f(x)\right]\right\} = -\frac{d}{ds}\left[F_c(s)\right]$

$F_s\left[xe^{-a^2x^2}\right] = -\frac{d}{ds}\left\{F_c\left[e^{-a^2x^2}\right]\right\}$

$= -\frac{d}{ds}\left[\frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}}\right]$

$= -\frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}} \left(\frac{-2s}{4a^2}\right)$

$F_s\left[xe^{-a^2x^2}\right] = \frac{s}{2\sqrt{2}a^3} e^{\frac{-s^2}{4a^2}}$

17. **Solve for *f(x)*, the integral equation** $\int_{0}^{\infty} f(x)\sin sx\, dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2. \\ 0, & s \ge 2 \end{cases}$

Solution:

Given $\int_{0}^{\infty} f(x)\sin sx\, dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2. \\ 0, & s \ge 2 \end{cases}$ ------ (1)

We know that

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$f(x) = \sqrt{\frac{2}{\pi}}\,F_s^{-1}\left\{\begin{array}{l}1, 0\le s<1\\ 2, 1\le s<2\\ 0,\ s\ge 2\end{array}\right\} \qquad F_s^{-1}[F(s)] = f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\,\sin sx\,ds$$

$$f(x) = \sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\left[\int_0^1 1\sin sx\,ds + \int_1^2 2\sin sx\,ds + \int_2^\infty 0\sin sx\,ds\right]$$

$$= \frac{2}{\pi}\left[\int_0^1 1\sin sx\,ds + \int_1^2 2\sin sx\,ds\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-\cos sx}{x}\right)_0^1 + 2\left(\frac{-\cos sx}{x}\right)_1^2\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos x}{x} + \frac{\cos 0}{x} + 2\left(\frac{-\cos 2x}{x} + \frac{\cos x}{x}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos x}{x} + \frac{1}{x} - 2\frac{\cos 2x}{x} + 2\frac{\cos x}{x}\right]$$

$$= \frac{2}{\pi x}[1 - \cos x - 2\cos 2x + 2\cos x]$$

$$f(x) = \frac{2}{\pi x}[1 + \cos x - 2\cos 2x]$$

18. **Find the Fourier cosine and sine transform of x^{n-1}. Hence show that $\frac{1}{x}$ is self reciprocal under Fourier cosine and sine transforms.**

Solution:

By definition of Gamma integral

$$\int_0^\infty e^{-ax}x^{n-1}\,dx = \frac{\Gamma n}{a^n},\ a > 0, n > 0$$

Put $a = is$

$$\int_0^\infty e^{-isx}x^{n-1}\,dx = \frac{\Gamma n}{(is)^n},\ a > 0, n > 0$$

$$\int_0^\infty x^{n-1}e^{-isx}\,dx = \frac{\Gamma n}{i^n s^n}$$

$$= \frac{\Gamma n}{s^n}(-i)^n$$

$$= \frac{\Gamma n}{s^n}\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^n \qquad \because e^{-i\frac{\pi}{2}} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i$$

$$= \frac{\Gamma n}{s^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right) \qquad \because \text{by Demorives theorem } (\cos\theta \pm i\sin\theta)^n = \cos n\theta \pm i\sin n\theta$$

$$\int_0^\infty x^{n-1}(\cos sx - i\sin sx)\,dx = \frac{\Gamma n}{s^n}\left(\cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}\right)$$

$$\int_0^\infty x^{n-1}\cos sx\,dx - i\int_0^\infty x^{n-1}\sin sx\,dx = \frac{\Gamma n}{s^n}\cos\frac{n\pi}{2} - i\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$$

	Equating real and imaginary parts on both sides $\int_0^\infty x^{n-1}\cos sx\,dx = \frac{\Gamma n}{s^n}\cos\frac{n\pi}{2}$ $\quad$ $\int_0^\infty x^{n-1}\sin sx\,dx = \frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$ $\sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\cos sx\,dx = \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\cos\frac{n\pi}{2}$ $\quad$ $\sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\sin sx\,dx = \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$ $\boxed{F_c\left(x^{n-1}\right) = \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\cos\frac{n\pi}{2}}$ $\quad$ $\boxed{F_s\left(x^{n-1}\right) = \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}}$ Deduction: To prove $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine and sine transforms. It is enough to prove that $F_c\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{s}}$ and $F_s\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{s}}$ We know that $F_c\left(x^{n-1}\right) = \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\cos\frac{n\pi}{2}$ $\quad$ $F_s\left(x^{n-1}\right) = \sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$ Put $n = \frac{1}{2}$ $F_c\left(x^{\frac{1}{2}-1}\right) = \sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\cos\frac{\pi}{4}$ $\quad$ $F_s\left(x^{\frac{1}{2}-1}\right) = \sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\sin\frac{\pi}{4}$ $F_c\left(x^{-\frac{1}{2}}\right) = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\frac{1}{\sqrt{2}}$ $\quad$ $F_s\left(x^{-\frac{1}{2}}\right) = \frac{\sqrt{2}}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\frac{1}{\sqrt{2}}$ $\quad$ $\because \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ $\boxed{F_c\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{s}}}$ $\quad$ $\boxed{F_s\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{s}}}$ $\therefore \frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine and sine transforms.
19.	**Find the function $f(x)$ if its sine transform is $\frac{e^{-as}}{s}$** **Solution:** Given $F_s\left[f(x)\right] = F_s(s) = \frac{e^{-as}}{s}$ $f(x) = F_s^{-1}\left[F_s(s)\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds$ $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx\,ds$ Taking diff on both sides w.r.to x $\frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx\,ds\right)$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-as}}{s} \frac{\partial}{\partial x}(\sin sx)\, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-as}}{\not{s}} \cos sx \times \not{s}\, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-as} \cos sx\, ds$$

$$\frac{d}{dx}[f(x)] = \sqrt{\frac{2}{\pi}} \left(\frac{a}{a^2+x^2} \right) \qquad \because \int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2+s^2} \text{ here } a = a, b = x$$

Integrating on w.r.to x

$$f(x) = \sqrt{\frac{2}{\pi}} a \int \frac{1}{a^2+x^2} \qquad \because \int_0^\infty \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$= \sqrt{\frac{2}{\pi}} a \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right)}$$