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DEPARTMENT OF MATHEMATICS

NAME OF THE SUBJECT : TRANSFORMS & PARTIAL DIFFERENTIAL EQUATION

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UNIT – IV : FOURIER TRANSFORMS

UNIT – IV FOURIER TRANSFORMS

	IMPORTANT FORMULAE
1.	**Fourier transform pair:** i) The Fourier Transform of *f(x)* is $F\left[f(x)\right]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx$ ii) The Inverse Fourier Transform of *F(s)* is $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$ Here *F(s) & f(x)* are called Fourier transform pair.
2.	**Fourier Cosine transform pair:** i) The Fourier Cosine Transform of *f(x)* is $F_c\left[f(x)\right]=F_c(s)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\cos sx\,dx$ ii) The Inverse Fourier Cosine Transform of $F_c(s)$ is $f(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}F_c(s)\cos sx\,ds$ Here $F_c(s)$ *& f(x)* are called Fourier cosine transform pair.
3.	**Fourier Sine transform pair:** i) The Fourier Sine Transform of *f(x)* is $F_s\left[f(x)\right]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\sin sx\,dx$ ii) The Inverse Fourier Sine Transform of $F_s(s)$ is $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F_s(s)\sin sx\,ds$ Here $F_s(s)$ *& f(x)* are called Fourier sine transform pair.
4.	Parsevals Identity for Fourier transform: $\int_{-\infty}^{\infty}\left\|F(s)\right\|^2 ds=\int_{-\infty}^{\infty}\left\|f(x)\right\|^2 dx$
5.	Parsevals Identity for Fourier Cosine transform: $\int_{0}^{\infty}\left\|F_c(s)\right\|^2 ds=\int_{0}^{\infty}\left\|f(x)\right\|^2 dx$
6.	Parsevals Identity for Fourier Sine transform: $\int_{0}^{\infty}\left\|F_s(s)\right\|^2 ds=\int_{0}^{\infty}\left\|f(x)\right\|^2 dx$
7.	1) $\int_{0}^{\infty}e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}$ 2) $\int_{0}^{\infty}e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}$ 3) $\int_{0}^{\infty}e^{-u^2}du=\frac{\sqrt{\pi}}{2}$

	4) $F\left[xf(x)\right]=(-i)\frac{d}{ds}\left\{F\left[f(x)\right]\right\}=(-i)\frac{d}{ds}\left[F(s)\right]$ 5) $F_s\left[xf(x)\right]=-\frac{d}{ds}\left\{F_c\left[f(x)\right]\right\}=-\frac{d}{ds}\left[F_c(s)\right]$ 6) $F_c\left[xf(x)\right]=\frac{d}{ds}\left\{F_s\left[f(x)\right]\right\}=\frac{d}{ds}\left[F_s(s)\right]$ 7) If *f(x)* and *g(x)* are any two functions and $F_c(s)$ & $G_c(s)$ are there Fourier cosine transforms then $\int_0^\infty f(x)g(x)dx=\int_0^\infty F_c(s)G_c(s)ds$ holds. 8) If *f(x)* and *g(x)* are any two functions and $F_s(s)$ & $G_s(s)$ are there Fourier sine transforms then $\int_0^\infty f(x)g(x)dx=\int_0^\infty F_s(s)G_s(s)ds$ holds.
	PART -A
1.	**State Fourier integral theorem.** **Solution :** If $f(x)$ is piecewise continuous, differentiable and absolutely integrable in $(-\infty,\infty)$ then $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(t)\,e^{is(x-t)}\,dtds$
2.	**If F(s) is the Fourier transform of f (x), then show that** $F\{f(x-a)\}=e^{ias}\,F(s)$ **Solution :** Given $F\left[f(x)\right]=F(s)$ The Fourier Transform of *f(x)* is $F\left[f(x)\right]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx$ $F\left[f(x-a)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-a)e^{isx}dx$ Let $x-a=t \Rightarrow dx=dt$ $=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{is(t+a)}dt$ $=e^{isa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{ist}dt$ $\boxed{F\left[f(x-a)\right]=e^{ias}F\left[f(x)\right]}$
3.	**State Convolution theorem in Fourier Transform.** **Solution :** The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms . i.e. $F\left[f(x)*g(x)\right]=F\left[f(x)\right]F\left[g(x)\right]=F(s).G(s)$
4.	**If** $F\left\{f(x)\right\}=F(s),$ **then find** $F\left\{e^{iax}f(x)\right\}$**.** **Solution :**

	$F[f(x)]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$ $F\left[e^{iax}f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}f(x)e^{isx}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx+iax}dx$ $=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}dx$ $\boxed{F\left[e^{iax}f(x)\right]=F(s+a)}$
5.	**State and prove the change of scale property of Fourier Transform.** **Statement:** If $F[f(x)]=F(s)$ then $F[f(ax)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$ **Solution :** Given $F[f(x)]=F(s)$ The Fourier Transform of *f(x)* is $F[f(x)]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$ $F[f(ax)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$, If a > 0 $\; Put\ ax=t \Rightarrow adx=dt \Rightarrow dx=\frac{dt}{a}\quad when\ x=-\infty \Rightarrow t=-\infty\ and\ x=\infty \Rightarrow t=\infty$ $F[f(ax)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}\frac{dt}{a}$ $F[f(ax)]=\frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}dt=\frac{1}{a}F\left(\frac{s}{a}\right)$. -(1) If a < 0 $\; Put\ ax=t,\ adx=dt, dx=\frac{dt}{a}$ $when\ x=-\infty \Rightarrow t=\infty\ and\ x=\infty \Rightarrow t=-\infty$ $\Rightarrow F[f(ax)]=\frac{-1}{\sqrt{2\pi}}\int_{\infty}^{-\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}\frac{dt}{a}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\left(\frac{s}{a}\right)t}\frac{dt}{a}=\frac{1}{a}F\left(\frac{s}{a}\right)$. ---(2) From (1) & (2) we get $F(f(ax))=\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right),\ a\neq 0$
6.	**Find the Fourier Sine transform of** $\frac{1}{x}$**.**
	Solution : The Fourier Sine Transform of *f(x)* is $F_s[f(x)]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\, dx$

$$F_s\left(\frac{1}{x}\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin sx}{x}dx$$

$$F_s\left(\frac{1}{x}\right)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\sin t}{t}dt=\sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right)=\sqrt{\frac{\pi}{2}} \qquad \because \int_0^{\infty}\frac{\sin mx}{x}dx=\frac{\pi}{2}$$

PART-B

1. **Find the Fourier transforms of** $f(x)=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$ **and hence evaluate** $\int_0^{\infty}\frac{\sin x}{x}dx$. **Using Parseval's identity, prove that** $\int_0^{\infty}\frac{\sin^2 t}{t^2}dt=\frac{\pi}{2}$.

Solution: Given $f(x)=\begin{cases}1, & -a<x<a\\ 0, & \text{otherwise}\end{cases}$

The Fourier transform f(x) is

$$F[f(x)]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx.$$

$$F(s)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a}0\,e^{isx}dx+\int_{-a}^{a}1e^{isx}dx+\int_{a}^{\infty}0\,e^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}\cos sx\,dx+i\int_{-a}^{a}\sin sx\,dx\right] \quad \because \text{sinsx is an even fn} \therefore \int_{-a}^{a}\sin sx\,dx=0$$

$$=\frac{1}{\sqrt{2\pi}}2\int_0^{a}\cos sx\,dx$$

$$=\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[\frac{\sin sx}{s}\right]_0^{a} \qquad =\sqrt{\frac{2}{\pi}}\left[\frac{\sin as}{s}-0\right]$$

$$\boxed{F(s)=\sqrt{\frac{2}{\pi}}\left[\frac{\sin as}{s}\right]}$$

Deduction: 1

By inverse Fourier transform of *F(s)*.

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\frac{\sin as}{s}\right)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)(\cos sx-i\sin sx)\,ds$$

$$=\frac{\sqrt{2}}{\sqrt{2}\sqrt{\pi}\sqrt{\pi}}\left[\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)(\cos sx)\,ds-i\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)(\sin sx)\,ds\right]$$

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\left(\frac{\sin as}{s}\right)\cos sx\,ds \qquad \because \left(\frac{\sin as}{s}\right)\sin sx \textit{ is an odd function}$$

$$\int_0^{\infty}\left(\frac{\sin as}{s}\right)\cos sx\,ds=\frac{\pi}{2}f(x)$$

Put x=0

$$\int_0^{\infty}\left(\frac{\sin as}{s}\right)\cos 0\,ds=\frac{\pi}{2}f(0)$$

$$\int_0^{\infty}\left(\frac{\sin as}{s}\right)ds=\frac{\pi}{2}(1) \qquad \because f(x)=1\Rightarrow f(0)=1$$

Put $a=1$ and $s=x$ we get

$$\boxed{\therefore \int_0^{\infty}\left(\frac{\sin x}{x}\right)dx=\frac{\pi}{2}}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty}[F(s)]^2\,dx=\int_{-\infty}^{\infty}[f(x)]^2\,ds$$

$$\int_{-\infty}^{\infty}\left[\sqrt{\frac{2}{\pi}}\frac{\sin sa}{s}\right]^2 ds=\int_{-a}^{a}1^2\,dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{\sin sa}{s}\right]^2 ds=[x]_{-a}^{a}$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds=[a-(-a)]$$

$$\frac{2}{\pi}2\int_0^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds=2a$$

$$\int_0^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds=\frac{2\pi a}{4}$$

Put $a=1$ & $s=t$ we get,

$$\boxed{\int_0^{\infty}\frac{\sin^2 t}{t^2}dt=\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}}.$$

2. **Find the Fourier transform of** $f(x)=\begin{cases} x\,; & if\ |x|<a \\ 0\,; & if\ |x|>a \end{cases}$.

Solution: Given $f(x)=\begin{cases} x, & -a<x<a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform f(x) is

$$F[f(x)]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx\,.$$

$$F(s)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a}0\,e^{isx}dx+\int_{-a}^{a}x\,e^{isx}dx+\int_{a}^{\infty}0\;e^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}x(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}x\cos sx\;dx+i\int_{-a}^{a}x\sin sx\,dx\right]\quad \because x\cos sx \text{ is an } odd \text{ fn } \therefore \int_{-a}^{a}x\cos sx\,dx=0$$

$$=i\frac{1}{\sqrt{2\pi}}2\int_{0}^{a}x\sin sx\;dx\qquad \because x\sin x \text{ is an even function } \therefore \int_{-a}^{a}x\sin sx\,dx=2\int_{0}^{a}x\sin sx\,dx$$

$$=i\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[(x)\left(\frac{-\cos sx}{s}\right)-(1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^a$$

$$=i\sqrt{\frac{2}{\pi}}\left[-\frac{x\cos sx}{s}+\frac{\sin sx}{s^2}\right]_0^a$$

$$=i\sqrt{\frac{2}{\pi}}\left[\left(-\frac{a\cos sa}{s}+\frac{\sin sa}{s^2}\right)-(0)\right]$$

$$\boxed{F(s)=i\sqrt{\frac{2}{\pi}}\left[\left(\frac{\sin sa-as\cos sa}{s^2}\right)\right]}$$

3. **Find the Fourier transform of** $f(x)=\begin{cases}a-|x|\,; & if\ |x|<a\\ 0\,; & if\ |x|>a>0\end{cases}$ **is** $\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s^2}\right)$**. Hence deduce that (i)** $\int_0^\infty\left(\frac{\sin t}{t}\right)^2dt=\frac{\pi}{2}$. **(ii)** $\int_0^\infty\left(\frac{\sin t}{t}\right)^4dt=\frac{\pi}{3}$.

Solution: Given $f(x)=\begin{cases}a-|x|, & -a<x<a\\ 0, & \text{otherwise}\end{cases}$

The Fourier transform f(x) is

$$F\left[f(x)\right]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx\,.$$

$$F(s)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a}0\,e^{isx}dx+\int_{-a}^{a}\left(a-|x|\right)e^{isx}dx+\int_{a}^{\infty}0\;e^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left(a-|x|\right)(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}\left(a-|x|\right)\cos sx\;dx+i\int_{-a}^{a}\left(a-|x|\right)\sin sx\,dx\right]$$

$$\because\left(a-|x|\right)\sin sx \text{ is an } odd \text{ fn} \therefore \int_{-a}^{a}\left(a-|x|\right)\sin sx\,dx=0$$

$$=\frac{1}{\sqrt{2\pi}}2\int_{0}^{a}\left(a-x\right)\cos sx\quad dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[(a-x)\left(\frac{\sin sx}{s}\right)-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{\cos sx}{s^2}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(-\frac{1}{s^2}\right)(\cos sa - \cos 0)\right]$$

$$\boxed{F(s) = \sqrt{\frac{2}{\pi}}\left[\frac{1-\cos sa}{s^2}\right]}$$

$$\boxed{F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin^2\left(\frac{as}{2}\right)}{s^2}\right]} \qquad \because \sin^2\theta = \frac{1-\cos 2\theta}{2} \Rightarrow 1-\cos\theta = 2\sin^2\left(\frac{\theta}{2}\right) \text{here } \theta = \frac{as}{2}$$

Deduction: 1

By inverse Fourier transform of *F(s)*.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(2\sqrt{\frac{2}{\pi}}\left[\frac{\sin^2\left(\frac{as}{2}\right)}{s^2}\right]\right)e^{-isx}ds$$

$$= \frac{2}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2(\cos sx - i\sin sx)\,ds$$

$$= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2(\cos sx)\,ds - i\int_{-\infty}^{\infty}\left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2(\sin sx)\,ds\right]$$

$$f(x) = \frac{4}{\pi}\left[\int_{0}^{\infty}\left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2(\cos sx)\,ds\right] \qquad \because \left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2(\sin sx) \text{ is an odd function}$$

$$\int_{0}^{\infty}\left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2\cos sx\,ds = \frac{\pi}{4}f(x)$$

Put x=0

$$\int_0^\infty \left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2 (\cos 0)\, ds = \frac{\pi}{4} f(0)$$

$$\int_0^\infty \left(\frac{\sin\left(\frac{as}{2}\right)}{s}\right)^2 ds = \frac{\pi a}{4} \qquad \because f(x) = a - |x| \Rightarrow f(0) = a$$

Put a=1 and s=t get

$$\int_0^\infty \left(\frac{\sin\left(\frac{s}{2}\right)}{s}\right)^2 ds = \frac{\pi}{4} \qquad put\, \frac{s}{2} = t \Rightarrow \frac{ds}{2} = dt$$

$$\int_0^\infty \left(\frac{\sin t}{2t}\right)^2 2dt = \frac{\pi}{4}$$

$$\boxed{\therefore \int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{\pi}{2}}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^\infty [F(s)]^2 dx = \int_{-\infty}^\infty [f(x)]^2 ds$$

$$\int_{-\infty}^\infty \left[2\sqrt{\frac{2}{\pi}}\left[\frac{\sin^2\left(\frac{as}{2}\right)}{s^2}\right]\right]^2 ds = \int_{-a}^a \left(a - |x|\right)^2 dx$$

$$\frac{8}{\pi} 2\int_0^\infty \left[\frac{\sin\left(\frac{as}{2}\right)}{s}\right]^4 ds = 2\int_0^a \left(a - x\right)^2 dx \quad \because \left(a - |x|\right)^2 \text{ and } \frac{\sin^2\left(\frac{as}{2}\right)}{s^2} \text{ are even functions}$$

$$\frac{8}{\pi} \int_0^\infty \left[\frac{\sin\left(\frac{as}{2}\right)}{s}\right]^4 ds = \int_0^a \left(a - x\right)^2 dx$$

$$\frac{8}{\pi} \int_0^\infty \left[\frac{\sin\left(\frac{as}{2}\right)}{s}\right]^4 ds = \left[\frac{\left(a - x\right)^3}{-3}\right]_0^a$$

	$\frac{8}{\pi}\int_0^\infty\left[\frac{\sin\left(\frac{as}{2}\right)}{s}\right]^4 ds=\left[(0)-\left(\frac{-a^3}{3}\right)\right]$ $\int_0^\infty\left[\frac{\sin\left(\frac{as}{2}\right)}{s}\right]^4 ds=\frac{a^3\pi}{3\times 8}$ Put *a =1* & *s=t* we get, $\int_0^\infty\left[\frac{\sin\left(\frac{s}{2}\right)}{s}\right]^4 ds=\frac{\pi}{24}$ $\quad put\frac{s}{2}=t\Rightarrow\frac{ds}{2}=dt$ $\int_0^\infty\left[\frac{\sin t}{2t}\right]^4 2dt=\frac{\pi}{24}$ $\boxed{\int_0^\infty\left[\frac{\sin t}{t}\right]^4 dt=\frac{\pi}{3}}$.
4.	**Find the Fourier transform of** $f(x)=\begin{cases}1-\lvert x\rvert, & \lvert x\rvert<1\\ 0, & \lvert x\rvert\ge 1\end{cases}$ **and hence find the value of** **(i)** $\int_0^\infty\frac{\sin^2 t}{t^2}dt$. **(ii)** $\int_0^\infty\frac{\sin^4 t}{t^4}dt$. **Sloution:** Hint in the previous problem *a=1*.
5.	**Find the Fourier transform of** $f(x)=\begin{cases}a^2-x^2, & \lvert x\rvert\le a\\ 0, & \lvert x\rvert\ge a\end{cases}$ **and hence evaluate** **(i)** $\int_0^\infty\left(\frac{\sin t-t\cos t}{t^3}\right)dt=\frac{\pi}{4}$ **(ii)** $\int_0^\infty\left(\frac{\sin t-t\cos t}{t^3}\right)^2 dt=\frac{\pi}{15}$ **Solution:** Given $f(x)=\begin{cases}a^2-x^2, & -a<x<a\\ 0, & \text{otherwise}\end{cases}$ The Fourier transform of f(x) is $F[f(x)]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx$. $F(s)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a}0\,e^{isx}dx+\int_{-a}^{a}\left(a^2-x^2\right)e^{isx}dx+\int_a^\infty 0\,e^{isx}dx\right]$ $=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left(a^2-x^2\right)(\cos sx+i\sin sx)\,dx$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}\left(a^2-x^2\right)\cos sx\ dx + i\int_{-a}^{a}\left(a^2-x^2\right)\sin sx\,dx\right]$$

$$\because \left(a^2-x^2\right)\sin sx \text{ is an } odd \text{ fn} \therefore \int_{-a}^{a}\left(a^2-x^2\right)\sin sx\,dx = 0$$

$$= \frac{1}{\sqrt{2\pi}}2\int_{0}^{a}\left(a^2-x^2\right)\cos sx\,dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[\left(a^2-x^2\right)\left(\frac{\sin sx}{s}\right)-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$= -2\sqrt{\frac{2}{\pi}}\left[\frac{x\cos sx}{s^2}-\frac{\sin sx}{s^3}\right]_0^a$$

$$= -2\sqrt{\frac{2}{\pi}}\left[\left(\frac{a\cos sa}{s^2}-\frac{\sin sa}{s^3}\right)-(0)\right]$$

$$= -2\sqrt{\frac{2}{\pi}}\left[\frac{as\cos sa-\sin sa}{s^3}\right]$$

$$\boxed{F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - as\cos sa}{s^3}\right]}$$

Deduction: 1

By inverse Fourier transform of *F(s)*.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa-as\cos sa}{s^3}\right]\right)e^{-isx}ds$$

$$= \frac{2}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left[\frac{\sin sa-as\cos sa}{s^3}\right]\left(\cos sx - i\sin sx\right)ds$$

$$= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left[\frac{\sin sa-as\cos sa}{s^3}\right]\left(\cos sx\right)ds - i\int_{-\infty}^{\infty}\left[\frac{\sin sa-as\cos sa}{s^3}\right]\left(\sin sx\right)ds\right]$$

$$f(x) = \frac{4}{\pi}\int_{0}^{\infty}\left[\frac{\sin sa-as\cos sa}{s^3}\right]\cos sx\,ds \qquad \because \left[\frac{\sin sa-as\cos sa}{s^3}\right]\left(\sin sx\right) \text{is an odd function}$$

$$\int_{0}^{\infty}\left[\frac{\sin sa-as\cos sa}{s^3}\right]\cos sx\,ds = \frac{\pi}{4}f(x)$$

Put x=0

$$\int_{0}^{\infty}\left[\frac{\sin sa-as\cos sa}{s^3}\right]\left(\cos 0\right)ds = \frac{\pi}{4}f(0)$$

$$\int_{0}^{\infty}\left[\frac{\sin sa-as\cos sa}{s^3}\right]ds = \frac{\pi a^2}{4} \qquad \because f(x) = a^2-x^2 \Rightarrow f(0) = a^2$$

Put *a*=1 and s=t get

$$\boxed{\int_0^\infty \left[\frac{\sin t - t\cos t}{t^3}\right] dt = \frac{\pi}{4}}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2\, ds = \int_{-\infty}^{\infty} [f(x)]^2 dx$$

$$\int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa - as\cos sa}{s^3}\right]\right]^2 ds = \int_{-a}^{a} \left(a^2 - x^2\right)^2 dx$$

$$\frac{8}{\pi} 2\int_0^\infty \left[\frac{\sin sa - as\cos sa}{s^3}\right]^2 ds = 2\int_0^a \left(a^4 - 2a^2x^2 + x^4\right) dx$$

$$\because \left(a^2 - x^2\right)^2 \text{ and} \left[\frac{\sin sa - as\cos sa}{s^3}\right]^2 \text{ are even functions}$$

$$\frac{8}{\pi}\int_0^\infty \left[\frac{\sin sa - as\cos sa}{s^3}\right]^2 ds = \left(a^4x - \frac{2a^2x^3}{3} + \frac{x^5}{5}\right)_0^a$$

$$\frac{8}{\pi}\int_0^\infty \left[\frac{\sin sa - as\cos sa}{s^3}\right]^2 ds = \left(a^5 - \frac{2a^5}{3} + \frac{a^5}{5}\right)$$

$$\frac{8}{\pi}\int_0^\infty \left[\frac{\sin sa - as\cos sa}{s^3}\right]^2 ds = \left(\frac{15a^5 - 10a^5 + 3a^5}{15}\right)$$

$$\int_0^\infty \left[\frac{\sin sa - as\cos sa}{s^3}\right]^2 ds = \left(\frac{8a^5}{15}\right) \times \frac{\pi}{8}$$

Put *a =1* & *s=t* we get,

$$\boxed{\int_0^\infty \left[\frac{\sin t - t\cos t}{t^3}\right]^2 dt = \frac{\pi}{15}}$$

6.

Find the Fourier transform of $f(x) = \begin{cases} 1 - x^2\ ;\ if\ |x| < 1 \\ 0\ ; \quad if\ |x| \geq 1 \end{cases}$.

Hence show that $(i)\int_0^\infty \left[\frac{\sin s - s\cos s}{s^2}\right]\cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$ and $(ii)\int_0^\infty \frac{(x\cos x - \sin x)^2}{x^6} dx = \frac{\pi}{15}$

Solution: Given $f(x) = \begin{cases} 1 - x^2, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform f(x) is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx\ .$$

$$F(s) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1} 0\, e^{isx} dx + \int_{-1}^{1} \left(1 - x^2\right) e^{isx} dx + \int_1^\infty 0\ e^{isx} dx\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \left(1 - x^2\right)(\cos sx + i\sin sx)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1}(1-x^2)\cos sx\; dx + i\int_{-1}^{1}(1-x^2)\sin sx\, dx\right]$$

$\because (1-x^2)\sin sx$ is an *odd* fn $\therefore \int_{-1}^{1}(1-x^2)\sin sx\, dx = 0$

$$= \frac{1}{\sqrt{2\pi}}\, 2\int_{0}^{1}(1-x^2)\cos sx\, dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[(1-x^2)\left(\frac{\sin sx}{s}\right) - (-2x)\left(\frac{-\cos sx}{s^2}\right) + (-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^1$$

$$= -2\sqrt{\frac{2}{\pi}}\left[\frac{x\cos sx}{s^2} - \frac{\sin sx}{s^3}\right]_0^1$$

$$= -2\sqrt{\frac{2}{\pi}}\left[\left(\frac{\cos s}{s^2} - \frac{\sin s}{s^3}\right) - (0)\right]$$

$$= -2\sqrt{\frac{2}{\pi}}\left[\frac{s\cos s - \sin s}{s^3}\right]$$

$$\boxed{F(s) = 2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]}$$

Deduction: 1

By inverse Fourier transform of *F(s)*.

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]\right)e^{-isx}ds$$

$$= \frac{2}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right](\cos sx - i\sin sx)\,ds$$

$$= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right](\cos sx)\,ds - i\int_{-\infty}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right](\sin sx)\,ds\right]$$

$$f(x) = \frac{4}{\pi}\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]\cos sx\, ds \qquad \because \left[\frac{\sin s - s\cos s}{s^3}\right](\sin sx) \text{ is an odd function}$$

$$\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]\cos sx\, ds = \frac{\pi}{4}\, f(x)$$

Put $x = \frac{1}{2}$

$$\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]\cos\left(\frac{s}{2}\right)ds = \frac{\pi}{4}\, f\left(\frac{1}{2}\right)$$

$$\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]\cos\left(\frac{s}{2}\right)ds = \frac{\pi}{4}\times\frac{3}{4} \qquad \because f(x) = 1 - x^2 \Rightarrow f\left(\frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

	$\boxed{\int_0^\infty \left[\frac{\sin s - s\cos s}{s^3}\right]\cos\left(\frac{s}{2}\right)ds = \frac{3\pi}{16}}$ (ii) By Parseval's identity, $\int_{-\infty}^{\infty}[F(s)]^2\,ds = \int_{-\infty}^{\infty}[f(x)]^2dx$ $\int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]\right]^2 ds = \int_{-1}^{1}\left(1-x^2\right)^2 dx$ $\frac{8}{\pi}2\int_0^\infty\left[\frac{\sin sa - as\cos sa}{s^3}\right]^2 ds = 2\int_0^1\left(1-2x^2+x^4\right)dx$ $\because \left(1-x^2\right)^2$ and $\left[\frac{\sin s - s\cos s}{s^3}\right]^2$ are an even functions $\frac{8}{\pi}\int_0^\infty\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = \left(x-\frac{2x^3}{3}+\frac{x^5}{5}\right)_0^1$ $\frac{8}{\pi}\int_0^\infty\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = \left(1-\frac{2}{3}+\frac{1}{5}\right)$ $\frac{8}{\pi}\int_0^\infty\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = \left(\frac{15-10+3}{15}\right)$ $\int_0^\infty\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = \left(\frac{8}{15}\right)\times\frac{\pi}{8}$ Put $s=t$ we get, $\boxed{\int_0^\infty \frac{\left(\sin t - t\cos t\right)^2}{t^6}\,dx = \frac{\pi}{15}}$
7.	**Find the Fourier cosine and sine transform of** $f(x)=\begin{cases} x, & for\ 0<x<1 \\ 2-x, & for\ 1<x<2 \\ 0, & for\ x>2 \end{cases}$. **Solution:** Given $f(x)=\begin{cases} x, & for\ 0<x<1 \\ 2-x, & for\ 1<x<2 \\ 0, & for\ x>2 \end{cases}$ The Fourier Cosine transform of f(x) is $F_c\left[f(x)\right]=F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$. $=\sqrt{\frac{2}{\pi}}\left[\int_0^1 x\cos sx\,dx+\int_1^2(2-x)\cos sx\,dx+\int_2^\infty 0\cos sx\,dx\right]$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[(x)\left(\frac{\sin sx}{s}\right)-(1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1+\left[(2-x)\left(\frac{\sin sx}{s}\right)-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_1^2\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[\frac{x\sin sx}{s}+\frac{\cos sx}{s^2}\right]_0^1+\left[(2-x)\left(\frac{\sin sx}{s}\right)-\frac{\cos sx}{s^2}\right]_1^2\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[\left(\frac{\sin s}{s}+\frac{\cos s}{s^2}\right)-\left(0+\frac{1}{s^2}\right)\right]+\left[\left(0-\frac{\cos 2s}{s^2}\right)-\left(\frac{\sin s}{s}-\frac{\cos s}{s^2}\right)\right]\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left[\cancel{\frac{\sin s}{s}}+\frac{\cos s}{s^2}-\frac{1}{s^2}-\frac{\cos 2s}{s^2}-\cancel{\frac{\sin s}{s}}+\frac{\cos s}{s^2}\right]$$

$$\boxed{F_c(s)=\sqrt{\frac{2}{\pi}}\left[\frac{2\cos s-\cos 2s-1}{s^2}\right]}$$

The Fourier sine transform of f(x) is

$$F_s\left[f(x)\right]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx\,.$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^1 x\sin sx\,dx+\int_1^2(2-x)\sin sx\,dx+\cancel{\int_2^\infty 0\sin sx\,dx}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[(x)\left(\frac{-\cos sx}{s}\right)-(1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^1+\left[(2-x)\left(\frac{-\cos sx}{s}\right)-(-1)\left(\frac{-\sin sx}{s^2}\right)\right]_1^2\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[\frac{-x\cos sx}{s}+\frac{\sin sx}{s^2}\right]_0^1+\left[-(2-x)\left(\frac{\cos sx}{s}\right)-\frac{\sin sx}{s^2}\right]_1^2\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\left[\left(-\frac{\cos s}{s}+\frac{\sin s}{s^2}\right)-(0)\right]+\left[\left(0-\frac{\sin 2s}{s^2}\right)-\left(-\frac{\cos s}{s}-\frac{\sin s}{s^2}\right)\right]\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left[-\cancel{\frac{\cos s}{s}}+\frac{\sin s}{s^2}-\frac{\sin 2s}{s^2}+\cancel{\frac{\cos s}{s}}+\frac{\sin s}{s^2}\right]$$

$$\boxed{F_s(s)=\sqrt{\frac{2}{\pi}}\left[\frac{2\sin s-\sin 2s}{s^2}\right]}$$

8. **Find Fourier transform of $e^{-a|x|}$ and hence deduce that**

(a) $\int_0^\infty \frac{\cos xt}{a^2+t^2}dt=\frac{\pi}{2a}e^{-a|x|}$ **(b)** $F\left[xe^{-a|x|}\right]=i\sqrt{\frac{2}{\pi}}\frac{2as}{\left(s^2+a^2\right)^2}$.

The Fourier transform of f(x) is

$$F\left[f(x)\right]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\,.$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|}\left(\cos sx+i\sin sx\right)dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} e^{-a|x|} \cos sx\, dx + i \int_{-\infty}^{\infty} e^{-a|x|} \sin sx\, dx \right]$$

$\because e^{-a|x|} \sin sx$ is an *odd* fn $\therefore \int_{-\infty}^{\infty} e^{-a|x|} \sin sx\, dx = 0$

$$= \frac{1}{\sqrt{2\pi}} 2\int_0^{\infty} e^{-ax} \cos sx\, dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \int_0^{\infty} e^{-ax} \cos sx\, dx$$

$$F(s) = F\left[e^{-a|x|}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right] \qquad \because \int_0^{\infty} e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2} \text{ here } a = a; b = s$$

Deduction (a):

By inverse Fourier transform of *F(s)* is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2+s^2}\right] e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \left[\frac{a}{a^2+s^2}\right] (\cos sx - i \sin sx)\, ds$$

$$= \frac{a}{\pi} \left[\int_{-\infty}^{\infty} \left[\frac{1}{a^2+s^2}\right] (\cos sx)\, ds - ia \int_{-\infty}^{\infty} \left[\frac{1}{a^2+s^2}\right] (\sin sx)\, ds \right]$$

$$f(x) = \frac{2a}{\pi} \int_0^{\infty} \left[\frac{1}{a^2+s^2}\right] \cos sx\, ds \qquad \because \left(\frac{1}{a^2+s^2}\right)(\sin sx) \text{ is an odd function}$$

$$\int_0^{\infty} \left(\frac{1}{a^2+s^2}\right) \cos sx\, ds = \frac{\pi}{2a} f(x)$$

$$\boxed{\int_0^{\infty} \frac{\cos sx}{a^2+s^2} ds = \frac{\pi}{2a} e^{-a|x|}}$$

Put $s=t$

$$\boxed{\int_0^{\infty} \frac{\cos tx}{t^2+a^2} dt = \frac{\pi}{2a} e^{-a|x|}}$$

Deduction (b):

By Property

$$F\left[x\, f(x)\right] = -i \frac{d}{ds} [F(s)]$$

$$F\left[x e^{-a|x|}\right] = -i \frac{d}{ds} F\left(e^{-a|x|}\right)$$

$$= -i \frac{d}{ds} \left(\sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2} \right)$$

	$=-ia\sqrt{\frac{2}{\pi}}\left(\frac{-1}{\left(a^2+s^2\right)^2}(0+2s)\right)=i\sqrt{\frac{2}{\pi}}\frac{2as}{\left(a^2+s^2\right)^2}$ $\boxed{F\left[xe^{-a\|x\|}\right]=i\sqrt{\frac{2}{\pi}}\left(\frac{2as}{\left(s^2+a^2\right)^2}\right)}$
9.	**Find the Fourier sine and cosine transform of** $e^{-ax}, a>0$ **and deduce that** **i)** $\int_0^\infty \frac{s}{s^2+a^2}\sin sx\,dx=\frac{\pi}{2}e^{-ax}$**.** **ii)** $\int_0^\infty \frac{1}{s^2+a^2}\cos sx\,dx=\frac{\pi}{2a}e^{-ax}$ **Solution:** The Fourier sine transform of f(x) is $F_s\left[f(x)\right]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$. $=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$ $\boxed{F_s(s)=F_s\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]}$ $\because\int_0^\infty e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}$ here $a=a;b=s$ The Fourier cosine transform of f(x) is $F_c\left[f(x)\right]=F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$. $=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$ $\boxed{F_c(s)=F_c\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]}$ $\because\int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}$ here $a=a;b=s$ The inverse Fourier sine transform of $F_s(s)$ is $f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$ $=\sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\sin sx\,dx$ $=\frac{2}{\pi}\int_0^\infty\left[\frac{s}{a^2+s^2}\right]\sin sx\,dx$ $\int_0^\infty\left[\frac{s}{a^2+s^2}\right]\sin sx\,dx=\frac{\pi}{2}f(x)$ $\boxed{\int_0^\infty\left[\frac{s}{a^2+s^2}\right]\sin sx\,dx=\frac{\pi}{2}e^{-ax}}$

	The inverse Fourier Cosine transform of $F_c(s)$ is $f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$ $=\sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\cos sx\,dx$ $=\frac{2a}{\pi}\int_0^\infty \left[\frac{1}{a^2+s^2}\right]\cos sx\,dx$ $\int_0^\infty \left[\frac{a}{a^2+s^2}\right]\cos sx\,dx=\frac{\pi}{2}f(x)$ $\boxed{\int_0^\infty \left[\frac{a}{a^2+s^2}\right]\cos sx\,dx=\frac{\pi}{2a}e^{-ax}}$
10.	**Find the Fourier sine and cosine transform of** $e^{-ax}, a>0$ **and hence find** $F_c\left[xe^{-ax}\right]$ and $F_s\left[xe^{-ax}\right]$. **Solution:** The Fourier sine transform f(x) is $F_s\left[f(x)\right]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$. $=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$ $\boxed{F_s(s)=F_s\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]}$ $\because \int_0^\infty e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}$ here $a=a; b=s$ The Fourier cosine transform f(x) is $F_c\left[f(x)\right]=F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$. $=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$ $\boxed{F_c(s)=F_c\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]}$ $\because \int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}$ here $a=a; b=s$ We know that **i)** $F_s\left[xf(x)\right]=-\frac{d}{ds}\left\{F_c\left[f(x)\right]\right\}=-\frac{d}{ds}\left[F_c(s)\right]$ $F_s\left[xe^{-ax}\right]=-\frac{d}{ds}\left\{F_c\left[e^{-ax}\right]\right\}=-\frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\right\}$ $=-a\sqrt{\frac{2}{\pi}}\frac{d}{ds}\left\{\frac{1}{a^2+s^2}\right\}$ $=-a\sqrt{\frac{2}{\pi}}\left[\frac{-1}{\left(a^2+s^2\right)^2}(0+2s)\right]$

$$\boxed{F_s\left[xe^{-ax}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{2as}{\left(a^2+s^2\right)^2}\right]}$$

ii) $F_c\left[xf(x)\right]=\frac{d}{ds}\left\{F_s\left[f(x)\right]\right\}=\frac{d}{ds}\left[F_s(s)\right]$

$$F_s\left[xe^{-ax}\right]=\frac{d}{ds}\left\{F_c\left[e^{-ax}\right]\right\}=\frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\frac{\left(a^2+s^2\right)(1)-s(0+2s)}{\left(a^2+s^2\right)^2}\right\}$$

$$=\sqrt{\frac{2}{\pi}}\left\{\frac{a^2+s^2-2s^2}{\left(a^2+s^2\right)^2}\right\}$$

$$\boxed{F_s\left[xe^{-ax}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{a^2-s^2}{\left(a^2+s^2\right)^2}\right]}$$

11. **Find the Fourier sine transform of** $\frac{e^{-ax}}{x}, a>0$ **and hence find** $F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$.

Solution:

The Fourier sine transform of f(x) is

$$F_s\left[f(x)\right]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

$$F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx$$

Taking diff. on both sides w.r.to s

$$\frac{d}{ds}\left\{F_s\left[\frac{e^{-ax}}{x}\right]\right\}=\frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin sx\,dx\right\}$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{x}\frac{\partial}{\partial s}\left(\sin sx\right)dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-ax}}{\cancel{x}}\left(\cos sx\right)\cancel{x}\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\frac{d}{ds}\left\{F_s\left[\frac{e^{-ax}}{x}\right]\right\}=\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]$$

Integrating on on both sides w.r.to s

$$F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int\left[\frac{a}{a^2+s^2}\right]ds$$

	$$\boxed{F_s\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)}\qquad \because \int\frac{a}{x^2+a^2}dx=\tan^{-1}\left(\frac{x}{a}\right)$$ Similarly $F_s\left[\frac{e^{-bx}}{x}\right]=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{b}\right)$ Deduction: $$F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right]=F_s\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]$$ $$=F_s\left[\frac{e^{-ax}}{x}\right]-F_s\left[\frac{e^{-bx}}{x}\right]$$ $$=\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{a}\right)-\sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{s}{b}\right)$$ $$\boxed{F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right]=\sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{s}{a}\right)-\tan^{-1}\left(\frac{s}{b}\right)\right]}$$
12.	**Find the Fourier cosine transform of** $\frac{e^{-ax}}{x}, a>0$ **and hence find** $F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right]$ **Solution:** The Fourier cosine transform f(x) is $$F_c\left[f(x)\right]=F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx$$ $$F_c\left[\frac{e^{-ax}}{x}\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}\cos sx\,dx$$ Taking diff. on both sides w.r.to s $$\frac{d}{ds}\left\{F_c\left[\frac{e^{-ax}}{x}\right]\right\}=\frac{d}{ds}\left\{\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}\cos sx\,dx\right\}$$ $$=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{x}\frac{\partial}{\partial s}(\cos sx)\,dx$$ $$=\sqrt{\frac{2}{\pi}}\int_0^\infty\frac{e^{-ax}}{\not x}(-\sin sx)\not x\,dx$$ $$=-\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$ $$\frac{d}{ds}\left\{F_c\left[\frac{e^{-ax}}{x}\right]\right\}=-\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]$$ Integrating on on both sides w.r.to s $$F_c\left[\frac{e^{-ax}}{x}\right]=-\sqrt{\frac{2}{\pi}}\int\left[\frac{s}{a^2+s^2}\right]ds$$

$$= -\sqrt{\frac{2}{\pi}}\int\left[\frac{s}{a^2+s^2}\right]ds$$

$$= -\sqrt{\frac{2}{\pi}}\frac{1}{2}\int\left[\frac{2s}{a^2+s^2}\right]ds$$

$$= -\sqrt{\frac{2}{\pi}}\frac{1}{2}\log\left(s^2+a^2\right) \qquad \because \int\frac{f'(x)}{f(x)}dx = \log\left[f(x)\right]$$

$$= -\frac{1}{\sqrt{2\pi}}\log\left(s^2+a^2\right)$$

$$= \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+a^2}\right)$$

$$\boxed{F_c\left[\frac{e^{-ax}}{x}\right] = \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+a^2}\right)}$$

Similarly $F_c\left[\frac{e^{-bx}}{x}\right] = \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+b^2}\right)$

Deduction:

$$F_c\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = F_c\left[\frac{e^{-ax}}{x}-\frac{e^{-bx}}{x}\right]$$

$$= F_c\left[\frac{e^{-ax}}{x}\right] - F_c\left[\frac{e^{-bx}}{x}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+a^2}\right) - \frac{1}{\sqrt{2\pi}}\log\left(\frac{1}{s^2+b^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$\boxed{F_s\left[\frac{e^{-ax}-e^{-bx}}{x}\right] = \frac{1}{\sqrt{2\pi}}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)}$$

13. **Using Parseval'sidentity evaluate the following integrals.**

1) $\int_0^\infty \frac{dx}{(x^2+a^2)^2}$

2) $\int_0^\infty \frac{x^2}{(x^2+a^2)^2}dx$, where $a>0$.

Solution:

Assume $f(x) = e^{-ax}$

The Fourier sine transform f(x) is

$$F_s\left[f(x)\right] = F_s(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$\boxed{F_s(s) = F_s\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2} \text{ here } a = a; b = s$$

The Fourier cosine transform f(x) is

$$F_c\left[f(x)\right] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx \, .$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos x \, dx$$

$$\boxed{F_c(s) = F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2} \text{ here } a = a; b = s$$

(i) The Parseval's identity for Fourier cosine transform is

$$\int_0^\infty \left|F_c(s)\right|^2 ds = \int_0^\infty \left|f(x)\right|^2 dx$$

$$\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\right)^2 ds = \int_0^\infty \left(e^{-ax}\right)^2 dx$$

$$\frac{2a^2}{\pi} \int_0^\infty \frac{1}{\left(a^2+s^2\right)^2} ds = \int_0^\infty e^{-2ax} dx$$

$$\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2} ds = \frac{\pi}{2a^2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2} ds = \frac{-\pi}{4a^3}\left[e^{-\infty} - e^{-0}\right]$$

$$\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2} ds = \frac{-\pi}{4a^3}\left[0-1\right] \qquad \because e^{-\infty} = 0; \; e^{-0} = 1$$

$$\int_0^\infty \frac{1}{\left(a^2+s^2\right)^2} ds = \frac{\pi}{4a^3}$$

Put s=x we get

$$\boxed{\int_0^\infty \frac{1}{\left(a^2+x^2\right)^2} dx = \frac{\pi}{4a^3}}$$

(ii) The Parseval's identity for Fourier sine transform is

$$\int_0^\infty \left|F_s(s)\right|^2 ds = \int_0^\infty \left|f(x)\right|^2 dx$$

$$\int_0^\infty \left(\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\right)^2 ds = \int_0^\infty \left(e^{-ax}\right)^2 dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}\,ds=\int_0^\infty e^{-2ax}dx$$

$$\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}\,ds=\frac{\pi}{2}\left[\frac{e^{-2ax}}{-2a}\right]_0^\infty$$

$$\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}\,ds=\frac{-\pi}{4a}\left[e^{-\infty}-e^{-0}\right]$$

$$\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}\,ds=\frac{-\pi}{4a}\left[0-1\right] \qquad \because e^{-\infty}=0;\ e^{-0}=1$$

$$\int_0^\infty \frac{s^2}{\left(a^2+s^2\right)^2}\,ds=\frac{\pi}{4a}$$

Put s=x we get

$$\boxed{\int_0^\infty \frac{x^2}{\left(a^2+x^2\right)^2}\,dx=\frac{\pi}{4a}}$$

14. **Evaluate (a)** $\int_0^\infty \frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx$ **(b)** $\int_0^\infty \frac{1}{\left(x^2+a^2\right)\left(x^2+b^2\right)}dx$ **using Fourier transforms.**

Solution:

(a) Assume $f(x)=e^{-ax}\quad ;g(x)=e^{-bx}$

The Fourier sine transform f(x) is

$$F_s\left[f(x)\right]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx\,.$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\boxed{F_s(s)=F_s\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}\ \text{ here } a=a; b=s$$

Similarly

$$\boxed{G_s(s)=F_s\left[e^{-bx}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{s}{b^2+s^2}\right]}$$

We know that

$$\int_0^\infty F_s(s)G_s(s)ds=\int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{s}{b^2+s^2}\right]ds=\int_0^\infty e^{-ax}e^{-bx}dx$$

$$\frac{2}{\pi}\int_0^\infty \left[\frac{s^2}{\left(a^2+s^2\right)\left(b^2+s^2\right)}\right]ds = \int_0^\infty e^{-ax-bx}dx$$

$$\int_0^\infty \left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]ds = \frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

$$= \frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

$$= \frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \frac{-\pi}{2(a+b)}\left[e^{-\infty}-e^{-0}\right]_0^\infty$$

$$= \frac{-\pi}{2(a+b)}\left[0-1\right] \qquad \because e^{-\infty}=0; e^{-0}=1$$

$$\int_0^\infty \left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]ds = \frac{\pi}{2(a+b)}$$

Put s=x we get

$$\boxed{\int_0^\infty \left[\frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)}\right]dx = \frac{\pi}{2(a+b)}}$$

(b) Assume $f(x)=e^{-ax} \quad ; g(x)=e^{-bx}$

The Fourier cosine transform f(x) is

$$F_c\left[f(x)\right]=F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx\,.$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\boxed{F_c(s)=F_c\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2} \quad \text{here } a=a; b=s$$

Similarly

$$\boxed{G_c(s)=F_c\left[e^{-bx}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right]}$$

We know that

$$\int_0^\infty F_c(s)G_c(s)ds=\int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right]ds=\int_0^\infty e^{-ax}e^{-bx}dx$$

$$\frac{2ab}{\pi}\int_0^\infty\left[\frac{1}{(a^2+s^2)(b^2+s^2)}\right]ds=\int_0^\infty e^{-ax-bx}dx$$

$$\int_0^\infty\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]ds=\frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}dx$$

$$=\frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}dx$$

$$=\frac{\pi}{2ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$=\frac{-\pi}{2ab(a+b)}\left[e^{-\infty}-e^{-0}\right]_0^\infty$$

$$=\frac{-\pi}{2ab(a+b)}[0-1] \qquad \because e^{-\infty}=0; e^{-0}=1$$

$$\int_0^\infty\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]ds=\frac{\pi}{2ab(a+b)}$$

Put s=x we get

$$\boxed{\int_0^\infty\left[\frac{1}{(x^2+a^2)(x^2+b^2)}\right]dx=\frac{\pi}{2ab(a+b)}}$$

Evaluate (a) $\int_0^\infty\frac{x^2}{(x^2+9)(x^2+16)}dx$ **, (b)** $\int_0^\infty\frac{1}{(x^2+1)(x^2+4)}dx$ **using Fourier transforms.**

Solution:

(a) Assume $f(x)=e^{-ax}\quad ; g(x)=e^{-bx}$

The Fourier sine transform f(x) is

$$F_s\left[f(x)\right]=F_s(s)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx\,.$$

$$=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\sin sx\,dx$$

$$\boxed{F_s(s)=F_s\left[e^{-ax}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]} \qquad \because\int_0^\infty e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}\ \text{ here } a=a; b=s$$

Similarly

$$\boxed{G_s(s)=G_s\left[e^{-bx}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{s}{b^2+s^2}\right]}$$

We know that

$$\int_0^\infty F_s(s)G_s(s)ds=\int_0^\infty f(x)g(x)dx$$

$$\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{s}{a^2+s^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{s}{b^2+s^2}\right]ds = \int_0^\infty e^{-ax}e^{-bx}dx$$

$$\frac{2}{\pi}\int_0^\infty \left[\frac{s^2}{\left(a^2+s^2\right)\left(b^2+s^2\right)}\right]ds = \int_0^\infty e^{-ax-bx}dx$$

$$\int_0^\infty \left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]ds = \frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

$$= \frac{\pi}{2}\int_0^\infty e^{-(a+b)x}dx$$

$$= \frac{\pi}{2}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

$$= \frac{-\pi}{2(a+b)}\left[e^{-\infty} - e^{-0}\right]_0^\infty$$

$$= \frac{-\pi}{2(a+b)}\left[0-1\right] \qquad \because e^{-\infty}=0; e^{-0}=1$$

$$\boxed{\int_0^\infty \left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]ds = \frac{\pi}{2(a+b)}} ------(1)$$

Put a=3 & b=4 and s=x we get

$$(1) \Rightarrow \int_0^\infty \frac{x^2}{\left(x^2+9\right)\left(x^2+16\right)}dx = \frac{\pi}{2(3+4)}$$

$$\boxed{\int_0^\infty \frac{x^2}{\left(x^2+9\right)\left(x^2+16\right)}dx = \frac{\pi}{14}}$$

(b) Assume $f(x)=e^{-ax} \quad ; g(x)=e^{-bx}$

The Fourier cosine transform f(x) is

$$F_c\left[f(x)\right] = F_c(s) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx\,.$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos sx\,dx$$

$$\boxed{F_c(s) = F_c\left[e^{-ax}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]} \qquad \because \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2} \text{ here } a=a; b=s$$

Similarly

$$\boxed{G_c(s) = F_c\left[e^{-bx}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right]}$$

We know that

	$\int_0^\infty F_c(s)G_c(s)ds = \int_0^\infty f(x)g(x)dx$ $\int_0^\infty \sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+s^2}\right]\sqrt{\frac{2}{\pi}}\left[\frac{b}{b^2+s^2}\right]ds = \int_0^\infty e^{-ax}e^{-bx}dx$ $\frac{2ab}{\pi}\int_0^\infty\left[\frac{1}{(a^2+s^2)(b^2+s^2)}\right]ds = \int_0^\infty e^{-ax-bx}dx$ $\int_0^\infty\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]ds = \frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}dx$ $= \frac{\pi}{2ab}\int_0^\infty e^{-(a+b)x}dx$ $= \frac{\pi}{2ab}\left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$ $= \frac{-\pi}{2ab(a+b)}\left[e^{-\infty}-e^{-0}\right]_0^\infty$ $= \frac{-\pi}{2ab(a+b)}[0-1] \qquad \because e^{-\infty}=0; e^{-0}=1$ $\boxed{\int_0^\infty\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right]ds = \frac{\pi}{2ab(a+b)}}$ Put a=1 & b=2 s=x we get $(1) \Rightarrow \int_0^\infty \frac{1}{(x^2+1)(x^2+4)}dx = \frac{\pi}{2(1)(2)(1+2)}$ $\boxed{\int_0^\infty \frac{1}{(x^2+1)(x^2+4)}dx = \frac{\pi}{12}}$
Self reciprocal: If a transformation of a function *f(x)* is equal to *f(s)* then the function *f(x)* is called self reciprocal.	
14.	**Find the Fourier transform of $e^{-a^2x^2}$ Hence prove that $e^{\frac{-x^2}{2}}$ is self reciprocal with respect to Fourier Transforms.** **Solution:** The Fourier transform f(x) is $F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$ $F\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$ $= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2+isx}dx$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(a^2x^2-isx\right)}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[(ax)^2-isx+\left(\frac{is}{2a}\right)^2-\left(\frac{is}{2a}\right)^2\right]}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} e^{\left(\frac{is}{2a}\right)^2} dx$$

$$=\frac{1}{\sqrt{2\pi}} e^{\left(\frac{is}{2a}\right)^2}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} dx$$

$(A-B)^2 = A^2-2AB+B^2$

$2AB = isx$

Here $A=ax\ ,B=\frac{is}{2a}$

Let $u=ax-\frac{is}{2a}\ \Rightarrow du=adx\ \Rightarrow dx=\frac{du}{a}\ ;\ u:-\infty\, to\, \infty$

$$=\frac{1}{\sqrt{2\pi}} e^{\frac{i^2s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-u^2}\frac{du}{a}$$

$$=\frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-u^2}du \quad \because i^2=-1$$

$$=\frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}}2\int_{0}^{\infty} e^{-u^2}du \quad \because e^{-u^2} \text{ is an even function}$$

$$=\frac{1}{a\sqrt{2}\sqrt{\pi}} e^{\frac{-s^2}{4a^2}}2\frac{\sqrt{\pi}}{2} \qquad \because \int_{0}^{\infty} e^{-u^2}du=\frac{\sqrt{\pi}}{2}$$

$$\boxed{F\left[e^{-a^2x^2}\right]=\frac{1}{a\sqrt{2}}e^{\frac{-s^2}{4a^2}}} \qquad ------(1)$$

Deduction:

To prove $e^{\frac{-x^2}{2}}$ is self reciprocal

It is enough to prove that $F\left[e^{-\frac{x^2}{2}}\right]$ is $e^{-\frac{s^2}{2}}$

Put $a=\frac{1}{\sqrt{2}}$ in (1)

$$F\left[e^{-\left(\frac{1}{\sqrt{2}}\right)^2x^2}\right]=\frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}}e^{\frac{-s^2}{4\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$F\left[e^{-\frac{x^2}{2}}\right]=e^{\frac{-s^2}{\frac{4}{2}}}$$

$$\boxed{F\left[e^{-\frac{x^2}{2}}\right]=e^{\frac{-s^2}{2}}}$$

	$\therefore e^{\frac{-x^2}{2}}$ is self reciprocal.

15.

Find the Fourier transform of $e^{\frac{-x^2}{2}}$.

(or) Show that $e^{\frac{-x^2}{2}}$ is self reciprocal with respect to Fourier Transforms.

Solution:

Let $f(x) = e^{-\frac{x^2}{2}}$

Assume $f(x) = e^{-a^2x^2}$ where $a = \frac{1}{\sqrt{2}}$

The Fourier transform f(x) is

$$F\left[f(x)\right] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

$$F\left[e^{-a^2x^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$$

> $(A-B)^2 = A^2 - 2AB + B^2$
>
> $2AB = isx$
>
> Here $A = ax$, $B = \frac{is}{2a}$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2+isx}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(a^2x^2-isx\right)}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[(ax)^2 - isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} e^{\left(\frac{is}{2a}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\left(\frac{is}{2a}\right)^2}\int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} dx$$

Let $u = ax - \frac{is}{2a} \Rightarrow du = adx \Rightarrow dx = \frac{du}{a}$; $u : -\infty\, to\, \infty$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{i^2s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-u^2}\frac{du}{a}$$

$$= \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}}\int_{-\infty}^{\infty} e^{-u^2}du \quad \because i^2 = -1$$

$$= \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}}2\int_{0}^{\infty} e^{-u^2}du \quad \because e^{-u^2} \text{ is an even function}$$

$$= \frac{1}{a\sqrt{2}\sqrt{\pi}} e^{\frac{-s^2}{4a^2}}2\frac{\sqrt{\pi}}{2} \qquad \because \int_{0}^{\infty} e^{-u^2}du = \frac{\sqrt{\pi}}{2}$$

$$\boxed{F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}}e^{\frac{-s^2}{4a^2}}} \qquad ------(1)$$

Deduction:

	To prove $e^{\frac{-x^2}{2}}$ is self reciprocal It is enough to prove that $F\left[e^{-\frac{x^2}{2}}\right]$ is $e^{-\frac{s^2}{2}}$ Put $a=\frac{1}{\sqrt{2}}$ in (1) $F\left[e^{-\left(\frac{1}{\sqrt{2}}\right)^2 x^2}\right]=\frac{1}{\left(\frac{1}{\sqrt{2}}\right)\sqrt{2}}e^{\frac{-s^2}{4\left(\frac{1}{\sqrt{2}}\right)^2}}$ $F\left[e^{-\frac{x^2}{2}}\right]=e^{\frac{\frac{-s^2}{4}}{2}}$ $\boxed{F\left[e^{-\frac{x^2}{2}}\right]=e^{\frac{-s^2}{2}}}$ $\therefore e^{\frac{-x^2}{2}}$ is self reciprocal.
16.	**Find the Fourier cosine transform of** $e^{-a^2x^2}$ **Hence find** $F_s\left[xe^{-a^2x^2}\right]$. **Solution:** **Let** $f(x)=e^{-a^2x^2}$ The Fourier cosine transform f(x) is $F_c\left[f(x)\right]=F_c(s)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos sx\,dx$ $\qquad \because \int_0^{\infty} f(x)dx=\frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$ $=\sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-a^2x^2}\cos sx\,dx$ $=\sqrt{\frac{2}{\pi}}\frac{1}{2}\int_{-\infty}^{\infty} e^{-a^2x^2}\cos sx\,dx$ $F_c\left[f(x)\right]=\text{R.P.of }\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$ $\qquad \because \cos sx=\text{R.P.of } e^{isx}$ $F_c\left[f(x)\right]=\text{R.P.of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$ $=\text{R.P.of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2}e^{isx}dx$ $=\text{R.P.of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2x^2+isx}dx$ $=\text{R.P.of }\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(a^2x^2-isx\right)}dx$

$(a-b)^2=a^2-2ab+b^2$

$-2ab=isx$

Here $a=ax$

$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[(ax)^2 - isx + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]} dx$$

$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} e^{\left(\frac{is}{2a}\right)^2} dx$$

$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} e^{\left(\frac{is}{2a}\right)^2} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{is}{2a}\right)^2} dx$$

Let $u = ax - \frac{is}{2a} \;\Rightarrow du = a dx \;\Rightarrow dx = \frac{du}{a}\;$; $\; u : -\infty \, to \, \infty$

$$= \text{R.P.of } \frac{1}{\sqrt{2\pi}} e^{\frac{i^2 s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a}$$

$$= \text{R.P.of } \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du \quad \because i^2 = -1$$

$$= \text{R.P.of } \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} 2\int_{0}^{\infty} e^{-u^2} du \quad \because e^{-u^2} \text{ is an even function}$$

$$= \text{R.P.of } \frac{1}{a\sqrt{2}\sqrt{\pi}} e^{\frac{-s^2}{4a^2}} 2\frac{\sqrt{\pi}}{2} \qquad \because \int_{0}^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

$$\boxed{F\left[e^{-a^2x^2}\right] = \frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}}} \qquad ------(1)$$

Deduction:

$$F_s\left[xf(x)\right] = -\frac{d}{ds}\left\{F_c\left[f(x)\right]\right\} = -\frac{d}{ds}\left[F_c(s)\right]$$

$$F_s\left[xe^{-a^2x^2}\right] = -\frac{d}{ds}\left\{F_c\left[e^{-a^2x^2}\right]\right\}$$

$$= -\frac{d}{ds}\left[\frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}}\right]$$

$$= -\frac{1}{a\sqrt{2}} e^{\frac{-s^2}{4a^2}} \left(\frac{-2s}{4a^2}\right)$$

$$F_s\left[xe^{-a^2x^2}\right] = \frac{s}{2\sqrt{2}a^3} e^{\frac{-s^2}{4a^2}}$$

17. **Solve for *f(x)*, the integral equation** $\int_0^{\infty} f(x)\sin sx dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2. \\ 0, & s \ge 2 \end{cases}$

Solution:

Given $\int_0^{\infty} f(x)\sin sx dx = \begin{cases} 1, & 0 \le s < 1 \\ 2, & 1 \le s < 2. \\ 0, & s \ge 2 \end{cases} \quad -----(1)$

We know that

	$F_s\left[f(x)\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin sx\,dx$ $f(x)=\sqrt{\frac{2}{\pi}}F_s^{-1}\left[\begin{cases}1, 0\le s<1\\ 2, 1\le s<2\\ 0, s\ge 2\end{cases}\right]$ $\qquad F^{-1}\left[F_s(s)\right]=f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} F_s(s)\sin sx\,ds$ $f(x)=\sqrt{\frac{2}{\pi}}\sqrt{\frac{2}{\pi}}\left[\int_0^1 1\sin sx\,ds+\int_1^2 2\sin sx\,ds+\int_2^{\infty} 0\sin sx\,ds\right]$ $=\frac{2}{\pi}\left[\int_0^1 1\sin sx\,ds+\int_1^2 2\sin sx\,ds\right]$ $=\frac{2}{\pi}\left[\left(\frac{-\cos sx}{x}\right)_0^1+2\left(\frac{-\cos sx}{x}\right)_1^2\right]$ $=\frac{2}{\pi}\left[\left(\frac{-\cos x}{x}+\frac{\cos 0}{x}\right)+2\left(\frac{-\cos 2x}{x}+\frac{\cos x}{x}\right)\right]$ $=\frac{2}{\pi}\left[\frac{-\cos x}{x}+\frac{1}{x}-2\frac{\cos 2x}{x}+2\frac{\cos x}{x}\right]$ $=\frac{2}{\pi x}\left[1-\cos x-2\cos 2x+2\cos x\right]$ $f(x)=\frac{2}{\pi x}\left[1+\cos x-2\cos 2x\right]$
18.	**Find the Fourier cosine and sine transform of x^{n-1}. Hence show that $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine and sine transforms.** **Solution:** By definition of Gamma integral $\int_0^{\infty} e^{-ax}x^{n-1}\,dx=\frac{\Gamma n}{a^n},\quad a>0, n>0$ Put $a=is$ $\int_0^{\infty} e^{-isx}x^{n-1}\,dx=\frac{\Gamma n}{(is)^n},\quad a>0, n>0$ $\int_0^{\infty} x^{n-1}e^{-isx}\,dx=\frac{\Gamma n}{i^n s^n}$ $=\frac{\Gamma n}{s^n}(-i)^n$ $=\frac{\Gamma n}{s^n}\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right)^n \qquad \because e^{-i\frac{\pi}{2}}=\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}=-i$ $=\frac{\Gamma n}{s^n}\left(\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}\right) \qquad \because$ by Demorives theorem $\left(\cos\theta\pm i\sin\theta\right)^n=\cos n\theta\pm i\sin n\theta$ $\int_0^{\infty} x^{n-1}(\cos sx-i\sin sx)\,dx=\frac{\Gamma n}{s^n}\left(\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}\right)$ $\int_0^{\infty} x^{n-1}\cos sx\,dx-i\int_0^{\infty} x^{n-1}\sin sx\,dx=\frac{\Gamma n}{s^n}\cos\frac{n\pi}{2}-i\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$

Equating real and imaginary parts on both sides

$$\int_0^\infty x^{n-1}\cos sx\,dx=\frac{\Gamma n}{s^n}\cos\frac{n\pi}{2}\qquad\qquad \int_0^\infty x^{n-1}\sin sx\,dx=\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\cos sx\,dx=\sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\cos\frac{n\pi}{2}\qquad\qquad \sqrt{\frac{2}{\pi}}\int_0^\infty x^{n-1}\sin sx\,dx=\sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$$

$$\boxed{F_c\left[x^{n-1}\right]=\sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\cos\frac{n\pi}{2}}\qquad\qquad \boxed{F_s\left[x^{n-1}\right]=\sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}}$$

Deduction:

To prove $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine and sine transforms.

It is enough to prove that $F_c\left[\frac{1}{\sqrt{x}}\right]=\frac{1}{\sqrt{s}}$ and $F_s\left[\frac{1}{\sqrt{x}}\right]=\frac{1}{\sqrt{s}}$

We know that

$$F_c\left[x^{n-1}\right]=\sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\cos\frac{n\pi}{2}\qquad\qquad F_s\left[x^{n-1}\right]=\sqrt{\frac{2}{\pi}}\frac{\Gamma n}{s^n}\sin\frac{n\pi}{2}$$

Put $n=\frac{1}{2}$

$$F_c\left[x^{\frac{1}{2}-1}\right]=\sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\cos\frac{\pi}{4}\qquad\qquad F_s\left[x^{\left(\frac{1}{2}-1\right)}\right]=\sqrt{\frac{2}{\pi}}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}\sin\frac{\pi}{4}$$

$$F_c\left[x^{-\frac{1}{2}}\right]=\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\frac{1}{\sqrt{2}}\qquad\qquad F_s\left[x^{-\frac{1}{2}}\right]=\frac{\sqrt{2}}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\frac{1}{\sqrt{2}}\qquad \because \cos\frac{\pi}{4}=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}\text{ and }\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

$$\boxed{F_c\left[\frac{1}{\sqrt{x}}\right]=\frac{1}{\sqrt{s}}}\qquad\qquad \boxed{F_s\left[\frac{1}{\sqrt{x}}\right]=\frac{1}{\sqrt{s}}}$$

$\therefore$ $\frac{1}{\sqrt{x}}$ is self reciprocal under Fourier cosine and sine transforms.

19. **Find the function $f(x)$ if its sine transform is $\frac{e^{-as}}{s}$**

Solution:

Given $F_s\left[f(x)\right]=F_s(s)=\frac{e^{-as}}{s}$

$$f(x)=F^{-1}\left[F_s(s)\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(s)\sin sx\,ds$$

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx\,ds$$

Taking diff on both sides w.r.to x

$$\frac{d}{dx}\left[f(x)\right]=\frac{d}{dx}\left[\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx\,ds\right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-as}}{s} \frac{\partial}{\partial x} (\sin sx)\, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-as}}{\not{s}} \cos sx \times \not{s}\; ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-as} \cos sx\; ds$$

$$\frac{d}{dx}[f(x)] = \sqrt{\frac{2}{\pi}} \left[\frac{a}{a^2 + x^2} \right] \quad \because \int_0^\infty e^{-ax} \cos bx\; dx = \frac{a}{a^2 + s^2} \text{ here } a = a,\ b = x$$

Integrating on w.r.to x

$$f(x) = \sqrt{\frac{2}{\pi}} a \int \frac{1}{a^2 + x^2} \quad \because \int_0^\infty \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$= \sqrt{\frac{2}{\pi}} a \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{a}\right)}$$