UNIT-IV

FOURIER TRANSFORMS

Example 1

Find the F.T of f(x) defined by

$$f(x) = \begin{cases} 0 & x<a \\ 1 & a<x<b \\ 0 & x>b. \end{cases}$$

The F.T of f(x) is given by

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{isx} .dx .$$

$$= \frac{1}{\sqrt{2\pi}} \Big[\qquad\Big]_{a}^{b}$$

$$= \frac{1}{\sqrt{2\pi}} . \frac{e^{ibs} - e^{ias}}{is}$$

Example 2

Find the F.T of f(x) = x for $|x| \le a$
= 0 for $|x| > a$.

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} .x.dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} x \, .d \left(\frac{e^{isx}}{is} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{xe^{isx}}{is} - \frac{e^{isx}}{(is)^2} \right\}_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{ae^{isa}}{is} - \frac{e^{isa}}{(is)^2} + \frac{ae^{-isa}}{is} + \frac{e^{-isa}}{(is)^2} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{\;}{is} (e^{isa} + e^{-isa}) + \frac{1}{s^2} (e^{isa} - e^{-isa}) \right.$$

$$= \frac{1}{\sqrt{2\pi}} \quad \cos sa + \frac{2i}{s^2} \sin sa$$

$$= 2i \; . \; \frac{1}{\sqrt{2\pi}} [\sin sa - as \cos sa].s^2$$

$$= \sqrt{(2/\pi)} \; \frac{i\,[\sin sa - as \cos sa]}{s^2}$$

Example 3

Find the F.T of $f(x) = e^{iax}$, $0 < x < 1$

$= 0$ otherwise

The F.T of f(x) is given by

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x) \, dx.$$

$$= \frac{1}{\;} \int^{1} e^{isx} . e^{iax} \, dx.$$

$$\sqrt{2\pi} \qquad 0$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^1 e^{i(s+a)x}\,.dx\,.$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{i(s+a)x}}{i(s+a)} \right)_0^1$$

$$= \frac{1}{i\sqrt{2\pi}.(s+a)} \{e^{i(s+a)x} - 1\}$$

$$= \frac{i}{\sqrt{2\pi}.(s+a)} \{1 - e^{i(s+a)}\}$$

Example 4

Find the F.T of $e^{-a^2x^2}$, a>0 and hence deduce that the F.T of $e^{-x^2/2}$ is $e^{-s^2/2}$.

The F.T of f(x) is given by

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\,dx.$$

$$F\left(e^{-a^2x^2}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} . e^{isx}\,.dx.$$

$$= \frac{e^{-s^2/4a^2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-[ax - (is/2a)]^2}\,dx\,.$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2}\,dt, \text{ by putting } ax - (is/2a) = t$$

$$= \frac{e^{-s^2/4a^2}}{a\sqrt{2\pi}} . \sqrt{\pi}, \text{ since } \int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi} \text{ (using Gamma functions).}$$

$$= \frac{1}{\sqrt{2}.a} e^{-s^2/4a^2} \text{ . ---------------------- (i)}$$

To find $F\{e^{-x^2/2}\}$

Putting a = 1/ √2 in (1), we get

$$F\{e^{-x^2/2}\} = e^{-s^2/2} .$$

Note:

If the F.T of f(x) is f(s), the function f(x) is called self-reciprocal. In the above example $e^{-x^2/2}$ is self-reciprocal under F.T.

Example 5

Find the F.T of

f(x) = 1 for |x|<1.

= 0 for |x|>1.

Hence evaluate $\int_0^\infty \frac{\sin x}{x} dx$.

The F.T of f(x),

i.e., $$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{isx} .(1).dx .$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} . \frac{e^{is} - e^{-is}}{is}$$

$$= \sqrt{(2/\pi)} \frac{\sin s}{s} , \quad s \neq 0s$$

Thus, $F\{f(x)\}= F(s) = \sqrt{(2/\pi)}. \frac{\sin s}{s} , \quad s \neq 0$

s

Now by the inversion formula , we get

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s).\ e^{-isx} .ds.$$

or

$$= \int_{-\infty}^{\infty} \sqrt{(2/\pi)} \frac{\sin s}{s} .\ e^{-isx} .ds. = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|>1. \end{cases}$$

i.e,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} e^{-isx} .\ ds. = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|>1. \end{cases}$$

Putting x = 0, we get

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin s}{s} ds = 1$$

i.e,

$$\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin s}{s} ds = 1,$$ since the integrand is even.

$$\Rightarrow \int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2}$$

Hence,

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Exercises

(1) Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } |x|<a \\ 0 & \text{for } |x|>a. \end{cases}$$

(2) Find the Fourier transform of

$$f(x) = \begin{cases} x^2 & \text{for } |x|\le a \\ 0 & \text{for } |x|>a. \end{cases}$$

(3) Find the Fourier transform of

$$f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0. \end{cases}$$