MA8353 TRANSFORM AND PARTIAL DIFFERENTIAL EQUATION

# UNIT– I

## PARTIAL DIFFERENTIAL EQUATIONS OF HIGHER ORDER WITHCONSTANT COEFFICIENTS.

### Homogeneous Linear Equations with constant Coefficients.

A homogeneous linear partial differential equation of the $n^{th}$ order is of the form

$$c_0 \frac{\partial^n z}{\partial x^n} + c_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots\ldots + c_n \frac{\partial^n z}{\partial y^n} = F(x,y) \quad \text{------------} (1)$$

where $c_0, c_1$,---------, $c_n$ are constants and F is a function of „x" and „y". It is homogeneous because all its terms contain derivatives of the same order.

Equation (1) can be expressed as

$$(c_0 D^n + c_1 D^{n-1} D' + \ldots.. + c_n D'^n)\, z = F(x,y) \text{ or } f(D,D')\, z = F(x,y) \quad \text{------------} (2),$$

where, $\frac{\partial}{\partial x} \equiv D$ and $\frac{\partial}{\partial y} \equiv D'$.

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely, the complementary function and the particular integral.

The complementary function is the complete solution of $f(D,D')z = 0$ ---------------- (3), which must contain n arbitrary functions as the degree of the polynomial $f(D,D')$. The particular integral is the particular solution of equation (2).

## Finding the complementary function

Let us now consider the equation $f(D,D')z = F(x,y)$

The auxiliary equation of (3) is obtained by replacing D by m and D' by 1.i.e, $c_0$

$$m^n + c_1 m^{n-1} + \ldots + c_n = 0 \text{ ---------------------------- (4)}$$

Solving equation (4) for „m", we get „n" roots. Depending upon the nature of the roots,the Complementary function is written as given below:

| Roots of the auxiliary equation | Nature of the roots | Complementary function(C.F) |
|---|---|---|
| $m_1, m_2, m_3, \ldots, m_n$ | distinct roots | $f_1(y+m_1x)+f_2(y+m_2x) + \ldots+f_n(y+m_nx)$. |
| $m_1 = m_2 = m, m_3, m_4, \ldots, m_n$ | two equal roots | $f_1(y+m_1x)+xf_2(y+m_1x) + f_3(y+m_3x) + \ldots + f_n(y+m_nx)$. |
| $m_1 = m_2 = \ldots = m_n = m$ | all equal roots | $f_1(y+mx)+xf_2(y+mx) + x^2f_3(y+mx)+\ldots + \ldots+x^{n-1} f_n(y+mx)$ |

## Finding the particular Integral

Consider the equation $f(D,D')z = F(x,y)$.

Now, the P.I is given by $\frac{1}{f(D,D')} F(x,y)$

**Case (i) :** When $F(x,y) = e^{ax+by}$

$$P.I = \frac{1}{f(D,D')} e^{ax+by}$$

Replacing D by „a" and D' by „b", we have1

$$P.I = \text{-----------}\ e^{ax+by}, \quad \text{where } f(a,b) \neq 0.$$

f (a,b)

**Case (ii) :** When F(x,y) = sin(ax + by) (or) cos (ax +by)1

P.I = ----------------- sin (ax+by) or cos (ax+by)

$f(D^2,DD',D'^2)$

Replacing $D^2 = -a^2$, $DD'^2 = -ab$ and $D' = -b^2$, we get

1

P.I = ----------------- sin (ax+by) or cos (ax+by) , where $f(-a^2, -ab, -b^2) \neq 0$.$f(-a^2, -ab, -$

$b^2)$

**Case (iii) :** When $F(x,y) = x^m y^n$,

1

P.I = ---------- $x^m y^n = [f(D, D')]^{-1} x^m y^n$f(D,D')

Expand $[f(D,D')]^{-1}$ in ascending powers of D or D' and operate on $x^m y^n$ term by term.

**Case (iv) :** When F(x,y) is any function of x and y.

1

P.I =--------------- F (x,y).

f (D,D')

1

Resolve----------- into partial fractions considering f (D,D') as a function of D alone.f (D,D')

Then operate each partial fraction on F(x,y) in such a way that1

--------- F (x,y) = ∫ F(x,c-mx) dx ,D–

mD'

where c is replaced by y+mx after integration

**Example 26**

Solve$(D^3 - 3D^2D' + 4D'^3) z = e^{x+2y}$

The auxillary equation is $m=m^3 - 3m^2 + 4 = 0$

The roots are m = -1,2,2

Therefore the C.F is $f_1(y-x) + f_2(y+2x) + xf_3(y+2x)$.

$$P.I. = \frac{e^{x+2y}}{D^3 - 3D^2D' + 4D'^3} \text{ (Replace D by 1 and } D' \text{ by 2)}$$

$$= \frac{e^{x+2y}}{1 - 3(1)(2) + 4(2)^3}$$

$$= \frac{e^{x+2y}}{27}$$

Hence, the solution is z = C.F. + P.I

ie, $$z = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{e^{x+2y}}{27}$$

**Example 27**

Solve $(D^2 - 4DD' + 4D'^2) z = \cos(x - 2y)$The

auxiliary equation is $m^2 - 4m + 4 = 0$ Solving, we get

m = 2,2.

Therefore the C.F is $f_1(y+2x) + xf_2(y+2x)$.

$$\therefore P.I = \frac{1}{D^2 - 4DD' + 4D'^2} \cos(x-2y)$$

Replacing $D^2$ by – 1, $DD'$ by 2 and $D'^2$ by –4, we have

$$P.I = \frac{1}{(-1) - 4(2) + 4(-4)} \cos(x-2y)$$

$$= -\frac{\cos(x-2y)}{25}$$

$\therefore$ Solution is $z = f_1(y+2x) + xf_2(y+2x) - \dfrac{\quad}{25}$ .

**Example 28**

Solve $(D^2 - 2DD')\, z = x^3y + e^{5x}$ The

auxiliary equation is $m^2 - 2m = 0$. Solving,

we get m = 0,2.
Hence the C.F is $f_1(y) + f_2(y+2x)$.

$$P.I_1 = \frac{x^3y}{D^2 - 2DD'}$$

$$= \frac{1}{D^2\left(1 - \frac{2D'}{D}\right)}(x^3y)$$

$$= \frac{1}{D^2}\left(1 - \frac{2D'}{D}\right)^{-1}(x^3y)$$

$$= \frac{1}{D^2}\left(1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \ldots\ldots\right)(x^3y)$$

$$= \frac{1}{D^2}\left((x^3y) + \frac{2}{D}D'(x^3y) + \frac{4}{D^2}D'^2(x^3y) + \ldots\right)$$

$$= \frac{1}{D^2}\left((x^3y) + \frac{2}{D}(x^3) + \frac{4}{D^2}(0) + \ldots\right)$$

$$P.I_1 = \frac{1}{D^2}(x^3y) + \frac{2}{D^3}(x^3)$$

$$P.I_1 = \frac{x^5y}{20} + \frac{x^6}{60}$$

$$P.I_2 = \frac{e^{5x}}{D^2 - 2DD'} \text{ (Replace D by 5 and } D' \text{ by 0)}$$

$$= \frac{e^{5x}}{25}$$

$$\therefore \text{Solution is } Z = f_1(y) + f_2(y+2x) + \frac{x^5y}{20} + \frac{x^6}{60} + \frac{e^{5x}}{25}$$

**Example 29**

Solve $(D^2 + DD' - 6 D'^2) z = y \cos x$.

The auxiliary equation is $m^2 + m - 6 = 0$.
Therefore, $m = -3, 2$.

Hence the C.F is $f_1(y-3x) + f_2(y + 2x)$.

$$P.I = \frac{y \cos x}{D^2 + DD' - 6D'^2}$$

$$= \frac{y \cos x}{(D + 3D')(D - 2D')}$$

$$= \frac{1}{(D+3D')} \frac{1}{(D - 2D')} y \cos x$$

$$= \frac{1}{(D+3D')} \int (c - 2x) \cos x \, dx, \text{ where } y = c - 2x$$

$$= \frac{1}{(D+3D')} \int (c - 2x)\, d(\sin x)$$

$$= \frac{1}{(D+3D')} [(c - 2x)(\sin x) - (-2)(-\cos x)]$$

$$= \frac{1}{(D+3D')} [y \sin x - 2 \cos x]$$

$$= \int [(c + 3x) \sin x - 2 \cos x]\, dx, \text{ where } y = c + 3x$$

$= \int (c + 3x)\, d(-\cos x) - 2\int \cos x \; dx$

$= (c + 3x)(-\cos x) - (3)(-\sin x) - 2 \sin x$

$= -y \cos x + \sin x$ Hence

the complete solution is

$z = f_1(y - 3x) + f_2(y + 2x) - y \cos x + \sin x$

**Example 30**

Solve $r - 4s + 4t = e^{2x+y}$

Given equation is $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$

i.e, $(D^2 - 4DD' + 4D'^2)\, z = e^{2x+y}$

The auxiliary equation is $m^2 - 4m + 4 = 0$.

Therefore, $m = 2,2$

Hence the C.F is $f_1(y + 2x) + x f_2(y + 2x)$.

$$P.I. = \frac{e^{2x+y}}{D^2 - 4DD' + 4D'^2}$$

Since $D^2 - 4DD' + 4D'^2 = 0$ for $D = 2$ and $D' = 1$, we have to apply the general rule.

$$\therefore P.I. = \frac{e^{2x+y}}{(D - 2D')(D - 2D')}$$

$$= \frac{1}{(D - 2D')} \frac{1}{(D - 2D')} e^{2x+y}$$

$= \frac{1}{(D - 2D')} \int e^{2x+c-2x} dx$ , where $y = c - 2x$.

$$= \frac{1}{(D - 2D')} \int e^{c}\, dx$$

$$= \frac{1}{(D - 2D')} e^{c} . x$$

$$= \frac{1}{D - 2D'} x e^{y+2x}$$

$= \int xe^{c-2x+2x}\, dx$ , where $y = c - 2x$.

$= \int xe^{c}\, dx$

$= e^{c} . x^2/2$

$= \frac{x^2 e^{y+2x}}{2}$

Hence the complete solution is

$$z = f_1(y+2x) + f_2(y+2x) + \frac{1}{2} x^2 e^{2x+y}$$