1.5 LINEARLY INDEPENDENCE AND LINEARLY DEPENDENCE

Linearly dependent set

A subset S of a vector space is called linearly dependent if there is a finite number of distinct vectors $v_1, v_2, \dots, v_n$ in S and scalars $\alpha_1, \alpha_2, \dots$, zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

Linearly independent set

A subset S of a vector space that is not linearly dependent is called independent.

i.e., A subset S of a vector space is called linearly independent if there exis. number of distinct vectors $v_1, v_2, \dots, v_n$ in S and scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ such

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0. \text{ Implies } \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

Note:

- Any set of vectors which contains zero vectors is linearly dependen
- In R^2 any two straight lines which are not parallel are linearly indep
- In R^2 any two straight lines which are parallel are linearly dependen
- In R^2 any three vectors are linearly dependent therefore any set of n in the R^m are linearly dependent if $n > m$.

Theorem 1.16: $\{0\}$ is a dependent set

Proof: Let V be a vector space over F

Let $v_1 = 0$

Therefore $\alpha_1 v_1 = 0 \Rightarrow \alpha_1 \neq 0$

$\therefore$ $\{0\}$ is linearly dependent.

Theorem 1.17: A singleton non zero vector is linearly independent set

Proof: Let V be a vector space over F

Let $v_1 \neq 0 \in V$

Therefore $\alpha_1 v_1 = 0 \Rightarrow \alpha_1 = 0$

$\therefore \{v_1\}$ is linearly independent.

Theorem 1.18: Any subset of a linearly independent set is linearly independent.

Proof:

Let V be a vector space over a field F.

Let $S = \{v_1, v_2, \dots, v_n\}$ be a linearly independent set.

Let $S_1 = \{v_1, v_2, \dots, v_m\}$ be a subset of S, where $m < n$.

Suppose S_1 is a linearly dependent set. Then there exist $\alpha_1, \alpha_2, \dots \dots \alpha_m$ in F not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$$

Hence $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m + 0v_{m+1} + \cdots + 0v_n = 0$ with $\alpha_1, \alpha_2, \dots \dots \alpha_m$ in F
not all zero.

Therefore $\{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$ is a linearly dependent set of V

i.e., S is a linearly dependent set of V, which is a contradiction.

Therefore S_1 is linearly independent.

Theorem 1.19: Any set containing a linearly dependent set is also linearly dependent

OR

Any super set of a linearly dependent set is linearly dependent set

Proof: Let V a vector space over F.

et S be a linearly dependent set of V…

Then there exits scalar $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

now consider the super set $S_1 = \{v_1, v_2, \dots, v_n, v_{n+1}\}$

Then we have $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n + 0v_{n+1} = 0$ with at least one $\alpha_i \neq 0$

$\therefore S_1$ is linearly dependent.

Theorem 1.20: A finite set of vectors that contains the zero vector will be linearly dependent.

Proof: Let $S = \{0, v_1, v_2, \dots, v_n\}$ be any set of vectors that contains the zero vector. Consider

$$a_1(0) + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

Which implies $a_1 \neq 0$

Therefore $S = \{0, v_1, v_2, \dots, v_n\}$ linearly dependent.

Theorem 1.21: Let $S = \{v_1, v_2, \dots, v_n\}$ be a linearly independent set of vectors in

a vector space V over a field F. Then every element of $L(S)$ can be uniquely written in the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, where $v_i \in S$ and $\alpha_i \in F$.

Proof: By the definition, every element of $L(S)$ is of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

We prove that every element of $L(S)$ can be uniquely written in the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

If not suppose there is linear combination $\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$ of S such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n, \quad \text{where} \quad \beta_i \in F$$

$$\Rightarrow (\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \cdots + (\alpha_n - \beta_n)v_n = 0$$

Since S is a linearly independent set, $(\alpha_i - \beta_i) = 0$ for all i.

$$\alpha_i - \beta_i = 0 \text{ for all } i$$

$$\therefore \alpha_i = \beta_i \text{ for all } i$$

Hence every element of $L(S)$ can be uniquely written in the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots \ldots + \alpha_n v_n$$

Theorem 1.22: A set $S = \{v_1, v_2, \ldots, v_n\}; n \geq 2$ is a linearly dependent set of vectors in V if and only if there exists a vector $v_k \in S$ such that v_k is a linear combination of the preceding vectors $v_1, v_2, \ldots, v_{k-1}$.

1. **Determine whether the following sets of vectors v_3(R) are linearly dependent or linearly independent.**

   i. **$V_1 = (0,2,-4)$, $V_2=(1,-2,-1)$, $V_3 = (1,-4,3)$**

   ii. **$V_1 = (1,2,-3)$, $V_2=(1,-3,2)$, $V_3 = (2,-1,5)$**

   iii. **$V_1 = (1,2,3)$, $V_2=(3,1,5)$, $V_3 = (3,-4,7)$**

Solution:

(i) Let $av_1 + bv_2 + cv_3 = 0$, a, b, c ϵ R

$$a(0,2,-4) + b(1,-2,-1) + c(1,-4,3) = (0,0,0)$$

⇨ $(0, 2a, -4a) + (b, -2b, -b) + (c, -4c, -3c) = (0, 0, 0)$

⇨ $(b+c, 2a-2b-4c, -4a-b+3c) = (0,0,0)$

$$b + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$2a - 2b - 4c \Rightarrow a - b - 2c = 0$ ……………..(2)

$-4a - b + 3c = 0$ …………(3)

Subtracting (3) from (2)

$5a - 5c = 0 \Rightarrow a = c$

From (1) $b = -c$

If we choose c = k, then a=k and b=-k

Hence the system is linearly dependent

(ii) $a(1,2,-3)+b(1,-3,2)+c(2,-1,5) = (0,0,0)$

$a + b + 2c = 0$ ……(1)

$2a - 3b - c = 0$ ……..(2)

$-3a + 2b + 5c = 0$ …………(3)

Multiply (1) by 2,

$2a + 2b + 4c = 0$ ………….(4)

Subtracting (1) and (2),

We get $5b + 5c = 0$ ……..(5)

Multiply (1) by (3),

$3a + 3b + 6c = 0$ ………..(6)

Adding (3) and (6),

$5b = 11c = 0$ ……..(7)

Substituting c=0 in (5)

We get b=0

From (1), a=0

a = 0, b = 0,c = 0

The given system is linearly independent.

(iii) a (1,2,3)+b(3,1,5)+c(3,-4,7) =(0,0,0)

a+3b+3c =0 …….(1)

2a +b-4c =0 ……….(2)

a+ 5b+ 7c =0 …….(3)

Subtracting (3) and (1),

2b +4c =0 …….(4)

Multiply (1) by (2), 2a +6b+ 6c =0 …(5)

Subtracting (5) and (2),

5b + 10c =0

b+ 2c =0 ……..(6)

Multiplying (6) by 2,

2b +4c =0 …….(7)

From (4) and (7),

B= -2c

Substituting b in (2)

2a -2c -4c =0

2a = 6c

a= 3c

The given system is linearly dependent.

2.If V_1 =(2, -1,0), V_2 =(1,2,1) and V_3= (0,2,-1). Show V_1, V_2, V_3 are linearly independent. Is it possible (3,2,1) as a linear combination of V_1, V_2, V_3.

Solution:

Let $av_1 + bv_2 + cv_3 = 0$, a, b, c ϵ F

a(2,-1,0)+b(1,2,1)+c(0,2,-1) =(0,0,0)

2a+b =0 …….(1)

-a +2b+2c =0 ……….(2)

b-c =0 …….(3)

these equation can be put in the form AX =0

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Det } A = \det \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \det \begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix} \; C_1 \to C_2 + C_3$$

$$= -\det \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} = -9 \neq 0$$

a= b = c =0

hence the system is linearly independent.

Let $v = a_1v_1 + a_2v_2 + a_3v_3$ where a_1, a_2, a_3 ϵ F

(3,2,1) = a_1(2,-1,0)+a_2(1,2,1)+a_3(0,2,-1)

(3,2,1) = ($2a_1 + a_2$, $-a_1 + 2a_2 + 2a_3$, $a_2 - a_3$)

Comparing

$3 = 2a_1 + a_2$ ……..(4)

$2 = -a_1 + 2a_2 + a_3$ ……….(5)

$1 = a_2 - a_3$ ………..(6)

Multiplying (5) by 2,

$4 = 2 - a_1 + 4a_2 + 4a_3$.(7)

Adding (4) and (7)

$7 = 5a_2 + 4a_3$ ……(8)

Multiplying (6) by 5,

$5 = 5a_2 + 5a_3$ ….(9)

Subtracting (8) and (9)

$2 = 9a_3 \Rightarrow a_3 = 2/9$

Substituting a_3 in (6)

$1 = a_2 - 2/9 \Rightarrow 1 + 2/9$

$a_2 = \frac{11}{9}$

Substituting a_2 in (4)

$3 = 2a_1 + \frac{11}{9}$

$2a_1 = 3 - \frac{11}{9}$

$2a_1 = \frac{27-11}{9}$

=> $a_1 = \frac{16}{2*9} = \frac{8}{9}$

$a_1 = \frac{8}{9}$, $a_2 \frac{11}{9}$, $a_3 = \frac{2}{9}$

hence $(3,2,1) = \frac{8}{9}(2,-1,0) + \frac{11}{9}(1,2,1) + \frac{2}{9}(0,2,-1)$

which is the required linear combination.

1. If x,y,z are linearly independent vectors in a vector space V then prove that all linearly independent x+y,x-y,x-2y+2

Solution:

Let a, b, c ϵ F such that

A(x+y) +b(x-y)+c(x-2y-z)=0

⇨ (a+b+c)x+(a- b- 2c)y + cz =0, x+0, y+0,z

Comparing a + b+ c =0 … (1)

a – b-2c =0 … (2),

c=0 …. (3)

Note:

1. Any matrix with distint eigen values can be diagonalizable.
2. All matrices donot posses n linearly independent eigen vectors. Therefore all matrices are not diagonalizable.
3. Similar matrices have the same eigen values.
4. If A is diagonalizable then it has n linearly indebendent eigen vectors.
5. Symmetric matrices are always diagonalizable.
6. Let A be a square matirix, A is orthogonally diagonalizable iff it is a symmetric matrix.

Definition:

A square matrix A is said to be orthogonally diagonalizable if there exists an orthogonal matrix N such that D =N^TAN is a diagonal matrix.

1.Show that the following matrix A= $\begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ is diagonalizable hence find A^9.

Solution:

The characteristic equation I sgiven by $|A - \lambda I|$=0

(i.e.,) $\begin{vmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{vmatrix} = 0$

=>$(-4-\lambda)(5-\lambda)$ -3(-6) = 0

=> -20+4 λ-5 λ+ λ^2+18 =0

=> λ^2- λ -2 =0

(λ+1)(λ-2) =0

λ =-1,2

The eigen values are λ=-1,2

To find eigen vectors :

(A - λI)v =0

$$\begin{vmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad .....(1)$$

Case (i)

Substituting λ =2 in we get

$$\begin{vmatrix} -4-2 & -6 \\ 3 & 5-2 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} -6 & -6 \\ 3 & 3 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-6x_1 -6x_2 = 0$

$3x_1 + 3x_2 = 0 \Rightarrow 3x_1 = -3x_2$

$\Rightarrow x_1 = -x_2$

Let $x_2 = t$, then $x_1 = t$

$V_1 = t\binom{-1}{1}$

Case (ii)

Substituting λ =-1 in we get

$$\begin{vmatrix} -4+1 & -6 \\ 3 & 5+1 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} -3 & -6 \\ 3 & 6 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-3x_1 -6x_2 = 0$

$3x_1 + 6x_2 = 0 \Rightarrow 3x_1 = -6x_2$

$\Rightarrow x_1 = -2x_2$

Let $x_2 = s$, then $x_1 = -2s$

$V_2 = s\binom{-2}{1}$

Since A has two linearly indebendent eigen vectors it is diagonalizable.

Modal matrix is the column vectors of the diagonalizing matrix M.

$$M = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$M^{-1} AM = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$M^{-1} = \frac{1}{|M|}$ (cofactor matrix)^T

$$=\frac{1}{(-1+2)}\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}$$

Substituting M^{-1} in (2),

$$M^{-1}AM = \begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}^{-1}\begin{bmatrix}-4 & -6\\ 3 & 5\end{bmatrix}\begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-4+6 & -6+10\\ 4-3 & 6-5\end{bmatrix}\begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}4 & 4\\ 1 & 1\end{bmatrix}\begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}-2+4 & -4+4\\ -1+1 & -2+1\end{bmatrix}$$

$$=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}=D$$

$M^{-1}AM = D$ …………..(3)

Pre-multiply (3) by M and postmultiply (3) by M^{-1} on both

$MM^{-1}AMM^{-1} = MDM^{-1}$

$A = MDM^{-1}$

$A^9 = MD^9M^{-1}$ ……………….. (4)

$$D^9 = \begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}^9 == \begin{bmatrix}2^9 & 0\\ 0 & (-1)^9\end{bmatrix}$$

$$== \begin{bmatrix}512 & 0\\ 0 & -1\end{bmatrix}$$

$A^9 = MD^9M^{-1}$

$$=\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$=\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} -512+0 & 0+2 \\ 512+0 & 0-1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} -514 & +1026 \\ 513 & 1025 \end{bmatrix}$$