GRADIENT – DIRECTIONAL DERIVATIVE

Vector differential operator

The vector differential operator ∇ (read as Del) is denoted by $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the three rectangular axes $OX, OY \ and \ OZ$.

It is also called Hamiltonian operator and it is neither a vector nor a scalar, but it behaves like a vector.

The gradient of a scalar function

If$\varphi(x, y, z)$ is a scalar point function continuously differentiable in a given region of space, then the gradient of φ is defined as $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

It is also denoted as Grad φ.

Note

(i) $\nabla\varphi$ is a vector quantity.

(ii) $\nabla\varphi = 0$ if φ is constant.

(iii) $\nabla(\varphi_1\varphi_2) = \varphi_1\nabla\varphi_2 + \varphi_2\nabla\varphi_1$

(iv) $\nabla\left(\frac{\varphi_1}{\varphi_2}\right) = \frac{\varphi_2\nabla\varphi_1 - \varphi_1\nabla\varphi_2}{{\varphi^2}_2}$ if $\varphi_2 \neq 0$

(v) $\nabla(\varphi \pm \chi) = \nabla\varphi \pm \nabla\chi$

Example: If $\boldsymbol{\varphi = \log(x^2 + y^2 + z^2)}$ then find $\boldsymbol{\nabla\varphi}$.

Solution:

Given $\varphi = \log(x^2 + y^2 + z^2)$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\left(\frac{2x}{x^2 + y^2 + z^2}\right) + \vec{j}\left(\frac{2y}{x^2 + y^2 + z^2}\right) + \vec{k}\left(\frac{2z}{x^2 + y^2 + z^2}\right)$$

$$= \frac{2}{x^2 + y^2 + z^2}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{2}{r^2}\vec{r}$$

Example: Find $\boldsymbol{\nabla(r), \nabla\left(\frac{1}{r}\right), \nabla(\log r)}$ where $\boldsymbol{r = |\vec{r}|}$ and $\boldsymbol{\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}}$.

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\Rightarrow |\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y, \quad 2r\frac{\partial r}{\partial z} = 2z$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

(i) $\nabla(r) = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$

$$= \vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}$$

$$= \frac{1}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{1}{r}\vec{r}$$

(ii) $\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial\left(\frac{1}{r}\right)}{\partial x} + \vec{j}\frac{\partial\left(\frac{1}{r}\right)}{\partial y} + \vec{k}\frac{\partial\left(\frac{1}{r}\right)}{\partial z}$

$$= \vec{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \vec{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \vec{k}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= \left(-\frac{1}{r^2}\right)\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= -\frac{1}{r^3}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = -\frac{1}{r^3}\vec{r}$$

(iii) $\nabla(\log r) = \sum \vec{i}\frac{\partial(\log r)}{\partial x}$

$$= \sum \vec{i}\frac{1}{r}\frac{\partial r}{\partial x}$$

$$= \sum \vec{i}\frac{1}{r}\frac{x}{r}$$

$$= \sum \vec{i}\frac{x}{r^2}$$

$$= \frac{1}{r^2}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{1}{r^2}\vec{r}$$

Example: Prove that $\nabla(r^n) = nr^{n-2}\,\vec{r}$

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\nabla(r^n) = \vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}$$

$$= \vec{i}\,nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\,nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\,nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left[\vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)\right]$$

$$= \frac{nr^{n-1}}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = nr^{n-2}\vec{r}$$

Example: Find $|\nabla\varphi|$ if $\varphi = 2xz^4 - x^2y$ at $(2,-2,-1)$

Solution:

Given $\varphi = 2xz^4 - x^2y$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Now $\frac{\partial\varphi}{\partial x} = 2z^4 - 2xy$, $\frac{\partial\varphi}{\partial y} = -x^2$, $\frac{\partial\varphi}{\partial z} = 8xz^3$

$$\therefore \nabla\varphi = \vec{i}(2z^4 - 2xy) + \vec{j}(-x^2) + \vec{k}(8xz^3)$$

$$\therefore (\nabla\varphi)_{(2,-2,-1)} = 10\vec{i} - 4\vec{j} - 16\vec{k}$$

$$|\nabla\varphi| = \sqrt{100+16+256} = \sqrt{372}$$

Directional Derivative (D.D) of a scalar point function

The derivative of a point function (scalar or vector) in a particular direction is called its directional derivative along the direction.

The directional derivative of a scalar function φ in a given direction $\vec{a}$ is the rate of change of φ in that direction. It is given by the component of $\nabla\varphi$ in the direction of $\vec{a}$.

The directional derivative of a scalar point function in the direction of $\vec{a}$ is given by

$$\textbf{D.D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

The maximum directional derivative is $|\nabla\varphi|$ or $|\text{grad }\varphi|$.

Example: Find the directional derivative of $\varphi = 4xz^2 + x^2yz$ at $(1,-2,1)$ in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

Given $\varphi = 4xz^2 + x^2yz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}(2xyz + 4z^2) + \vec{j}(x^2z) + \vec{k}(x^2y + 8xz)$$

$$\therefore (\nabla\varphi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

Given $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

$$|\vec{a}| = \sqrt{4+1+4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

$$= (8\vec{i} - \vec{j} - 10\vec{k})\cdot\frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{1}{3}(16+1+20) = \frac{37}{3}$$

Example: Find the directional derivative of $\varphi(x,y,z) = xy^2 + yz^3$ at the point P$(2,-1,1)$ in the direction of PQ where Q is the point $(3,1,3)$

Solution:

Given $\varphi = xy^2 + yz^3$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y^2) + \vec{j}\,(2xy + z^3) + \vec{k}\,(3yz^2)$$

$$\therefore (\nabla\varphi)_{(2,-1,\ 1)} = \vec{i} - 3\vec{j} - 3\vec{k}$$

Given $\vec{a} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= \left(3\vec{i} + \vec{j} + 3\vec{k}\right) - \left(2i - \vec{j} + \vec{k}\right)$$

$$= \vec{i} + 2\vec{j} + 2\vec{k}$$

$$|\vec{a}| = \sqrt{1+4+4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= \frac{\left(\vec{i} - 3\vec{j} - 3\vec{k}\right)\cdot\left(\vec{i} + 2\vec{j} + 2\vec{k}\right)}{3}$$

$$= \frac{1}{3}\,(1 - 6 - 6) = -\frac{11}{3}$$

Example: In what direction from $(-1, 1, 2)$ is the directional derivative of $\varphi = xy^2 z^3$ a maximum? Find also the magnitude of this maximum.

Solution:

Given $\varphi = xy^2 z^3$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y^2 z^3) + \vec{j}\,(2xy\,z^3) + \vec{k}\,(3xy^2 z^2)$$

$$\therefore (\nabla\varphi)_{(-1,\ 1,\ 2)} = 8\vec{i} - 16\vec{j} - 12\vec{k}$$

The maximum directional derivative occurs in the direction of $\nabla\varphi = 8\vec{i} - 16\vec{j} - 12\vec{k}$.

∴ The magnitude of this maximum directional derivative

$$|\nabla\varphi| = \sqrt{64 + 256 + 144} = \sqrt{464}$$

Example: Find the directional derivative of the scalar function $\varphi = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point $(3, 1, 3)$.

Solution:

Given $\varphi = xyz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(yz) + \vec{j}\,(xz) + \vec{k}\,(xy)$$

$$\therefore (\nabla\varphi)_{(3,\ 1,\ 3)} = 3\vec{i} + 9\vec{j} + 3\vec{k}$$

Given surface is $z = xy \Rightarrow z - xy = 0$

$$\nabla\chi = \vec{i}\frac{\partial\chi}{\partial x} + \vec{j}\frac{\partial\chi}{\partial y} + \vec{k}\frac{\partial\chi}{\partial z}$$

$$= \vec{i}\,(-y) + \vec{j}\,(-x) + \vec{k}\,(1)$$

Let $\vec{a} = \nabla\chi_{(3,1\,,3)} = -\vec{i} - 3\vec{j} + \vec{k}$

$$\Rightarrow |\vec{a}| = \sqrt{1+9+1} = \sqrt{11}$$

$$\text{D. D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= \frac{(3\vec{i}+9\vec{j}+3\vec{k})\cdot(-\vec{i}-3\vec{j}+\vec{k})}{\sqrt{11}}$$

$$= \frac{1}{\sqrt{11}}\,(-3-27+3) = -\frac{27}{\sqrt{11}}$$