Relation

A relation R is a well - defined rule, which tells whether given 2 elements x and y of A are related or not.

If x is related to y, we write xRy, otherwise x does not related to y.

Equivalence Relation

Let X be any set. R be a relation defined on X. If R satisfies Reflexive, Symmetric and Transitive then the relation R is said to be an Equivalence relation.

Partial Order Relation

Let X be any set. R be a relation defined on X. Then R is said to be a partial order relation if it satisfies reflexive, antisymmetric and transitive relation.

Example:

Subset relation ⊆ is a Partial order relation.

Solution:

Consider any three sets A, B, C

Since any set is a subset to itself, $A \subseteq A$, therefore ⊆ is reflexive.

If $A \subseteq B$ and $B \subseteq A$, then $A = B$, therefore ⊆ is antisymmetric.

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$, therefore ⊆ is transitive.

Hence ⊆ is a Partial order relation.

Example:2

Divides relation is a Partial order relation.

Solution:

For Z_+ be the set of positive integer $a, b, c \in Z_+$

Since a/a ,$/$ is reflexive.

Since $a/b \ and \ b/a \Rightarrow a = b$,$/$ is antisymmetric.

Since $a/b \ and \ b/c \Rightarrow a \ / \ c$ is transitive.

Therefore, Divides relation " $/$ " is a partial order relation.

Hence the proof.

Partially Ordered Set or Poset:

A set together with a partial order relation defined on it is called partially ordered set or Poset.

Usually, a partial order relation is defined by the symbol " $\leq$ ", this symbol does not necessarily mean "less than or equal to" as we use for real numbers.

For example,

Let $\mathbb{R}$ be the set of real numbers. The relation "less than or equal to" or " $\leq$ " is a partial order on $\mathbb{R}$. Therefore $(\mathbb{R}, \leq)$ is a Poset.

Comparable Property:

In a Poset for any 2 elements a, b either $a \leq b$ or $b \leq a$ is called comparable property. Otherwise it is called incomparable property.

Totally Ordered Set or Linearly Ordered Set or Chain:

A partially ordered set $(\rho, \leq)$ is said to be totally ordered set or linearly ordered set or chain if any 2 elements are comparable.

i.e., given any 2 elements x and y of a Poset either $x \leq y$ or $y \leq x$

Example:

aRb if $a \leq b$ is a total order.

aRb if a/b is not a total order.

For, Given elements 2 and 3 neither $2/3$ nor $3/2$.

(i.e., 2 and 3 are not comparable).

Problems:

1. Show that the "greater than or equal to" relation is a Partial ordering on the set of integers.

Solution:

Since $a \geq a$ for every integer a, $\geq$ is reflexive.

If $a \geq b$ and $b \geq a$ then $a = b$. Hence $\geq$ is antisymmetric.

Since $a \geq b$ and $b \geq c$ imply $a \geq c$. Hence $\geq$ is transitive.

Therefore, $\geq$ is a partial order relation on the set of integers.

2. In the Poset $(Z^+, /)$ are the integers 3 and 9 comparable? Are 5 and 7 are comparable?

Solution:

Since $3/9$, the integers 3 and 9 are comparable.

For 5, 7 neither $5/7$ nor $7/5$

Therefore, the integers 5 and 7 are not comparable (incomparable).

3. Check the following Posets are totally orders set (or linearly ordered set or chain) (i) $(Z, \leq)$ (ii) $(Z^+, /)$

Solution:

(i) Consider, the Poset $(Z, \leq)$

If a and b are integer then either $a \leq b$ or $b \leq a$, for all a, b

Therefore, the Poset $(Z, \leq)$ satisfies comparable property.

$(Z, \leq)$ is a totally ordered set.

(ii) Consider, the Poset $(Z^+, /)$

Take 5 and 7.

Since, neither $5/7$ nor $7/5$

$(Z^+, /)$ does not satisfies the comparable property.

Therefore, $(Z^+, /)$ is not a totally ordered set.

4. Show that $(N, \leq)$ is a partially ordered set where N is set of all positive integers and $\leq$ is defined by $m \leq n$ iff $n - m$ is a non – negative integer.

Solution:

Give N is the set of all positive integer.

The given relation is $m \leq n$ iff $n - m$ is a non – negative integer.

(i) To prove R is reflexive

Now, $\forall x \in N$, $x - x = 0$ is a non – negative integer.

Therefore, $xRx \ \forall x \in N$.

Therefore R is reflexive.

(ii) To prove R is Antisymmetric.

Consider xRy & yRx

Since $xRy \Rightarrow x - y$ is a non – negative integer.

$yRx \Rightarrow y - x$ is a non – negative integer.

$\Rightarrow -(x - y)$ is a non – negative integer.

$\Rightarrow x = y$

Therefore R is Antisymmetric.

(iii) To prove R is Transitive.

Assume xRy & yRz

Since $xRy \Rightarrow x - y$ is a non – negative integer.

$yRz \Rightarrow y - z$ is a non – negative integer.

$\Rightarrow (x - y) + (y - z)$ is a non – negative integer.

$\Rightarrow x - z$ is a non – negative integer.

$\Rightarrow xRz$

xRy & $yRz \Rightarrow xRz$

Therefore R is transitive.

Hence R is partial order relation.

5. Is the Poset $(Z^+,/)$ a lattice.

Solution:

Let a and b be any two positive integer.

Then LUB $\{a, b\}$ =LCM $\{a, b\}$

GLB $\{a, b\}$ =GCD $\{a, b\}$

Should exist in Z^+.

For, example let $a = 4, b = 20$

Then LUB $\{a, b\}$ =LCM $\{4, 20\} = 1$

GLB $\{a, b\}$ =GCD $\{4, 20\} = 4$

Hence both GLB and LUB exist.

Therefore, the Poset $(Z^+,/)$ a lattice.