5.4 SOLUTION OF STATE EQUATIONS

Solution of homogeneous state equations (solution of state equations without input or excitation)

Consider a first order differential equation, with initial condition, $x(0) = x_0$.

$\frac{dx}{dt} = ax; x(0) = x_0$

On rearranging equation we get $\frac{dx}{x} = a\,dt$ integrating the equation we get, $\log x = at + c$

$$x = e^{(at+c)} = e^{at}.e^{c}$$

When t=0, $x = x(0) = e^c$

Given that, $x(0) = x_0 \therefore e^c = x_0$

The solution of first order differential equation as,

$$x = e^{at}x_0$$

We know that, $e^x = \left[1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{i!}x^i\right]$

$$x = \left[1 + at + \frac{1}{2!}(at)^2 + \cdots + \frac{1}{i!}(at)^i\right]x_0$$

Consider the state equations without input vector,

$\dot{X}(t) = A\,X(t)$; $x(0) = x_0$.

The solution of the matrix or vector equation can be assumed

$$X(t) = A_0 + A_1 t + A_2 t^2 + \cdots$$

Where A_0, A_1, A_2, ….A_i are matrices and the elements of the matrices are constant.

$$X(t) = \left[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \frac{1}{i!}A^it^i\right]A_0$$

Where I is the unit matrix.

Matrix exponential may be written as,

$$e^{At} = \left[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \frac{1}{i!}A^it^i\right]$$

Hence the solution of the state equation is

$$X(t) = e^{At}x_0$$

The matrix e^{At} is alled state transition matrix and denoted by φ(t).

PROPERTIES OF STATE TRANSITION MATRIX

1. $\varphi(0) = e^{A\,x\,0} = I$
2. $\varphi(t) = e^{At} = (e^{-At})^{-1} = [\varphi(-t)]^{-1}$
3. $\varphi(t_1 + t_2) = e^{A(t_1+t_2)} = (e^{At_1})(e^{At_2}) = \varphi(t_1)\varphi(t_2)$

COMPUTATION OF STATE TRANSISTION MATRIX

Method 1: Computation of e^{At} using matrix exponential

In this method the e^{At} is computed using the matrix exponential of equation

$$e^{At} = \left[I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \frac{1}{i!}A^it^i\right]$$

Where,

e^{At}=state transition matrix of order n x n

A = System matrix of order n x n

I = Unit matrix of order n x n

Method 2: Computation of state transition matrix by Laplace transforms method

Consider the state equation without input vector $\dot{X}(t) = A\,X(t)$

On taking Laplace transform of above equation,

$$X(s) = (sI - A)^{-1}X(0)$$

On taking inverse Laplace transform

$$X(t) = L^{-1}[(sI - A)^{-1}]\,X(0)$$

On comparing above equation with state equation

$$X(t) = e^{At}x_0$$

$$e^{At} = L^{-1}[(sI - A)^{-1}]$$

$$L[e^{At}] = L[\varphi(t)] = \varphi(s)$$

Where $\varphi(s) = (sI - A)^{-1}$ it is called resolvant matrix