UNIT – III

NUMERICAL DIFFERENTIATION AND INTEGRATION

PROBLEMS BASED ON TRAPEZOIDAL RULE AND SIMPSON'S RULE

Trapezoidal rule

$$\int f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 \ldots\ldots y_{n-1})]$$

1. ***Evaluate*** $\int_0^1 \frac{1}{1+x^2}dx$ ***with*** $h = \frac{1}{6}$ ***by Trapezoidal Rule***

Solution : ***Let*** $f(x) = \frac{1}{1+x^2}$ ***and*** $h = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = f(x)$	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$

$$f(x) = \frac{1}{1+x^2}$$

$$f(0) = \frac{1}{1+0^2} = \frac{1}{1+0} = 1$$

$$f\left(\frac{1}{6}\right) = \frac{1}{1+\frac{1}{36}} = \frac{1}{\frac{36+1}{36}} = \frac{36}{37}$$

$$f\left(\frac{2}{6}\right) = \frac{1}{1+\frac{4}{36}} = \frac{1}{\frac{36+4}{36}} = \frac{36}{40} = \frac{9}{10}$$

By Trapezoidal Rule

$$\int f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 \ldots\ldots y_{n-1})]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1/6}{2}\left[\left(1+\frac{1}{2}\right) + 2\left(\frac{36}{37} + \frac{9}{10} + \frac{4}{5} + \frac{9}{13} + \frac{36}{61}\right)\right]$$

$$= \frac{1}{12}\left[\left(\frac{3}{2}\right) + 2(3.9554)\right]$$

$$= \frac{1}{12}\left[\left(\frac{3}{2}\right) + 7.9108\right]$$

$$= 0.7842$$

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2. Evaluate $\int_{1}^{2} \frac{1}{1+x^2}\,dx$ with four sub intervals by Trapezoidal Rule

Solution:

$$\text{Let } y = f(x) = \frac{1}{1+x^2} \quad \text{and } h = \frac{2-1}{4} = \frac{1}{4} = 0.25$$

x	1	1.25	1.5	1.75	2
$y = f(x)$	0.5	0.3902	0.3077	0.2462	0.2

$$f(x) = \frac{1}{1+x^2}$$

$$f(1) = \frac{1}{1+1^2} = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

$$f(1.25) = \frac{1}{1+(1.25)^2} = 0.3902$$

$$f(1.5) = \frac{1}{1+(1.5)^2} = 0.3077$$

Trapezoidal Rule

$$\int f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 \ldots\ldots y_{n-1})]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{0.25}{2}[(0.5 + 0.2) + 2(0.3902 + 0.3077 + 0.2462)]$$

$$= 0.125[(0.7) + 2(0.9441)]$$

$$= 0.125[2.5882] = 0.3235$$

Using Trapezoidal rule, evaluate $\int_{-1}^{1} \frac{1}{(1+x^2)} dx$ by taking eight equal intervals.

(MJ13)

Solution: Let $y = y(x) = f(x) = \frac{1}{1+x^2}$

Lenth of the given interval $[a, b] = b - a = 1 - (-1) = 2$

Lenth of the 8 equal intervals $= h = \frac{2}{8} = 0.25$

Form the table for the ordinates of the function $y = y(x) = f(x) = \frac{1}{1+x^2}$

x	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
$y = \frac{1}{1+x^2}$	0.5	0.64	0.8	0.94118	1	0.94118	0.8	0.64	0.5
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

$$\int_{x_0}^{x_0+nh} f(x)dx = \begin{cases} \frac{h}{2}[(\text{first term} + \text{last term}) + 2(\text{remaining terms})] \\ \text{(or)} \\ \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})] \end{cases}$$

$$= \frac{h}{2}[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= \frac{0.2}{2}[(0.5 + 0.5) + 2(0.64 + 0.8 + 0.94118 + 1 + 0.94118 + 0.8 + 0.64)]$$

$$\equiv 1.56559$$

$Simpson's\ \frac{1}{3}\ rule$

$$\int f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \cdots.) + 4(y_1 + y_3 + y_5 \ldots.)]$$

1. $Evaluate \int_0^1 \frac{1}{1+x^2}\,dx$ with $h = \frac{1}{6}$ by Simpson's $\frac{1}{3}$ rule

Solution :

Let $f(x) = \frac{1}{1+x^2}$ and $h = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
y	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$

$$f(x) = \frac{1}{1+x^2}$$

$$f(0) = \frac{1}{1+x^2} = \frac{1}{1+0} = 1$$

$$f\left(\frac{1}{6}\right) = \frac{1}{1+\frac{1}{36}} = \frac{1}{\frac{36+1}{36}} = \frac{36}{37}$$

$$f\left(\frac{2}{6}\right) = \frac{1}{1+\frac{4}{36}} = \frac{1}{\frac{36+4}{36}} = \frac{36}{40} = \frac{9}{10}$$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
y	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{1}{3}$ rule

$$\int f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \cdots) + 4(y_1 + y_3 + y_5 \ldots)]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{1/6}{2}\left[\left(1+\frac{1}{2}\right) + 2\left(\frac{9}{10}+\frac{9}{13}\right) + 4\left(\frac{36}{37}+\frac{4}{5}+\frac{36}{61}\right)\right]$$

$$= \frac{1}{12}\left[\left(\frac{3}{2}\right) + 2(1.5923) + 4(2.3632)\right]$$

$$= \frac{1}{12}\left[\left(\frac{3}{2}\right) + 3.1846 + 9.4528\right]$$

$$= \frac{1}{12}\left[\left(\frac{3}{2}\right) + 12.6374\right]$$

$$= 1.1781$$

2. Evaluate $\int_{1}^{2} \frac{1}{1+x^2}\, dx$ with Four sub interval by Simpson's $\frac{1}{3}$ rule

Solution :

Let $f(x) = \frac{1}{1+x^2}$ and $h = \frac{2-1}{4} = \frac{1}{4} = 0.25$

x	1	1.25	1.5	1.75	1
y	0.5	0.3902	0.3077	0.2462	0.2
	y_0	y_1	y_2	y_3	y_4

$$f(x) = \frac{1}{1+x^2}$$

$$f(1) = \frac{1}{1+x^2} = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

$$f(1.25) = \frac{1}{1+(1.25)^2} = 0.3902$$

$$f(1.5) = \frac{1}{1+(1.5)^2} = 0.3077$$

Simpson's $\frac{1}{3}$ rule

$$\int f(x)dx = \frac{h}{2}[(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \cdots.) \quad + 4(y_1 + y_3 + y_5 \ldots.)]$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{0.25}{2}[((0.5 + 0.2)) + 2(0.3077) + 4(0.3902 + 0.2462)]$$

$$= 0.125[(0.7) + 0.6154 + 2.5456]$$

$$= 0.125[3.861]$$

$$= 0.4826$$

A curve passes through the points (1, 2), (1.5, 2.4), (2, 2.7), (2.5, 2.8), (3, 3), (3.5, 2.6) & (4, 2.1). Obtain Area bounded by the curve, x axis between $x = 1$ and $x = 4$. Also find the volume of solids of revolution by revolving this area about x– axis.

Solution:

$$\text{WKT, Area } = \int_a^b y\,dx = \int_1^4 y\,dx$$

$$\text{Simpson's 1/3 rule, Area } = \int_1^4 y\,dx = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= 7.783$$

$$\text{Volume } = \pi\int_a^b y^2\,dx = \pi\int_1^4 y^2\,dx$$

<u>To find y^2:</u>

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1
y^2	4	5.76	7.29	7.84	9	6.76	4.41

$$\therefore \text{ Volume } = \pi\int_1^4 y^2\,dx = \pi\left\{\frac{h}{3}\left[\left(y_0^2 + y_6^2\right) + 4\left(y_1^2 + y_3^2 + y_5^2\right) + 2\left(y_2^2 + y_4^2\right)\right]\right\} \qquad \text{[by Simpson's 1/3 rule]}$$

$$= 64.07 \text{ cubic units}$$

Numerical integration using Simpson's 3/8 rule

Simpson's three eighth rule (Simpson's 3/8 rule) ($n = 3$ in quadratic form)

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \begin{cases} \dfrac{3h}{8}\left| \begin{array}{l} (y_0 + y_n) \\ +2(y_3 + y_6 + \cdots) \\ +3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots) \end{array} \right| \\ \text{(or)} \\ \dfrac{3h}{8}\left| \begin{array}{l} (\text{first term} + \text{last term}) \\ +2(\text{suffices with a multiple of } 3) \\ +3(\text{remaining terms}) \end{array} \right| \end{cases}$$

Compute the value of $\int_1^2 \frac{dx}{x}$ using Simpson's 3/8 rule

Solution: Let $y = f(x) = \frac{1}{x}$,

$h = 1/3$

The tabulated values of $y = f(x)$ are

	x_0	x_1	x_2	x_3
x	1	$4/3$	$5/3$	$6/3 = 2$
$f(x)$	$\frac{1}{1} = 1$	$\frac{1}{4/3} = 3/4 = .75$	$\frac{1}{5/3} = 0.6$	$\frac{1}{2} = 0.5$
	y_0	y_1	y_2	y_3

WKT, Simpson's 3/8 rule is

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_4) + 3\,(y_1 + y_2) + 2\,(y_3)\right]$$

$$\text{i.e.,}\quad \int_1^2 \left(\frac{1}{x}\right) dx = \frac{3\,(1/3)}{8}\left[(1 + 0.5) + 3\,(0.75 + 0.6) + 2\,(0)\right]$$

$$= 0.69375$$

By actual integration,

$$\int_1^2 \frac{1}{x}\,dx = (\log_e x)_1^2 = (\ln 2 - \ln 1) = 0.69315$$

Single integrals by Trapezoidal, Simpson 1/3 & 3/8

Compute the value of $\int_1^2 \frac{dx}{x}$ using

(a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule

Solution: Here $h = 0.25$,

$$y = f(x) = \frac{1}{x}$$

The tabulated values of $y = f(x)$ are

	x_0	x_1	x_2	x_3	x_4
x	1	1.25	1.50	1.75	2
$f(x)$	$\frac{1}{1} = 1$	$\frac{1}{1.25} = 0.8$	$\frac{1}{1.5} = 0.66667$	$\frac{1}{1.75} = 0.57143$	$\frac{1}{1.2} = 0.5$
	y_0	y_1	y_2	y_3	y_4

(a) WKT, Trapezoidal rule is

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$\Rightarrow \int_1^2 \left(\frac{1}{x}\right) dx = \frac{0.25}{2}[(1 + 0.5) + 2(0.8 + 0.66667 + 0.57143)]$$

$$= \frac{0.25}{2}[1.5 + 4.0762] = \frac{0.25}{2}[5.5762]$$

$$= 0.697025$$

(b) WKT, Simpson's 1/3 rule is

$$\int_a^b f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3) + 2(y_2)]$$

$$\Rightarrow \int_1^2 \left(\frac{1}{x}\right) dx = \frac{0.25}{3}[(1 + 0.5) + 4(0.8 + 0.57143) + 2(0.66667)]$$

$$= 0.693255$$

(c) WKT, Simpson's 3/8 rule is

$$\int_a^b f(x)\,dx = \frac{3h}{8}[(y_0 + y_4) + 3(y_1 + y_2) + 2(y_3)]$$

$$\text{i.e., } \int_1^2 \left(\frac{1}{x}\right) dx = \frac{3(0.25)}{8}[(1 + 0.5) + 3(0.8 + 0.66667) + 2(0.57143)]$$

$$= 0.66024$$

By actual integration,

$$\int_1^2 \frac{1}{x}\, dx = (\log_e x)_1^2 = (\ln 2 - \ln 1) = 0.69315$$

Evaluate $\int_0^{10} \frac{dx}{1+x}$ by dividing the range into 8 equal parts by (a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule

Solution: Here $h = 1.25$, $y = f(x) = \frac{1}{1+x}$

(a) I = 2.51368 , (b) I = 2.42200 , (c) I = 2.41838 , Actual integration, $\int_0^{10} \frac{dx}{1+x} = 2.39790.$

Example 4.31. **Evaluate $\int_0^1 x\, e^x\, dx$ taking 4 equal intervals by (a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule**

Solution: (a) 1.02307 , (b) 1.00017 , (c) 0.87468 , AI = 1

Example 4.32. **Calculate $\int_0^{\pi} \sin^3 x\,dx$ taking 7 ordinates (6 intervals) using a) Trapezoidal rule (b) Simpson's 1/3 rule (c) Simpson's 3/8 rule**

Solution: (a)1.33467 , (b)1.32612 , (c)1.30516 , AI = 1.3333.

Anna University Questions

1. The velocity v of a particle at a distance S from a point on its path is given by the table below:

S (meter)	0	10	20	30	40	50	60
v (m / sec)	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by Simpson's $1/3^{rd}$ rule and Simpson's $3/8^{th}$ rule. (AM10)

Solution: [S(1/3) : $I = 1.06338$, S(3/8) : $I = 1.06425$]

2. Evaluate $I = \int_0^6 \frac{1}{1+x}dx$ by using (i) direct integration (ii) Trapezoidal rule (iii) Simpson's one-third rule (iv) Simpson's three-eighth rule. (ND11)

Solution: [By Dir. Int. : $I = 1.9459$, Trap. : $I = 2.022$, S(1/3) : $I = 1.9587$, S(3/8) : $I = 1.966$]

3. Compute $\int_0^{\pi/2} \sin x dx$ using Simpson's 3/8 rule. (ND12)

Solution: [S(3/8) : $I = .9999988 \approx 1$]

4. The velocities of a car running on a straight road at intervals of 2 minutes are given below:

Time(min):	0	2	4	6	8	10	12
Velocity(km/hr):	0	22	30	27	18	7	0

Using Simpson's $\frac{1}{3}$–rd rule find the distance covered by the car. (ND13)

Solution: [By S(1/3) rule, the distance covered by the car : $I = 3.55$ km,]

5. Taking $h = 0.05$ evaluate $\int_1^{1.3} \sqrt{x}dx$ using Trapezoidal rule and Simpson's three-eighth rule. (AM14)

Solution: [By Trap. : $I = 0.32147$, S(3/8) : $I = 0.321485354$]

6. The velocity v of a particle at a distance α from a point on its path is given by the table: (ND14)

$s(ft)$	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38

Estimate the time taken to travel 60 feet by using Simpson's $\frac{3}{8}$ rule.