Half Range Expansions:

In many Engineering problems it is required to expand a function $f(x)$ in the range $(0, \pi)$

In a Fourier series of period 2π or in the range $(0, l)$ in a Fourier series of period $2l$. If it is required to expand $f(x)$ in the interval $(0, l)$, then it is immaterial what the function may be outside the range $0 < x < l$.

If we extend the function $f(x)$ by reflecting it in the Y – axis so that $f(-x) = f(x)$, then the extended function is even for which $b_n = 0$. The Fourier expansion of $f(x)$ will contain only cosine terms.

If we extend the function $f(x)$ by reflecting it in the origin so that $f(-x) = -f(x)$, then the extended function is odd for which $a_0 = a_n = 0$. The Fourier expansion of $f(x)$ will contain only sine terms.

Here a function $f(x)$ defined over the interval $0 < x < l$ is capable of two distinct half range series.

(i) Sine Series

(ii) Cosine Series

Problems under Half Range Sine series and Cosine series

1. Expand $f(x) = x$ as a cosine series in $0 < x < l$ and deduce the value of

(i) $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$ $\quad (ii)$ $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$

Solution:

Given $f(x) = x$

The cosine series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$ ……..(1)

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$= \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l$$

$$= \frac{1}{l}[l^2] = l$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx \quad = \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[x\left(\sin\frac{n\pi x}{l}\right)\frac{l}{n\pi} - \left(-\cos\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^2}{n^2\pi^2}(-1)^n - \frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{2l}{n^2\pi^2}[(-1)^n - 1]$$

$$= 0 \; if\; n\; is\; even$$

$$= \frac{-4l}{n^2\pi^2}\; if\; n\; is\; odd$$

Substituting in equation (1) we get

$$f(x) = \frac{l}{2} - \frac{4l}{\pi^2} \sum_{n=odd} \frac{1}{n^2} \cos \frac{n\pi x}{l} \qquad \ldots\ldots(2)$$

Deduction (i)

By Parseval's identity

$$\frac{2}{l} \int_0^l [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\Rightarrow \frac{2}{l} \int_0^l x^2 \, dx = \frac{l^2}{2} + \sum_{n=odd} \left(\frac{-4l}{n^2\pi^2} \right)^2$$

$$\Rightarrow \frac{2}{l} \left[\frac{x^3}{3} \right]_0^l = \frac{l^2}{2} + \sum_{n=odd} \frac{16\, l^2}{n^4 \pi^4}$$

$$\Rightarrow \frac{2}{l} \left[\frac{l^3}{3} \right] = \frac{l^2}{2} + \frac{16\, l^2}{\pi^4} \sum_{n=odd} \frac{1}{n^4}$$

$$\Rightarrow \frac{2l^2}{3} = \frac{l^2}{2} + \frac{16\, l^2}{\pi^4} \sum_{n=odd} \frac{1}{n^4}$$

$$\Rightarrow \frac{16\, l^2}{\pi^4} \sum_{n=odd} \frac{1}{n^4} = \frac{2l^2}{3} - \frac{l^2}{2} = l^2 \left[\frac{2}{3} - \frac{1}{2} \right] = \frac{l^2}{6}$$

$$\sum_{n=odd} \frac{1}{n^4} = \frac{l^2}{6} \left[\frac{\pi^4}{16\, l^2} \right] = \frac{\pi^4}{96}$$

$$(i.e.) \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

Deduction (ii)

$$Let\ S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

$$= \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right] + \left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \cdots\right]$$

$$= \frac{\pi^4}{96} + \frac{1}{2^4}\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots\right] \qquad by\ (i)$$

$$(i.e.)\ S = \frac{\pi^4}{96} + \frac{1}{2^4}S \ = \frac{\pi^4}{96} + \frac{1}{16}S$$

$$S - \frac{1}{16}S \ = \frac{\pi^4}{96}$$

$$S\left(1 - \frac{1}{16}\right) \ = \frac{\pi^4}{96}$$

$$S\left(\frac{15}{16}\right) \ = \frac{\pi^4}{96}$$

$$S \ = \frac{\pi^4}{96}\left(\frac{16}{15}\right) = \frac{\pi^4}{90}$$

$$(i.e.)\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$$

2. Obtain the Sine series for $f(x) = x$ in $0 < x < \pi$ and hence deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

Solution:

Given $f(x) = x$

The Sine series is $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ ………..(1)

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \ dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin nx \ dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\left(\frac{-1}{n}\right)^{n}\right] = \frac{-2(-1)^n}{n}$$

Substitute in equation (1) we get

$$f(x) = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

Deduction:

By Parseval's identity

$$\frac{2}{\pi}\int_0^{\pi} [f(x)]^2 \, dx = \sum_{n=1}^{\infty} b_n^2$$

$$\frac{2}{\pi}\int_0^{\pi} x^2 \, dx = \sum_{n=1}^{\infty} \left(\frac{-2(-1)^n}{n}\right)^2$$

$$\frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\Rightarrow \frac{2}{\pi}\left[\frac{\pi^3}{3}\right] = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$(i.e.)\ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2\pi^2}{3}\left(\frac{1}{4}\right) = \frac{\pi^2}{6}$$

Complex or Exponential Form of Fourier series:

1. Find the complex form of the Fourier series of $f(x) = e^{-x}$ in $-1 \leq x \leq 1$

Solution:

The complex form of the Fourier series in $(-1, 1)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x} \qquad \dots\dots..(1)$$

Where $c_n = \frac{1}{2}\int_{-1}^{1} f(x)\, e^{-in\pi x}\, dx$

$$= \frac{1}{2}\int_{-1}^{1} e^{-x}\, e^{-in\pi x}\, dx = \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1}$$

$$= \frac{e^{1+in\pi} - e^{-(1+in\pi)}}{2(1+in\pi)}$$

$$= \frac{e(\cos n\pi + i\sin n\pi) - e^{-1}(\cos n\pi - i\sin n\pi)}{2(1+in\pi)}$$

$$= \frac{e(-1)^n - e^{-1}(-1)^n}{2(1+in\pi)}$$

$$c_n = \frac{(e - e^{-1})(-1)^n}{2}\left(\frac{1 - in\pi}{1 + n^2\pi^2}\right)$$

$$= \frac{(-1)^n(1 - in\pi)}{1 + n^2\pi^2}\sinh 1$$

Hence (1) becomes

$$e^{-x} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1 - in\pi)}{1 + n^2\pi^2}\sinh 1\ e^{in\pi x}$$

2. Find the complex form of the Fourier series of $f(x) = \cos ax$ in $(-\pi, \pi)$ where 'a' is neither zero nor an integer.

Solution:

Here $2c = 2\pi$ or $c = \pi$

Let the complex form of the Fourier series be

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \ldots\ldots..(1)$$

$$where\ c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{-inx}}{a^2 - n^2}(-in\cos ax + a\sin ax)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(a^2 - n^2)}\left[e^{-in\pi}(-in\cos a\pi + a\sin a\pi) - e^{in\pi}(-in\cos a\pi + a\sin a\pi)\right]$$

$$= \frac{1}{2\pi(a^2 - n^2)}\left[in\cos a\pi\left(e^{in\pi} - e^{-in\pi}\right) + a\sin a\pi\left(e^{in\pi} + e^{-in\pi}\right)\right]$$

$$= \frac{1}{2\pi(a^2 - n^2)}\left[in\cos a\pi\,(2i\sin n\pi) + a\sin a\pi(2\cos n\pi)\right]$$

$$c_n = \frac{1}{2\pi(a^2 - n^2)}(-1)^n 2\,\mathrm{a}\sin a\pi$$

Hence (1) becomes $\cos ax = \frac{a\sin a\pi}{\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{(a^2-n^2)}e^{inx}$

Harmonic Analysis:

The process of finding the Fourier series for a function given by numerical values is known as harmonic analysis.

In harmonic analysis the Fourier coefficients a_0, a_n and b_n of the function $y = f(x)$ in $(0, 2\pi)$ are given by

$$a_0 = 2[mean\ value\ of\ y\ in(0, 2\pi)]$$

$$a_n = 2[mean\ value\ of\ y\cos nx\ in(0, 2\pi)]$$

$$b_n = 2[mean\ value\ of\ y\sin nx\ in(0, 2\pi)]$$