5.4 Solved Problems Boundary Layer Thickness

PROBLEM 1:The velocity distribution in the boundary layer is given by

$$\frac{u}{U} = \frac{y}{\sigma}$$

where u is the velocity y from the plate and u=U at , $y = \delta$,δ being boundary layer thickness.Find

i. The displacement thickness
ii. The momentum thickness
iii. The energy thickness and
iv. The value of δ */ θ.

Solution:

Velocity distribution:

$$\frac{u}{U} = \frac{y}{\sigma}$$

(i) The displacement thickness δ^*

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy$$

$$= \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy$$

$$= \left[y - \frac{y^2}{2\delta} \right]_0^\delta$$

$$= \left(\delta - \frac{\delta^2}{2\delta} \right) = \delta - \frac{\delta}{2}$$

$$= \frac{\delta}{2}$$

(ii) The momentum thickness

$$\theta = \int_o^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

$$= \int_o^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy$$

$$= \int_o^\delta \left(\frac{y}{\delta} - \frac{y^2}{\delta^2} \right) dy$$

or

$$\theta = \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^{\sigma} = \frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2} = \frac{\delta}{2} - \frac{\delta}{3} = \frac{\delta}{6}$$

iii. The energy thickness

$$\delta_e = \int_o^{\delta} \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right)dy$$

$$= \int_o^{\delta} \frac{y}{\delta}\left(1 - \frac{y^2}{\delta^2}\right)dy = \int_o^{\delta}\left(\frac{y}{\delta} - \frac{y^3}{\delta^3}\right)dy$$

$$= \left[\frac{y^2}{2\delta} - \frac{y^4}{4\delta^3}\right]_0^{\delta} = \frac{\delta^2}{2\delta} - \frac{\delta^4}{4\delta^3}$$

$$= \frac{\delta}{2} - \frac{\delta}{4}$$

$$= \frac{\delta}{4}$$

iv. The value of δ */ θ.

$$\frac{\delta *}{\theta} = \frac{\delta/2}{\delta/6}$$

$$= 3.0$$

PROBLEM 2: The velocity distribution in the boundary layer is given by ,

$$\frac{u}{U} = \frac{3}{2}\frac{y}{\sigma} - \frac{1}{2}\frac{y^2}{\sigma^2}$$

σ being the boundary layer thickness.Calculate the following

(i) The ratio of displacement thickness to boundary layer thickness $\left(\frac{\delta *}{\delta}\right)$

(ii) The ratio of momentum thickness to boundary layer thickness $\left(\frac{\theta}{\delta}\right)$

Solution

Velocity distribution: $\frac{u}{U} = \frac{3}{2}\frac{y}{\sigma} - \frac{1}{2}\frac{y^2}{\sigma^2}$

(i) $\dfrac{\delta *}{\delta}$:

$$\delta * = \int_0^{\delta}\left(1-\frac{u}{U}\right)dy = \int_0^{\delta}\left(1-\frac{3}{2}\frac{y}{\sigma}+\frac{1}{2}\frac{y^2}{\sigma^2}\right)dy$$

$$=\left[y-\frac{3}{2}\times\frac{y^2}{3\sigma}+\frac{1}{2}\times\frac{y^3}{3\sigma^2}\right]_0^{\delta}$$

$$\left[\sigma-\frac{3}{4}\frac{\sigma^2}{\sigma}+\frac{1}{2}\frac{\sigma^2}{3\sigma^2}\right]$$

$$=\left(\sigma-\frac{3}{4}\sigma+\frac{\sigma}{6}\right)$$

$$\sigma * = \frac{5}{12}\sigma$$

$$\therefore \frac{\sigma *}{\sigma} = \frac{5}{12}\sigma.$$

(ii) θ/σ

$$\theta = \int_0^{\sigma}\frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

$$= \int_0^{\sigma}\left(\frac{3}{2}\frac{y}{\sigma}-\frac{1}{2}\frac{y^2}{\sigma^2}\right)\left(1-\frac{3}{2}\frac{y}{\sigma}+\frac{1}{2}\frac{y^2}{\sigma^2}\right)dy$$

$$=\int_0^{\sigma}\left(\frac{3}{2}\frac{y}{\sigma}-\frac{9}{4}\frac{y^2}{\sigma^2}+\frac{3}{4}\frac{y^3}{\sigma^3}-\frac{1}{2}\frac{y^2}{\sigma^2}+\frac{3}{4}\frac{y^3}{\sigma^3}-\frac{1}{4}\frac{y^4}{\sigma^4}\right)dy$$

$$=\int_0^{\sigma} 3/2\,\frac{y}{\sigma}-\left(\frac{9}{4}\frac{y^2}{\sigma^2}+\frac{1}{2}\frac{y^2}{\sigma^2}\right)+\left(\frac{3}{4}\frac{y^3}{\sigma^3}+\frac{3}{4}\frac{y^3}{\sigma^3}\right)-\frac{1}{4}\frac{y^4}{\sigma^4}\Bigg]dy$$

$$=\int_0^{\sigma}\left(\frac{3}{2}\frac{y}{\sigma}-\frac{11}{4}\frac{y^2}{\sigma^2}+\frac{3}{4}\frac{y^3}{\sigma^3}-\frac{1}{4}\frac{y^4}{\sigma^4}\right)dy$$

$$=\left[\frac{3}{2}\frac{y^2}{2\sigma}-\frac{11}{4}\frac{y^3}{3\sigma^2}+\frac{3}{2}\times\frac{y^4}{4\sigma^3}-\frac{1}{4}\times\frac{y^5}{4\sigma^4}\right]_0^{\delta}$$

$$=\left[\frac{3}{2}\times\frac{y^2}{2\sigma}\times\frac{11}{4}\times\frac{y^3}{3\sigma^2}+\frac{3}{2}\times\frac{\sigma^4}{4\sigma^3}-\frac{1}{4}\times\frac{\sigma^5}{5\sigma^4}\right]_0^\delta$$

$$\theta=\left(\frac{3}{4}\sigma-\frac{11}{12}\sigma+\frac{3}{8}\sigma-\frac{1}{20}\sigma\right)=\frac{19}{120}\sigma$$

PROBLEM 3 : Find the displacement thickness ,the momentum thickness and energy thickness for the velocity distribution in the boundary layer is given by

$$\frac{u}{U}=2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2.$$

Solution. Given :

Velocity distribution $\frac{u}{U}=2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2$

(*i*) Displacement thickness δ^* is given by equation

$$\delta^*=\int_0^\delta\left(1-\frac{u}{U}\right)dy$$

Substituting the value of $\frac{u}{U}=2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2$, we have

$$\delta^*=\int_0^\delta\left\{1-\left[2\left(\frac{y}{\delta}\right)-\left(\frac{y}{\delta}\right)^2\right]\right\}dy$$

$$=\int_0^\delta\left\{1-2\left(\frac{y}{\delta}\right)+\left(\frac{y}{\delta}\right)^2\right\}dy=\left[y-\frac{2y^2}{2\delta}+\frac{y^3}{3\delta^2}\right]_0^\delta$$

$$=\delta-\frac{\delta^2}{\delta}+\frac{\delta^3}{3\delta^2}=\delta-\delta+\frac{\delta}{3}=\frac{\delta}{3}.\quad\textbf{Ans.}$$

(*ii*) Momentum thickness θ, is given by equation

$$\theta=\int_0^\delta\frac{u}{U}\left\{1-\frac{u}{U}\right\}dy=\int_0^\delta\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\left[1-\left(\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right)\right]dy$$

$$=\int_0^\delta\left[\frac{2y}{\delta}-\frac{y^2}{\delta^2}\right]\left[1-\frac{2y}{\delta}+\frac{y^2}{\delta^2}\right]dy$$

$$=\int_0^\delta\left[\frac{2y}{\delta}-\frac{4y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^2}{\delta^2}+\frac{2y^3}{\delta^3}-\frac{y^4}{\delta^4}\right]dy$$

$$=\int_0^\delta\left[\frac{2y}{\delta}-\frac{5y^2}{\delta^2}+\frac{4y^3}{\delta^3}-\frac{y^4}{\delta^4}\right]dy=\left[\frac{2y^2}{2\delta}-\frac{5y^3}{3\delta^2}+\frac{4y^4}{4\delta^3}-\frac{y^5}{5\delta^4}\right]_0^\delta$$

$$= \left[\frac{\delta^2}{\delta} - \frac{5\delta^3}{3\delta^2} + \frac{\delta^4}{\delta^3} - \frac{\delta^5}{5\delta^4}\right] = \delta - \frac{5\delta}{3} + \delta - \frac{\delta}{5}$$

$$= \frac{15\delta - 25\delta + 15\delta - 3\delta}{15} = \frac{30\delta - 28\delta}{15} = \frac{\mathbf{2\delta}}{\mathbf{15}}. \quad \textbf{Ans.}$$

(*iii*) Energy thickness δ^{**} is given by equation

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left[1 - \frac{u^2}{U^2}\right] dy = \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \left[\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right]^2\right) dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \left[\frac{4y^2}{\delta^2} + \frac{y^4}{\delta^4} - \frac{4y^3}{\delta^3}\right]\right) dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{4y^2}{\delta^2} - \frac{y^4}{\delta^4} + \frac{4y^3}{\delta^3}\right) dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)\left(1 - \frac{4y^2}{\delta^2} - \frac{y^4}{\delta^4} + \frac{4y^3}{\delta^3}\right) dy$$

$$= \int_0^\delta \left(\frac{2y}{\delta} - \frac{8y^3}{\delta^3} - \frac{2y^5}{\delta^5} + \frac{8y^4}{\delta^4} - \frac{y^2}{\delta^2} + \frac{4y^4}{\delta^4} + \frac{y^6}{\delta^6} - \frac{4y^5}{\delta^5}\right) dy$$

$$= \int_0^\delta \left[\frac{2y}{\delta} - \frac{y^2}{\delta^2} - \frac{8y^3}{\delta^3} + \frac{12y^4}{\delta^4} - \frac{6y^5}{\delta^5} + \frac{y^6}{\delta^6}\right] dy$$

$$= \left[\frac{2y^2}{2\delta} - \frac{y^3}{3\delta^2} - \frac{8y^4}{4\delta^3} + \frac{12y^5}{5\delta^4} - \frac{6y^6}{6\delta^5} + \frac{y^7}{7\delta^6}\right]_0^\delta$$

$$= \frac{\delta^2}{\delta} - \frac{\delta^3}{3\delta^2} - \frac{2\delta^4}{\delta^3} + \frac{12\delta^5}{5\delta^4} - \frac{\delta^6}{\delta^5} + \frac{\delta^7}{7\delta^6} = \delta - \frac{\delta}{3} - 2\delta + \frac{12}{5}\delta - \delta + \frac{\delta}{7}$$

$$= -2\delta - \frac{\delta}{3} + \frac{12}{5}\delta + \frac{\delta}{7} = \frac{-210\delta - 35\delta + 252\delta + 15\delta}{105}$$

$$= \frac{-245\delta + 267\delta}{105} = \frac{\mathbf{22\delta}}{\mathbf{105}}. \quad \textbf{Ans.}$$

PROBLEM 4: For the velocity for the laminar boundary layer flows given as

$$\frac{u}{U} = 2\left(\frac{y}{\sigma}\right) - \left(\frac{y}{\sigma}\right)^2$$

find out the expression for boundary layer thickness (δ), shear stress (τ_0), co-efficient of drag (C_D) in terms of Reynolds number.

Solution. Given :

(*i*) The velocity distribution $\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ ...(*i*)

Substituting this value of $\frac{u}{U}$, we get

$$\frac{\tau_o}{\rho U^2} = \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)\left(1-\left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)\right)dy\right]$$

$$= \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)\left(1-\frac{2y}{\sigma} + \frac{y^2}{\sigma^2}\right)dy\right]$$

$$= \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{2y}{\sigma} - \frac{4y^2}{\sigma^2} + \frac{2y^3}{\sigma^3} - \frac{y^2}{\sigma^2} + \frac{2y^3}{\sigma^3} - \frac{y^4}{\sigma^2}\right)dy\right]$$

$$= \frac{d}{dx}\left[\int_0^{\sigma}\left(\frac{2y}{\sigma} - \frac{5y^2}{\sigma^2} + \frac{4y^3}{\sigma^3} - \frac{y^4}{\sigma^4}\right)dy\right]$$

$$= \frac{d}{dx}\left[\frac{2}{2}\frac{y^2}{\sigma} - \frac{5}{3}\frac{y^2}{\sigma^2} + \frac{4}{4}\frac{y^4}{\sigma^3} - \frac{1}{5}\frac{y^5}{\sigma^4}\right]_0^{\sigma}$$

$$= \frac{d}{dx}\left[\sigma - \frac{5}{3}\sigma + \sigma\frac{1}{5}\sigma\right] = \frac{d}{dx}\left(\frac{2}{15}\sigma\right)$$

$$\therefore \tau_o = \rho U^2 \times \frac{d}{dx}\left(\frac{2}{15}\sigma\right) = \frac{2}{15}\rho U^2\frac{d\delta}{dx} ------------(ii)$$

Also, according to Newton's law of viscosity

$$\tau_o = \mu\left(\frac{dy}{dx}\right)_{y=0} ------------(iii)$$

$$But \quad u = U\left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)$$

$$But \quad u = U\left(\frac{2y}{\sigma} - \frac{y^2}{\sigma^2}\right)$$

$$and \quad \frac{du}{dx} = U\left(\frac{2}{\sigma} - \frac{2y}{\sigma^2}\right), U \text{ being constant}$$

$$\therefore \left(\frac{du}{dx}\right)_{y=0} = U\left(\frac{2}{\sigma} - 0\right) = \frac{2U}{8}$$

Substituting this value in (iii), we get

$$\tau_o = \frac{2\mu U}{\sigma} ------------(iv)$$

Equating the values of τ_o given by equations (ii) and iv, we get

$$\frac{2}{15}\rho U^2 \frac{d\sigma}{dx} = \frac{2\mu U}{\sigma}$$

$$or \quad \sigma.\frac{d\sigma}{dx} = \frac{15\mu U}{\rho U^2} = \frac{15\mu}{\rho U^2}$$

$$or \quad \sigma.d\sigma = \frac{15\mu}{\rho U}dx$$

Integrating both sides, we get

$$\frac{\delta^2}{2} = \frac{15\mu}{\rho U}x + c \quad (where\ C = Constant\ of\ integration)$$

$$At\ x = 0,\ \delta = 0 \therefore C = 0$$

$$\therefore \frac{\delta^2}{2} = \frac{15\mu}{\rho U}$$

$$or\ \delta = \sqrt{\frac{2 \times 15\mu x}{\rho U}} = 5.48\sqrt{\frac{\mu x}{\rho U}}$$

$$= 5.48\sqrt{\frac{\mu x \times x}{\rho U \times x}} = 5.48 \sqrt{\frac{x^2}{Re_x}}$$

$$\left(where\ Re_x = \frac{\rho U x}{\varpi},\right)$$

$$or\ \sigma = 5.48 = \frac{x}{\sqrt{Re_x}} ------(v)$$

(ii) Shear stress τ_o :

From equation (iv), we have

$$\tau_o = \frac{2\mu U}{\sigma}$$

$$But\ \sigma = 5.48\frac{x}{\sqrt{Re_x}}$$

$$\therefore \tau_o = \frac{2\mu U}{5.48\frac{x}{\sqrt{Re_x}}} = \frac{2\mu U\sqrt{Re_x}}{5.48x} = 0.365\frac{\mu U}{x}\sqrt{Re_x} --------------(vi)$$

(iii) Local Co-efficient of drag, C_D^*

$$\tau_o = \frac{0.365\,\mu U}{x} = \sqrt{\text{Re}_x}$$

$$also\ \tau_o = C_D^* \frac{\rho U^2}{2} ----------(vii)\left(where\ C_D^* = local\ coefficient\ of\ drag\right)$$

Equating the two of τ_o, given by equation (vi) and (vii), we get

$$C_D^* = \frac{0.365\mu U}{x}\sqrt{\text{Re}_x} \quad or \quad C_D^* = 0.365 \times 2 \times \frac{\sqrt{\text{Re}_x}}{\frac{\rho U x}{\mu}}$$

$$= \frac{0.73}{\sqrt{\text{Re}_x}}$$

(iv) Co-efficient of drag, C_D:

We know that $C_D = \dfrac{F_D}{\frac{1}{2}\rho A U^2}$

Where, $F_D = \int_0^L \tau_o \times B \times dx$

$$= \int_0^l 0.365 \frac{\mu U}{x}\sqrt{\text{Re}_x} \times B \times dx$$

$$= 0.365 \int_0^l \frac{\mu U}{x}\sqrt{\frac{\rho U x}{\mu}} \times B \times dx \left(\because \text{Re}_x = \frac{\rho U x}{\mu}\right)$$

$$= 0.365 \int_0^l \frac{\mu U}{x}\sqrt{\frac{\rho U x}{\mu}} \times \frac{1}{\sqrt{x}} \times B \times dx$$

$$= 0.365\ \mu U \sqrt{\frac{\rho U}{\mu}} \times B \int_0^l x^{-\frac{1}{2}} \times dx$$

$$= 0.365\ \mu U \sqrt{\frac{\rho U}{\mu}} \times B \left[\frac{x^{-\frac{1}{2}}}{\frac{1}{2}}\right]$$

$$= 0.365 \times 2\mu U B \sqrt{\frac{\rho U}{\mu}} \times B\sqrt{L}$$

$$\therefore C_D = \frac{0.73\mu UB\sqrt{\frac{\rho U}{\mu}}}{\frac{1}{2}\rho A U^2}$$

(Where A – area of plate = L x B, L and B being length and width of the plate respectively)

$$\therefore C_D = \frac{0.73\mu UB\sqrt{\frac{\ell UL}{\mu}}}{\frac{1}{2}\ell \times L \times B \times U^2} = \frac{1.46\mu}{\ell L.U}\sqrt{\frac{\ell UL}{\mu}}$$

$$= \frac{1.46\sqrt{\mu}}{\sqrt{\ell L.U}} = 1.46\sqrt{\frac{\mu}{\ell L.U}} = \frac{1.46}{\sqrt{\mathrm{Re}_l}}$$