UNIT-IV FORCED VIBRATION

4.1 INTRODUCTION:

When a system is subjected continuously to time varying disturbances, the vibrations resulting under the presence of the external disturbance are referred to as forced vibrations.

Forced vibration is when an alternating force or motion is applied to a mechanical system. Examples of this type of vibration include a shaking washing machine due to an imbalance, transportation vibration (caused by truck engine, springs, road, etc), or the vibration of a building during an earthquake. In forced vibration the frequency of the vibration is the frequency of the force or motion applied, with order of magnitude being dependent on the actual mechanical system.

When a vehicle moves on a rough road, it is continuously subjected to road undulations causing the system to vibrate (pitch, bounce, roll etc). Thus the automobile is said to undergo forced vibrations. Similarly whenever the engine is turned on, there is a resultant residual unbalance force that is transmitted to the chassis of the vehicle through the engine mounts, causing again forced vibrations of the vehicle on its chassis. A building when subjected to time varying ground motion (earthquake) or wind loads, undergoes forced vibrations. Thus most of the practical examples of vibrations are indeed forced vibrations.

4.2 CAUSES RESONANCE:

Resonance is simple to understand if you view the spring and mass as energy storage elements – with the mass storing kinetic energy and the spring storing potential energy. As discussed earlier, when the mass and spring have no force acting on them they transfer energy back and forth at a rate equal to the natural frequency. In other words, if energy is to be efficiently pumped into both the mass and spring the energy source needs to feed the energy in at a rate equal to the natural frequency. Applying a force to the mass and spring is similar to pushing a child on swing, you need to push at the correct moment if you want the swing to get higher and higher. As in the case of the swing, the force applied does not necessarily have to be high to get large motions; the pushes just need to keep adding energy into the system.

The damper, instead of storing energy, dissipates energy. Since the damping force is proportional to the velocity, the more the motion, the more the damper dissipates the energy. Therefore a point will come when the energy dissipated by the damper will equal the energy being fed in by the force. At this point, the system has reached its maximum amplitude and will continue to vibrate at this level as long as the force applied stays the same. If no damping exists, there is nothing to dissipate the energy and therefore theoretically the motion will continue to grow on into infinity.

4.3 FORCED VIBRATION OF A SINGLE DEGREE-OF-FREEDOM SYSTEM:

We saw that when a system is given an initial input of energy, either in the form of an initial displacement or an initial velocity, and then released it will, under the right conditions, vibrate freely. If there is damping in the system, then the oscillations die away. If a system is given a continuous input of energy in the form of a continuously applied force or a continuously applied displacement, then the consequent vibration is called forced vibration. The energy input can overcome that dissipated by damping mechanisms and the oscillations are sustained.

We will consider two types of forced vibration. The first is where the ground to which the system is attached is itself undergoing a periodic displacement, such as the vibration of a building in an earthquake. The second is where a periodic force is applied to the mass, or object performing the motion; an example might be the forces exerted on the body of a car by the forces produced in the engine. The simplest form of periodic force or displacement is sinusoidal, so we will begin by considering forced vibration due to sinusoidal motion of the ground. In all real systems, energy will be dissipated, i.e. the system will be damped, but often the damping is very small. So let us first analyze systems in which there is no damping.

4.4 STEADY STATE RESPONSE DUE TO HARMONIC OSCILLATION:

Consider a spring-mass-damper system as shown in figure 4.1. The equation of motion of this system subjected to a harmonic force $F \sin \omega t$ can be given by

$$m\ddot{x} + kx + c\dot{x} = F \sin \omega t \tag{4.1}$$

where, m , k and c are the mass, spring stiffness and damping coefficient of the system, F is the amplitude of the force, w is the excitation frequency or driving frequency.

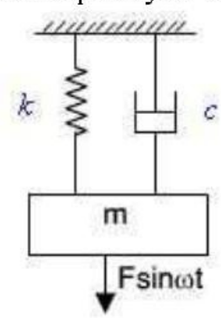


Figure 4.1 Harmonically excited system

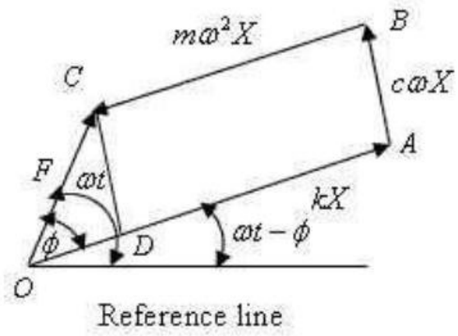


Figure 4.2: Force polygon

The steady state response of the system can be determined by solving equation(4.1) in many different ways. Here a simpler graphical method is used which will give physical understanding to this dynamic problem. From solution of differential equations it is known that the steady state solution (particular integral) will be of the form

$$x = X\sin(\omega t - \phi) \tag{4.2}$$

As each term of equation (4.1) represents a forcing term viz., first, second and third terms, represent the inertia force, spring force, and the damping forces. The term in the right hand side of equation (4.1) is the applied force. One may draw a close polygon as shown in figure 4.2 considering the equilibrium of the system under the action of these forces. Considering a reference line these forces can be presented as follows.

- Spring force = $kx = kX\sin(\omega t - \phi)$ (This force will make an angle $\omega t - \phi$ with the reference line, represented by line OA).
- Damping force = $c\dot{x} = c\omega X\cos(\omega t - \phi)$ (This force will be perpendicular to the spring force, represented by line AB).
- Inertia force = $m\ddot{x} = -m\omega^2 X\sin(\omega t - \phi)$ (this force is perpendicular to the damping force and is in opposite direction with the spring force and is represented by line BC) .
- Applied force = $F\sin\omega t$ which can be drawn at an angle ωt with respect to the reference line and is represented by line OC.

From equation (1), the resultant of the spring force, damping force and the inertia force will be the applied force, which is clearly shown in figure 4.2.

It may be noted that till now, we don't know about the magnitude of X and ϕ which can be easily computed from

Figure 2. Drawing a line CD parallel to AB, from the triangle OCD of Figure 2,

$$F^2 = (c\omega X)^2 + (kX - m\omega^2 X)^2$$

$$X = \frac{F}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$

$$\phi = \tan^{-1}\frac{c\omega}{k - m\omega^2}$$

$$X = \frac{F/k}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c\omega}{k}\right)^2}}$$

From the previous module of free-vibration it may be recalled that

- Natural frequency $\omega_n = \sqrt{\frac{k}{m}}$
- Critical damping $c_c = 2m\omega_n$
- Damping factor or damping ratio $\zeta = \frac{c}{c_c}$

$$\frac{c\omega}{k} = \frac{c}{c_c}\frac{c_c\omega}{k} = \zeta\frac{2m\omega_n\omega}{k} = 2\zeta\frac{\omega}{\omega_n}$$

 Hence,

$$\Rightarrow \quad \frac{Xk}{F} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right] + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}}$$

$$\tan\phi = \frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}$$

or

$$\phi = \tan^{-1}\left(\frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right)$$

As the ratio $\frac{F}{k}$ is the static deflection (X_0) of the spring, $\frac{Xk}{F} = \frac{X}{X_0}$ is known as the magnification factor or amplitude ratio of the system

4.5 FORCED VIBRATION WITH DAMPING:

In this section we will see the behaviour of the spring mass damper model when we add a harmonic force in the form below. A force of this type could, for example, be generated by a rotating imbalance.

$$F = F_0 \cos(2\pi f t).$$

If we again sum the forces on the mass we get the following ordinary differential equation:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(2\pi f t).$$

The steady state solution of this problem can be written as:

$$x(t) = X \cos(2\pi f t - \phi).$$

The result states that the mass will oscillate at the same frequency, f, of the applied force, but with a phase shift ϕ.

The amplitude of the vibration —X|| is defined by the following formula.

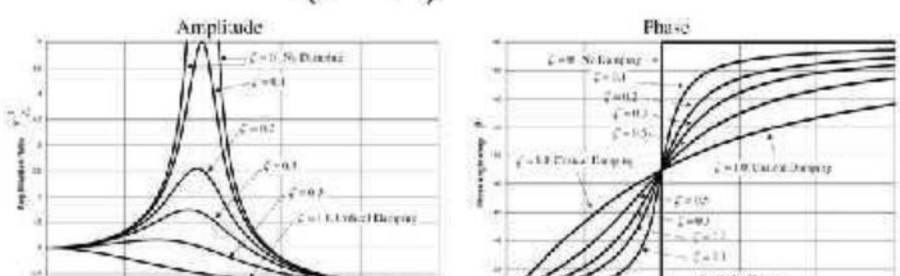
$$X = \frac{F_0}{k} \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}.$$

Where —r|| is defined as the ratio of the harmonic force frequency over the undamped natural frequency of the mass-spring-damper model.

$$r = \frac{f}{f_n}.$$

The phase shift , ϕ, is defined by the following formula.

$$\phi = \arctan\left(\frac{2\zeta r}{1 - r^2}\right).$$



The plot of these functions, called "the frequency response of the system", presents one of the most important features in forced vibration. In a lightly damped system when the forcing frequency nears the natural frequency ($r \approx 1$) the amplitude of the vibration can get extremely high. This phenomenon is called **resonance** (subsequently the natural frequency of a system is often referred to as the resonant frequency). In rotor bearing systems any rotational speed that excites a resonant frequency is referred to as a critical speed.

If resonance occurs in a mechanical system it can be very harmful – leading to eventual failure of the system. Consequently, one of the major reasons for vibration analysis is to predict when this type of resonance may occur and then to determine what steps to take to prevent it from occurring. As the amplitude plot shows, adding damping can significantly reduce the magnitude of the vibration. Also, the magnitude can be reduced if the natural frequency can be shifted away from the forcing frequency by changing the stiffness or mass of the system. If the system cannot be changed, perhaps the forcing frequency can be shifted (for example, changing the speed of the machine generating the force).

The following are some other points in regards to the forced vibration shown in the frequency response plots.

At a given frequency ratio, the amplitude of the vibration, X, is directly proportional to the amplitude of the force F_0 (e.g. if you double the force, the vibration doubles)

With little or no damping, the vibration is in phase with the forcing frequency when the frequency ratio $r < 1$ and 180 degrees out of phase when the frequency ratio $r > 1$

When $r \ll 1$ the amplitude is just the deflection of the spring under the static force F_0. This deflection is called the static deflection δ_{st}. Hence, when $r \ll 1$ the effects of the damper and the mass are minimal.

When $r \gg 1$ the amplitude of the vibration is actually less than the static deflection δ_{st}. In this region the force generated by the mass ($F = ma$) is dominating because the acceleration seen by the mass increases with the frequency. Since the deflection seen in the spring, X, is reduced in this region, the force transmitted by the spring ($F = kx$) to the base is reduced. Therefore the mass-spring-damper system is isolating the harmonic force from the mounting base – referred to as vibration isolation. Interestingly, more damping actually reduces the effects of vibration isolation when $r \gg 1$ because the damping force ($F = cv$) is also transmitted to the base.

4.6 ROTATING UNBALANCE FORCED VIBRATION:

One may find many rotating systems in industrial applications. The unbalanced force in such a system can be represented by a mass m with eccentricity e , which is rotating with angular velocity as shown in Figure 4.1.

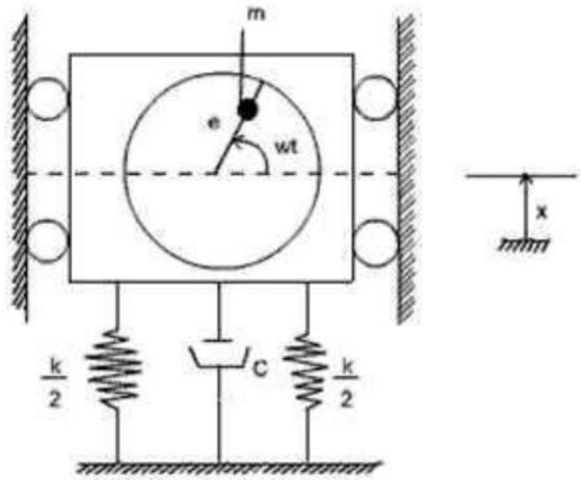


Figure 4.1 : Vibrating system with rotating unbalance

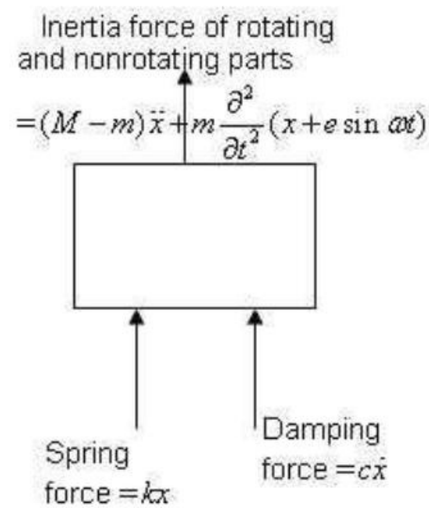


Figure 4.2. Freebody diagram of the system

Let x be the displacement of the nonrotating mass (M-m) from the static equilibrium position, then the displacement of the rotating mass m is $x + e\sin\omega t$

From the freebody diagram of the system shown in figure 4.2, the equation of motion is

$$(M-m)\ddot{x} + m\frac{\partial^2}{\partial t^2}(x + e\sin\omega t) + kx + c\dot{x} = 0 \tag{4.1}$$

or $$M\ddot{x} + k\dot{x} + cx = me\omega^2 \sin\omega t \tag{4.2}$$

This equation is same as equation (1) where F is replaced by $me\omega^2$. So from the force polygon as shown in figure 4.3

$$me\omega^2 = \sqrt{\left\{(-M\omega^2 + k)^2 + c\omega^2\right\}X^2} \tag{4.3}$$

or

$$X = \frac{me\omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} \tag{4.4}$$

or

$$\frac{X}{e} = \frac{\frac{m\omega}{M}}{\sqrt{\left(\frac{k}{M} - \omega^2\right)^2 + \left(\frac{c}{M}\omega\right)^2}} \tag{4.5}$$

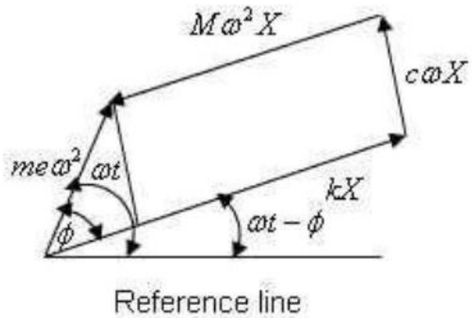


Figure 4.3: Force polygon

or

$$\frac{X}{e} = \frac{\frac{m\omega}{M}}{\sqrt{\left(\frac{k}{M} - \omega^2\right)^2 + \left(\frac{c}{M}\omega\right)^2}} \tag{4.6}$$

and

$$\tan\phi = \frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{4.7}$$

So the complete solution becomes

$$x(t) = x_1 e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi_1\right) + \frac{me\omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}\sin\left(\omega t - \phi\right) \tag{4.8}$$

4.7 VIBRATION ISOLATION AND TRANSMISSIBILITY:

When a machine is operating, it is subjected to several time varying forces because of which it tends to exhibit vibrations. In the process, some of these forces are transmitted to the foundation – which could undermine the life of the foundation and also affect the operation of any other machine on the same foundation. Hence it is of interest to minimize this force transmission. Similarly when a system is subjected to ground motion, part of the ground motion is transmitted to the system as we just discussed e.g., an automobile going on an uneven road; an instrument mounted on the vibrating surface of an aircraft etc. In these cases, we wish to minimize the motion transmitted from the ground to the system. Such considerations are used in the design of machine foundations and in order to understand some of the basic issues involved, we will study this problem based on the single d.o.f model discussed so far. we get the expression for force transmitted to the base as follows:

$$F_T = \sqrt{(kX_0)^2 + (c\Omega X_0)^2}$$

$$X_0 = X_e\sqrt{\frac{k^2 + (c\Omega)^2}{(k - (m\Omega)^2)^2 + (c\Omega)^2}}$$

4.7.1 Vibration Isolators:

Consider a vibrating machine; bolted to a rigid floor (Figure 2a).The force transmitted to the floor is equal to the force generated in the machine. The transmitted force can be decreased by adding a suspension and damping elements (often called vibration isolators) Figure 2b , or by adding what is called an inertia block, a large mass (usually a block of cast concrete), directly attached to the machine (Figure 2c).Another option is to add an additional level of mass (sometimes called a seismic mass, againa block ofcast concrete) and suspension(Figure2d).

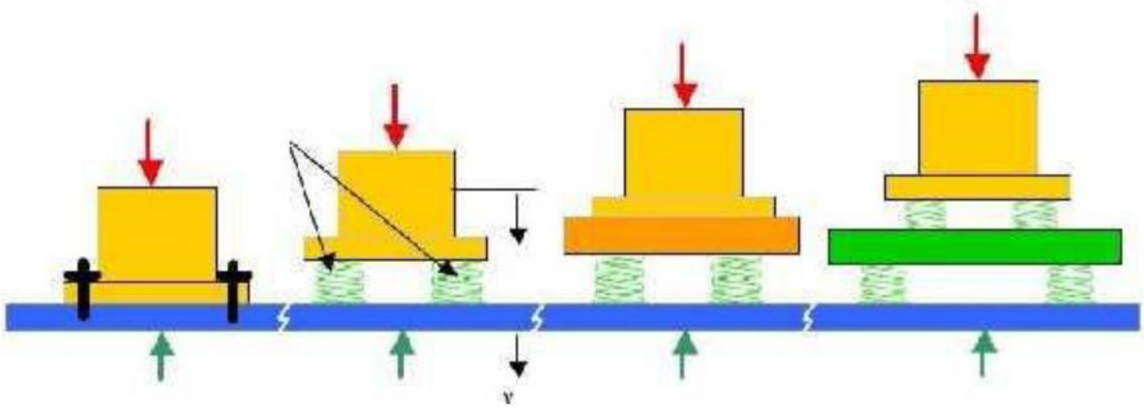


Figure 2.Vibration isolation systems: a) Machine bolted to a rigid foundation
b) Supported on isolation springs, rigid foundation c) machine attached to an inertial block. d) Supported on isolation springs, non-rigid foundation (such as a floor); or machine on isolation springs, seismic mass and second level of isolator springs

When oscillatory forces arise unavoidably in machines it is usually desired to prevent these forces from being transmitted to the surroundings. For example, some unbalanced forces are inevitable in a car engine, and it is uncomfortable if these are wholly transmitted to the car body. The usual solution is to mount the source of vibration on sprung supports. Vibration isolation is measured in terms of the motion or force transmitted to the foundation. The lesser the force or motion transmitted the greater the vibration isolation

Suppose that the foundation is effectively rigid and that only one direction of movement is effectively excited so that the system can be treated as having only one degree of freedom.

4.8 RESPONSE WITHOUT DAMPING:

The amplitude of the force transmitted to the foundations is Where ***k*** is the Stiffness of the support and ***x(t)*** is the displacement of the mass ***m***.

The governing equation can be determined by considering that the total forcing on the machine is equal to its mass multiplied by itsacceleration(Newton's second law)

The ratio (transmitted force amplitude) / (applied force amplitude) is called the **transmissibility**.

$$\text{Transmissibility} = \left|\frac{F_T}{F}\right| = \frac{1}{\left|1 - \frac{\omega^2}{{\omega_n}^2}\right|} = \frac{1}{\left|1 - \frac{f^2}{{f_n}^2}\right|}$$

The transmissibility can never be zero but will be less than 1 providing $\frac{\omega}{\omega_n} > \sqrt{2}$ or $\frac{f}{f_n} > \sqrt{2}$ otherwise it will be greater than 1.

4.9 SOLVED PROBLEMS

1. Derive the relation for the displacement of mass from the equilibrium position of the damped vibration system with harmonic forcing.

Consider a system consisting of spring, mass and damper as shown in Fig. 23.19. Let the system is acted upon by an external periodic (*i.e.* simple harmonic) disturbing force,

$$F_x = F \cos \omega . t$$

where F = Static force, and

ω = Angular velocity of the periodic disturbing force.

When the system is constrained to move in vertical guides, it has only one degree of freedom. Let at sometime t, the mass is displaced downwards through a distance x *from its mean position.*

The equation of motion may be written as,

$$m \times \frac{d^2x}{dt^2} = -c \times \frac{dx}{dt} - s.x + F \cos \omega t$$

or

$$m \times \frac{d^2x}{dt^2} + c \times \frac{dx}{dt} + s.x = F \cos \omega t$$

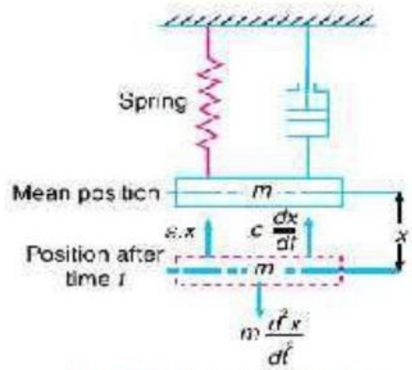


Fig. 23.19. Frequency of under damped forced vibrations.

This equation of motion may be solved either by differential equation method or by graphi-cal method as discussed below :

1. *Differential equation method*

The equation (*i*) is a differential equation of the second degree whose right hand side is some function in t. The solution of such type of differential equation consists of two parts ; one part is the complementary function and the second is particular integral. Therefore the solution may be written as

$$x = x_1 + x_2$$

where x_1 = Complementary function, and x_2 = Particular integral.

The complementary function is same as discussed in the previous article, *i.e.*

$x_1 = Ce^{-at}\cos(\omega_d t - \theta)$ . . . ***(ii)*** where C and θ are constants. Let us now find the value of particular integral as discussed below : Let the particular integral of equation ***(i)*** is given by

$$x_2 = B_1 \sin \omega.t + B_2 \cos \omega.t \qquad \text{. . (where } B_1 \text{ and } B_2 \text{ are constants)}$$

$$\therefore \qquad \frac{dx}{dt} = B_1.\omega \cos \omega.t - B_2.\omega \sin \omega.t$$

and

$$\frac{d^2x}{dt^2} = -B_1.\omega^2 \sin \omega.t - B_2.\omega^2 \cos \omega.t$$

Substituting these values in the given differential equation *(i)*, we get

$$m(-B_1.\omega^2 \sin \omega.t - B_2.\omega^2 \cos \omega.t) + c(B_1.\omega \cos \omega.t - B_2.\omega \sin \omega.t) + s(B_1 \sin \omega.t + B_2 \cos \omega.t)$$
$$= F \cos \omega.t$$

or

$$(-m.B_1.\omega^2 - c.\omega.B_2 + s.B_1)\sin \omega.t + (-m.\omega^2.B_2 + c.\omega.B_1 + s.B_2)\cos \omega.t$$
$$= F \cos \omega.t$$

or

$$\left[(s - m.\omega^2)B_1 - c.\omega.B_2\right]\sin \omega.t + \left[c.\omega.B_1 + (s - m.\omega^2)B_2\right]\cos \omega.t$$
$$= F \cos \omega.t + 0 \sin \omega.t$$

Comparing the coefficients of sin *ωt* and cos *ωt* on the left hand side and right hand side separately, we get

$$(s - m.\omega^2)B_1 - c.\omega.B_2 = 0 \qquad \text{. . . (iii)}$$

and

$$c.\omega.B_1 + (s - m.\omega^2)B_2 = F \qquad \text{. . . (iv)}$$

Now from equation *(iii)*

$$(s - m.\omega^2)B_1 = c.\omega.B_2$$

$$\therefore \qquad B_2 = \frac{s - m.\omega^2}{c.\omega} \times B_1 \qquad \text{. . . (v)}$$

Substituting the value of B_2 in equation *(iv)*

$$c.\omega.B_1 + \frac{(s - m.\omega^2)(s - m.\omega^2)}{c.\omega} \times B_1 = F$$

$$c^2.\omega^2.B_1 + (s - m.\omega^2)^2 B_1 = c.\omega.F$$

$$B_1\left[c^2.\omega^2 + (s - m.\omega^2)^2\right] = c.\omega.F$$

$$\therefore \qquad B_1 = \frac{c.\omega.F}{c^2.\omega^2 + (s - m.\omega^2)^2}$$

and $$B_2 = \frac{s - m\omega^2}{c.\omega} \times \frac{c.\omega.F}{c^2.\omega^2 + (s - m\omega^2)^2} \quad \text{... [From equation (v)]}$$

$$= \frac{F(s - m\omega^2)}{c^2.\omega^2 + (s - m\omega^2)^2}$$

$\therefore$ The particular integral of the differential equation (*i*) is

$$x_2 = B_1 \sin \omega.t + B_2 \cos \omega.t$$

$$= \frac{c.\omega.F}{c^2.\omega^2 + (s - m\omega^2)^2} \times \sin \omega t - \frac{F(s - m\omega^2)}{c^2.\omega^2 + (s - m\omega^2)^2} \times \cos \omega t$$

$$= \frac{F}{c^2.\omega^2 + (s - m\omega^2)^2} \left[c.\omega \sin \omega.t + (s - m\omega^2) \cos \omega.t \right] \quad \text{... (vi)}$$

Let $$c.\omega = X \sin \phi; \text{ and } s - m\omega^2 = X \cos \phi$$

$$\therefore \quad X = \sqrt{c^2.\omega^2 + (s - m\omega^2)^2} \quad \text{... (By squaring and adding)}$$

and $$\tan \phi = \frac{c.\omega}{s - m\omega^2} \quad \text{or} \quad \phi = \tan^{-1}\left(\frac{c.\omega}{s - m\omega^2}\right)$$

Now the equation (*vi*) may be written as

$$x_2 = \frac{F}{c^2.\omega^2 + (s - m.\omega^2)^2} \left[X \sin \phi \sin \omega t + X \cos \phi \cos \omega t \right]$$

$$= \frac{F.X}{c^2.\omega^2 + (s - m.\omega^2)^2} \times \cos(\omega t - \phi)$$

$$= \frac{F\sqrt{c^2.\omega^2 + (s - m\omega^2)^2}}{c^2.\omega^2 + (s - m\omega^2)^2} \times \cos(\omega t - \phi)$$

$$= \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi)$$

$\therefore$ The complete solution of the differential equation (*i*) becomes

$$x = x_1 + x_2$$

$$= C.e^{-at} \cos(\omega_d.t - \theta) + \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega t - \phi)$$

In actual practice, the value of the complementary function x_1 at any time t is much smaller as compared to particular integral x_2. Therefore, the displacement x, at any time t, is given by the particular integral x_2 only.

$$\therefore \quad x = \frac{F}{\sqrt{c^2.\omega^2 + (s - m\omega^2)^2}} \times \cos(\omega.t - \phi) \quad \text{... (vii)}$$

A little consideration will show that the frequency of forced vibration is equal to the angular velocity of the periodic force and the amplitude of the forced vibration is equal to the maximum displacement of vibration.

$\therefore$ Maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \quad \text{... (viii)}$$

This equation shows that motion is simple harmonic whose circular frequency is ω and the amplitude is $\frac{F}{\sqrt{c^2.\omega^2+(s-m.\omega^2)^2}}$.

2. A mass of 10 kg is suspended from one end of a helical spring, the other end being fixed. The stiffness of the spring is 10 N/mm. The viscous damping causes the amplitude to decrease to one-tenth of the initial value in four complete oscillations. If a periodic force of 150 cos 50 t N is applied at the mass in the vertical direction, find the amplitude of the forced vibrations. What is its value of resonance ?

Solution. Given : $m = 10$ kg ; $s = 10$ N/mm $= 10 \times 10^3$ N/m ; $x_5 = \frac{x_1}{10}$

Since the periodic force $F_x = F\cos \omega t = 150 \cos 50t$, therefore

Static force, $F = 150$ N

and angular velocity of the periodic disturbing force.

$$\omega = 50 \text{ rad/s}$$

We know that angular speed or natural circular frequency of free vibrations.

$$\omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{10\times10^3}{10}} = 31.6 \text{ rad/s}$$

Amplitude of the forced vibrations

Since the amplitude decreases to 1/10th of the initial value in four complete oscillations, therefore, the ratio of initial amplitude (x_1) to the final amplitude after four complete oscillations (x_5) is given by

$$\frac{x_1}{x_5} = \frac{x_1}{x_2}\times\frac{x_2}{x_3}\times\frac{x_3}{x_4}\times\frac{x_4}{x_5} = \left(\frac{x_1}{x_2}\right)^4 \quad \ldots \left(\because \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \frac{x_4}{x_5}\right)$$

$$\therefore \quad \frac{x_1}{x_2} = \left(\frac{x_1}{x_5}\right)^{1/4} = \left(\frac{x_1}{x_1/10}\right)^{1/4} = (10)^{1/4} = 1.78 \quad \ldots \left(x_5 = \frac{x_1}{10}\right)$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = a\times\frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$\log_e 1.78 = a\times\frac{2\pi}{\sqrt{(31.6)^2 - a^2}} \quad \text{or} \quad 0.576 = \frac{a\times 2\pi}{\sqrt{1000 - a^2}}$$

We know that amplitude of the forced vibrations,

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left[1 - \frac{\omega^2}{(\omega_n)^2}\right]^2}}$$

$$= \frac{0.015}{\sqrt{\frac{(57.74)^2(50)^2}{(10\times10^3)^2} + \left[1 - \left(\frac{50}{31.6}\right)^2\right]^2}} = \frac{0.015}{\sqrt{0.083 + 2.25}}$$

Squaring both sides and rearranging,

$$39.832\ a^2 = 332 \quad \text{or} \quad a^2 = 8.335 \quad \text{or} \quad a = 2.887$$

We know that $a = c/2m$ or $c = a \times 2m = 2.887 \times 2 \times 10 = 57.74$ N/m/s

and deflection of the system produced by the static force F,

$$x_o = F/s = 150/10 \times 10^3 = 0.015 \text{ m}$$

3. The mass of an electric motor is 120 kg and it runs at 1500 r.p.m. The armature mass is 35 kg and its C.G. lies 0.5 mm from the axis of rotation. The motor is mounted on five springs of negligible damping so that the force transmitted is one-eleventh of the impressed force. Assume that the mass of the motor is equally distributed among the five springs.

Determine : 1. stiffness of each spring; 2. dynamic force transmitted to the ba se at the operating speed; and 3. natural frequency of the system.

Solution. Given $m_1 = 120$ kg ; $m_2 = 35$ kg; $r = 0.5$ mm $= 5 \times 10^{-4}$ m; $\varepsilon = 1/11$; $N = 1500$ r.p.m. or $\omega = 2\pi \times 1500/60 = 157.1$ rad/s ;

1. Stiffness of each spring

Let s = Combined stiffness of the spring in N-m, and

ω_n = Natural circular frequency of vibration of the machine in rad/s.

$$= \frac{0.015}{1.53} = 9.8\times10^{-3} \text{ m} = 9.8 \text{ mm } \textbf{Ans.}$$

Amplitude of forced vibrations at resonance

We know that amplitude of forced vibrations at resonance,

$$x_{max} = x_0 \times \frac{s}{c.\omega_n} = 0.015 \times \frac{10\times10^3}{57.54\times31.6} = 0.0822 \text{ m} = 82.2 \text{ mm } \textbf{Ans.}$$

We know that transmissibility ratio (ε),

$$\frac{1}{11} = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{(\omega_n)^2}{\omega^2 - (\omega_n)^2} = \frac{(\omega_n)^2}{(157.1)^2 - (\omega_n)^2}$$

or $$(157.1)^2 - (\omega_n)^2 = 11(\omega_n)^2 \quad \text{or} \quad (\omega_n)^2 = 2057 \quad \text{or} \quad \omega_n = 45.35 \text{ rad/s}$$

4. What do you understand by transmissibility? Describe the method of finding the transmissibility ratio from unbalanced machine supported with foundation.

2. Dynamic force transmitted to the base at the operating speed (i.e. 1500 r.p.m. or 157.1 rad/s)

We

We know that maximum unbalanced force on the motor due to armature mass,

$$F = m_2 \omega^2 . r = 35 (157.1)^2 5 \times 10^{-4} = 432 \text{ N}$$

Sin

∴ Dynamic force transmitted to the base,

$$F_1 = \varepsilon . F = \frac{1}{11} \times 432 = 39.27 \text{ N} \quad \textbf{Ans}$$

3. Natural frequency of the system

We have calculated above that the natural frequency of the system,

$$\omega_n = 45.35 \text{ rad/s} \quad \textbf{Ans.}$$

A little consideration will show that when an unbalanced machine is installed on the foundation, it produces vibration in the foundation. In order to prevent these vibrations or to minimize the transmission of forces to the foundation, the machines are mounted on springs and dampers or on some vibration isolating material, as shown in Fig. 23.22. The arrangement is assumed to have one degree of freedom, *i.e.* it can move up and down only.

It may be noted that when a periodic (*i.e.* simple harmonic) disturbing force $F \cos \omega t$ is applied to a machine of mass m supported by a spring of stiffness s, then the force is transmitted by means of the spring and the damper or dashpot to the fixed support or foundation.

The ratio of the force transmitted (F_T) to the force applied (F) is known as the ***isolation factor*** or ***transmissibility ratio*** of the spring support.

We have discussed above that the force transmitted to the foundation consists of the following two forces :

1.Spring force or elastic force which is equal to $s. x_{max}$, and

2.Damping force which is equal to $c. \omega .x_{max}$.

Since these two forces are perpendicular to one another, as shown in Fig.23.23, therefore the force transmitted,

$$F_T = \sqrt{(s.x_{max})^2 + (c.\omega.x_{max})^2}$$

$$= x_{max} \sqrt{s^2 + c^2 . \omega^2}$$

∴ Transmissibility ratio,

$$\varepsilon = \frac{F_T}{F} = \frac{x_{max} \sqrt{s^2 + c^2 . \omega^2}}{F}$$

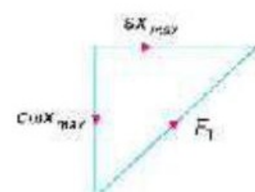


Fig. 23.23

We know that

$$x_{max} = x_o \times D = \frac{F}{s} \times D \qquad \ldots \left(\because x_o = \frac{F}{s} \right)$$

$$\therefore \qquad \varepsilon = \frac{D}{s}\sqrt{s^2 + c^2.\omega^2} = D\sqrt{1 + \frac{c^2.\omega^2}{s^2}}$$

$$= D\sqrt{1 + \left(\frac{2c}{c_c} \times \frac{\omega}{\omega_n} \right)^2} \qquad \ldots \left(\because \frac{c.\omega}{s} = \frac{2c}{c_c} \times \frac{\omega}{\omega_n} \right)$$

magnification factor,

$$D = \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n} \right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2} \right)^2}}$$

$$\therefore \qquad \varepsilon = \frac{\sqrt{1 + \left(\frac{2c.\omega}{c_c.\omega_n} \right)^2}}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n} \right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2} \right)^2}}$$

When the damper is not provided, then $c = 0$, and

$$\varepsilon = \frac{1}{1 - (\omega/\omega_n)^2}$$

From above, we see that when $\omega/\omega_n > 1$, ε is negative. This means that there is a phase difference of 180° between the transmitted force and the disturbing force $(F \cos \omega t)$. The value of ω/ω_n must be greater than $\sqrt{2}$ if ε is to be less than 1 and it is the numerical value of ε, independent of any phase difference between the forces that may exist which is important. It is therefore more convenient to use equation *(ii)* in the following form, *i.e.*

$$\varepsilon = \frac{1}{(\omega/\omega_n)^2 - 1} \quad \ldots (iii)$$

Fig. 23.24 is the graph for different values of damping factor c/c_c to show the variation of transmissibility ratio (ε) against the ratio ω/ω_n

1. When $\omega/\omega_n = \sqrt{2}$, then all the curves pass through the point $\varepsilon = 1$ for all values of damping factor c/c_c.

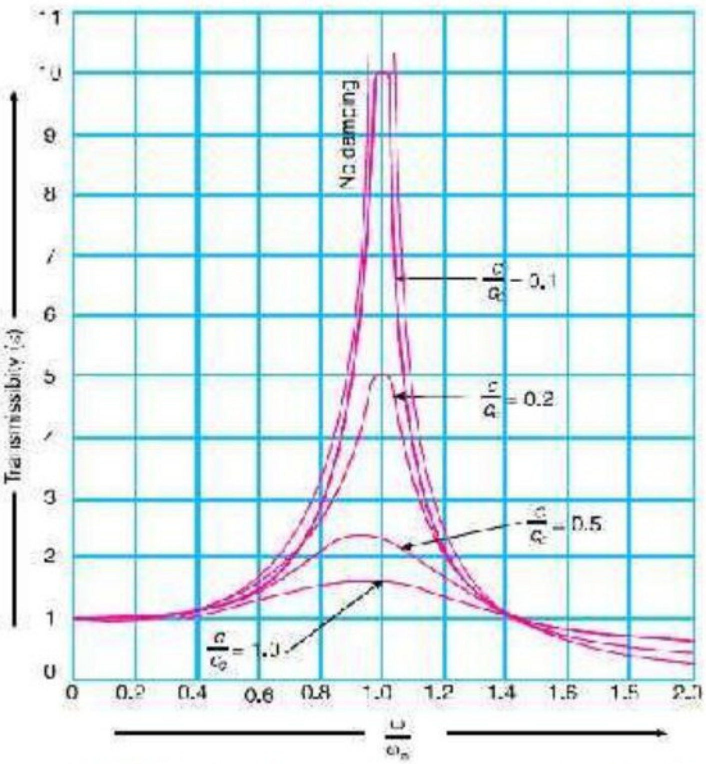


Fig. 23.24. Graph showing the variation of transmissibility ratio.

2. When $\omega/\omega_n < \sqrt{2}$, then $\varepsilon > 1$ for all values of damping factor c/c_c. This means that the force transmitted to the foundation through elastic support is greater than the force applied.

3. When $\omega/\omega_n > \sqrt{2}$, then $\varepsilon < 1$ for all values of damping factor c/c_c. This shows that the force transmitted through elastic support is less than the applied force. Thus vibration isolation is possible only in the range of $\omega/\omega_n > \sqrt{2}$.

5. A machine has a mass of 100 kg and unbalanced reciprocating parts of mass 2 kg which move through a vertical stroke of 80 mm with simple harmonic motion. The machine is mounted on four springs, symmetrically arranged with respect to centre of mass, in such a way that the machine has one degree of freedom and can undergo vertical displacements only.

Neglecting damping, calculate the combined stiffness of the spring in order that the force transmitted to the foundation is 1 / 25 th of the applied force, when the speed of rotation of ma-chine crank shaft is 1000 r.p.m.

When the machine is actually supported on the springs, it is found that the damping reduces the amplitude of successive free vibrations by 25%. Find : 1. the force transmitted to foundation at 1000 r.p.m., 2. the force transmitted to the foundation at reso nance, and 3. the amplitude of the forced vibration of the machine at resonance.

Solution. Given $m_1 = 100$ kg ; $m_2 = 2$ kg ; $l = 80$ mm $= 0.08$ m ; $\varepsilon = 1/25$; $N = 1000$ r.p.m. or $\omega = 2\pi \times 1000/60 = 104.7$ rad/s

Combined stiffness of springs

Let s = Combined stiffness of springs in N/m, and

ω_n = Natural circular frequency of vibration of the machine in rad/s

We know that transmissibility ratio (ε),

$$\frac{1}{25} = \frac{1}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} = \frac{(\omega_n)^2}{\omega^2 - (\omega_n)^2} = \frac{(\omega_n)^2}{(104.7)^2 - (\omega_n)^2}$$

or $(104.7)^2 - (\omega_n)^2 = 25(\omega_n)^2$ or $(\omega_n)^2 = 421.6$ or $\omega_n = 20.5$ rad/s

We know that $\omega_n = \sqrt{s/m_1}$

$\therefore$ $s = m_1 (\omega_n)^2 = 100 \times 421.6 = 42\,160$ N/m **Ans.**

1. Force transmitted to the foundation at 1000 r.p.m.

Let F_T = Force transmitted, and

x_1 = Initial amplitude of vibration.

Since the damping reduces the amplitude of successive free vibrations by 25%, therefore final amplitude of vibration,

$$x_2 = 0.75\, x_1$$

We know that

$$\log_e\left(\frac{x_1}{x_2}\right) = \frac{a \times 2\pi}{\sqrt{(\omega_n)^2 - a^2}} \quad \text{or} \quad \log_e\left(\frac{x_1}{0.75 x_1}\right) = \frac{a \times 2\pi}{\sqrt{421.6 - a^2}}$$

Squaring both sides,

$$(0.2877)^2 = \frac{a^2 \times 4\pi^2}{421.6 - a^2} \quad \text{or} \quad 0.083 = \frac{39.5\, a^2}{421.6 - a^2}$$

$$\ldots\left[\because \log_e\left(\frac{1}{0.75}\right) = \log_e 1.333 = 0.2877\right]$$

$$35 - 0.083\, a^2 = 39.5\, a^2 \quad \text{or} \quad a^2 = 0.884 \quad \text{or} \quad a = 0.94$$

We know that damping coefficient or damping force per unit velocity,

$$c = a \times 2m_1 = 0.94 \times 2 \times 100 = 188 \text{ N/m/s}$$

and critical damping coefficient,

$$c_c = 2m.\omega_n = 2 \times 100 \times 20.5 = 4100 \text{ N/m/s}$$

$\therefore$ Actual value of transmissibility ratio,

$$\varepsilon = \frac{\sqrt{1+\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2}}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2+\left(1-\frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$= \frac{\sqrt{1+\left(\frac{2\times188\times104.7}{4100\times20.5}\right)^2}}{\sqrt{\left(\frac{2\times188\times104.7}{4100\times20.5}\right)^2+\left[1-\left(\frac{104.7}{20.5}\right)^2\right]^2}} = \frac{\sqrt{1+0.22}}{\sqrt{0.22+629}}$$

$$= \frac{1.104}{25.08} = 0.044$$

We know that the maximum unbalanced force on the machine due to reciprocating parts,

$$F = m_2.\omega^2.r = 2(104.7)^2\,(0.08/2) = 877 \text{ N} \qquad \ldots (\because\ r = l/2)$$

∴ Force transmitted to the foundation,

$$F_T = \varepsilon.F = 0.044\times877 = 38.6 \text{ N } \textbf{Ans.} \qquad \ldots (\because\ \varepsilon = F_T / F)$$

2. Force transmitted to the foundation at resonance

Since at resonance, $\omega = \omega_n$, therefore transmissibility ratio,

$$\varepsilon = \frac{\sqrt{1+\left(\frac{2c}{c_c}\right)^2}}{\sqrt{\left(\frac{2c}{c_c}\right)^2}} = \frac{\sqrt{1+\left(\frac{2\times188}{4100}\right)^2}}{\sqrt{\left(\frac{2\times188}{4100}\right)^2}} = \frac{\sqrt{1+0.0084}}{0.092} = 10.92$$

3. Amplitude of the forced vibration of the machine at resonance

We know that amplitude of the forced vibration at resonance

$$= \frac{\text{Force transmitted at resonance}}{\text{Combined stiffness}} = \frac{367}{42\,160} = 8.7\times10^{-3} \text{ m}$$

$$= 8.7 \text{ mm } \textbf{Ans.}$$

6.(i) Derive the relation for magnification factor in case of forced vibration.

It is the ratio of *maximum displacement of the forced vibration (x_{max}) to the deflection due to the static force $F(x_o)$*. We have proved in the previous article that the maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{x_o}{\sqrt{\frac{c^2.\omega^2}{s^2}+\left(1-\frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

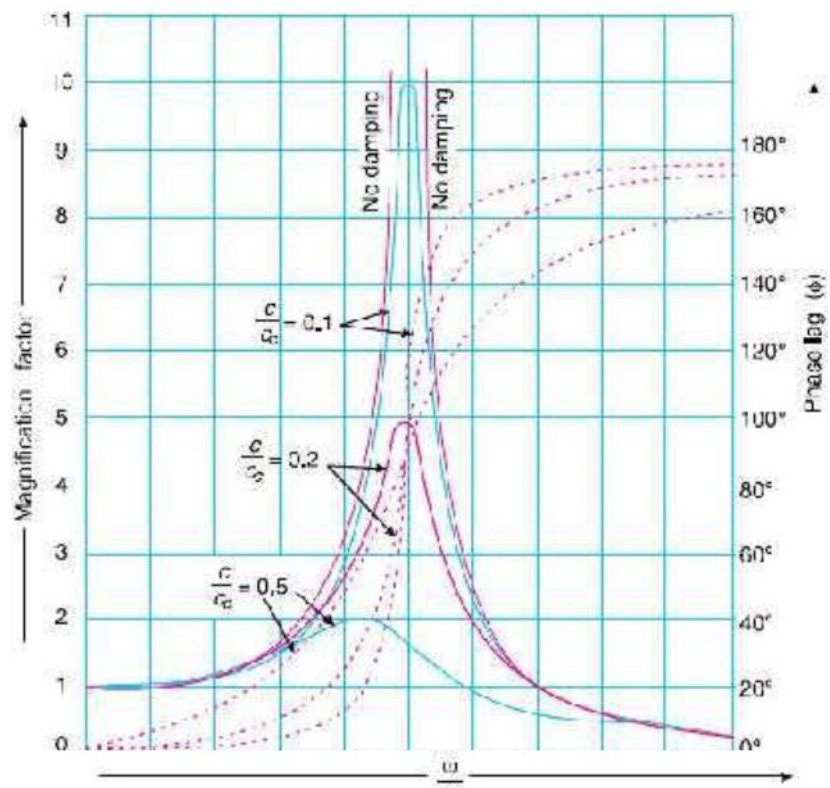


Fig. 23.21. Relationship between magnification factor and phase angle for different values of ω/ω_n.

∴ Magnification factor or dynamic magnifier,

$$D = \frac{x_{max}}{x_o} = \frac{1}{\sqrt{\frac{c^2.\omega^2}{s^2} + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}} \quad ...(i)$$

$$= \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$

$$\ldots \left[\because \frac{c.\omega}{s} = \frac{2c.\omega}{2m \times \frac{s}{m}} = \frac{2c.\omega}{2m(\omega_n)^2} = \frac{2c.\omega}{c_c.\omega_n}\right]$$

The magnification factor or dynamic magnifier gives the factor by which the static deflection produced by a force F (*i.e.* x_o) must be multiplied in order to obtain the maximum amplitude of the forced vibration (*i.e.* x_{max}) by the harmonic force $F \cos \omega t$

$$\therefore \quad x_{max} = x_o \times D$$

Fig. 23.21 shows the relationship between the magnification factor (D) and phase angle ϕ for different value of ω/ω_n and for values of damping factor c/c_c = 0.1, 0.2 and 0.5.

6.(ii) A single cylinder vertical petrol engine of total mass 300 kg is mounted upon a steel chassis frame and causes a vertical static deflection of 2 mm. The reciprocating parts of the engine has a mass of 20 kg and move through a vertical stroke of 150 mm with simple

harmonic motion. A dashpot is provided whose damping resistance is directly proportional to the velocity and amounts to 1.5 kN per metre per second.

Considering that the steady state of vibration is reached ; determine : 1. the amplitude of forced vibrations, when the driving shaft of the engine rotates at 480 r.p.m., and 2. the speed of the driving shaft at which resonance will occur.

Solution : Given. m = 300 kg; δ = 2 mm = 2 × 10 ^{3} m ; m_1 = 20 kg ; l = 150 mm = 0.15 m ; c = 1.5 kN/m/s = 1500 N/m/s ; N = 480 r.p.m. or ω □ 2□□ 480 / 60 = 50.3 rad/s

1. Amplitude of the forced vibrations

We know that stiffness of the frame,

$$s = m.g / \delta = 300 \times 9.81/2 \times 10^{-3} = 1.47 \times 10^{6} \text{ N/m}$$

Since the length of stroke (l) = 150 mm = 0.15 m, therefore radius of crank,

$$r = l / 2 = 0.15 / 2 = 0.075 \text{ m}$$

We know that the centrifugal force due to the reciprocating parts or the static force,

$$F = m_1 \omega^2 . r = 20 \, (50.3)^2 \, 0.075 = 3795 \text{ N}$$

∴ Amplitude of the forced vibration (maximum),

$$x_{max} = \frac{F}{\sqrt{c^2 \omega^2 + (s - m.\omega^2)^2}}$$

$$= \frac{3795}{\sqrt{(1500)^2 (50.3)^2 + [1.47 \times 10^6 - 300 (50.3)^2]^2}}$$

$$= \frac{3795}{\sqrt{5.7 \times 10^9 + 500 \times 10^9}} = \frac{3795}{710 \times 10^3} = 5.3 \times 10^{-3} \text{ m}$$

= 5.3 mm **Ans.**

2. Speed of the driving shaft at which the resonance occurs

Let N = Speed of the driving shaft at which the resonance occurs in r.p.m.

We know that the angular speed at which the resonance occurs,

$$\omega = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{1.47 \times 10^6}{300}} = 70 \text{ rad/s}$$

∴ $N = \omega \times 60 / 2\pi = 70 \times 60 / 2\pi$ = 668.4 r.p.m. **Ans.**

4.10 REVIEW QUESTIONS

1. Explain the term ‘dynamic magnifier’
2. What are the materials used for vibration isolation?
3. In vibration isolation system, if $\omega / \omega_n > 1$, then the phase difference between the transmitted force and the disturbing force is ?
4. In under damped vibrating system, if x_1 and x_2 are the successive values of the amplitude on the ________________
5. same side of the mean position, then the logarithmic decrement is equal to