LAPLACE TRANSFORM OF DERIVATIVES AND INTEGRALS

Problems using the formula

$$\boldsymbol{L[tf(t)] = \frac{-d}{ds} L[f(t)]}$$

Example: Find the Laplace transform for $tsin4t$

Solution:

$$L[tsin4t] = \frac{-d}{ds} L[tsin4t]$$

$$= \frac{-d}{ds}\left[\frac{4}{s^2+4}\right]$$

$$= \frac{-[(s^2+16)0-4(2s)]}{(s^2+16)^2}$$

$$\therefore L[tsin4t] = \frac{8s}{(s^2+16)^2}$$

Example: Find $L[tsin^2t]$

Solution:

$$L[tsin^2t] = \frac{-d}{ds} L[\sin^2 t] = \frac{-d}{ds} L\left[\frac{(1-cos2t)}{2}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}[L(1) - L(cos2t)]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{4}{s(s^2+4)}\right]$$

$$= -\frac{4}{2}\frac{d}{ds}\left[\frac{1}{s(s^2+4)}\right]$$

$$= -2\left[\frac{0-(3s^2+4)}{(s^3+4s)^2}\right]$$

$$\therefore L[tsin^2t] = \frac{2(3s^2+4)}{(s^3+4s)^2}$$

Example: Find the Laplace transform for $f(t) = sinat - atcosat$

Solution:

$$L[sinat - atcosat] = \mathrm{L}(\sin at) - a\,\mathrm{L(tcosat)}$$

$$= \frac{a}{s^2+a^2} - a\left(\frac{-d}{ds}L[cosat]\right)$$

$$= \frac{a}{s^2+a^2} + a\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{(s^2+a^2)1-s(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{s^2+a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{a(s^2+a^2)+a(a^2-s^2)}{(s^2+a^2)^2}$$

$$= \frac{as^2+a^3+a^3-as^2)}{(s^2+a^2)^2}$$

$$\therefore L[sinat - atcosat] = \frac{2a^3}{(s^2+a^2)^2}$$

Example: Find the Laplace transform for the following

(i) $te^{-3t}sin2t$ (ii) $te^{-t}cosat$ 9iii) $tsinhtcos2t$

Solution:

(i) $L[te^{-3t}sin2t] = L[tsin2t]_{s\to s+3} = \frac{-d}{ds}L[sin2t]_{s\to s+3}$

$$= \frac{-d}{ds}\left(\frac{2}{s^2+2^2}\right)_{s\to s+3}$$

$$= \left[\frac{(s^2+4)0-2(2s)}{(s^2+4)^2}\right]_{s\to s+3}$$

$$= \left[\frac{4s}{(s^2+4)^2}\right]_{s\to s+3}$$

$$\therefore L[te^{-3t}sin2t] = \frac{4(s+3)}{((s+3)^2+4\ )^2}$$

(ii) $L[te^{-t}cosat] = L[tcosat]_{s\to s+1} = \frac{-d}{ds}L[cosat]_{s\to s+1}$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+a^2}\right)_{s\to s+1}$$

$$= -\left[\frac{(s^2+a^2)1-s(2s)}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$= \left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$\therefore L[te^{-t}cosat] = \frac{(s+1)^2-a^2}{((s+1)^2+a^2\ )^2}$$

(iii) $L[tsinhtcos2t]$

$$L[tsinhtcos2t] = L\left[t\left(\frac{e^t-e^{-t}}{2}\right)cos2t\right]$$

$$= \frac{1}{2}[L(te^tcos2t) - L(te^{-t}cos2t)]$$

$$= \frac{1}{2}\left[\frac{-d}{ds}L[cos2t]_{s\to s-1} + \frac{d}{ds}L[cos2t]_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[\frac{-d}{ds}\left(\frac{s}{s^2+4}\right)_{s\to s-1} + \frac{d}{ds}\left(\frac{s}{s^2+4}\right)_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[-\left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s\to s-1} + \left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[-\left[\frac{4-s^2}{(s^2+4)^2}\right]_{s\to s-1} + \left[\frac{4-s^2}{(s^2+4)^2}\right]_{s\to s+1}\right]$$

$\therefore L[tsinhtcos2t] = \frac{1}{2}\left[\frac{(s-1)^2-4}{((s-1)^2+4)^2} + \frac{4-(s+1)^2}{((s+1)^2+4)^2}\right]$

Problems using the formula

$$\boldsymbol{L[t^2 f(t)] = \frac{d^2}{ds^2} L[f(t)]}$$

Example: Find the Laplace transform for (i) $\boldsymbol{t^2 sint}$ **(ii)** $\boldsymbol{t^2 cos2t}$

Solution:

(i) $L[t^2 sint] = \frac{d^2}{ds^2} L[sint]$

$= \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right]$

$= \frac{d}{ds}\left(\frac{[(s^2+1)0-1(2s)]}{(s^2+1)^2}\right)$

$= \frac{d}{ds}\left(\frac{-2s}{(s^2+1)^2}\right)$

$= -2\frac{d}{ds}\left(\frac{s}{(s^2+1)^2}\right)$

$= \frac{-2\left[(s^2+1)^2(1)-s(2)(s^2+1)(2s)\right]}{(s^2+1)^4}$

$= \frac{-2(s^2+1)[(s^2+1)-4s^2]}{(s^2+1)^4}$

$= \frac{-2[1-3s^2]}{(s^2+1)^3}$

$\therefore L[t^2 sint] = \frac{6s^2-2}{(s^2+1)^3}$

(ii) $L[t^2 cos2t] = \frac{d^2}{ds^2} L[cos2t]$

$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$

$= \frac{d}{ds}\left(\frac{[(s^2+4)1-s(2s)]}{(s^2+4)^2}\right)$

$= \frac{d}{ds}\left(\frac{4-s^2}{(s^2+4)^2}\right)$

$= \frac{\left[(s^2+4)^2(-2s)-(4-s^2)2(s^2+4)(2s)\right]}{(s^2+4)^4}$

$= \frac{2s(s^2+4)[(s^2+4)(-1)-(4-s^2)2]}{(s^2+4)^4}$

$= \frac{2s[s^2-12]}{(s^2+4)^3}$

$\therefore L[t^2 \cos 2t] = \frac{2s[s^2-12]}{(s^2+4)^3}$

Example: Find the Laplace transform for (i) $t^2e^{-2t}cost$ (ii) $t^2e^{4t}sin3t$

Solution:

(i) $L[t^2e^{-2t}cost] = L[t^2cost]_{s\to s+2} = \frac{d^2}{ds^2}L[cost]_{s\to s+2}$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)_{s\to s+2}$$

$$= \frac{d}{ds}\left[\frac{(s^2+1)1-s(2s)}{(s^2+1)^2}\right]_{s\to s+2}$$

$$= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right]_{s\to s+2}$$

$$= \left[\frac{\left[(s^2+1)^2(-2s)-(1-s^2)2(s^2+1)(2s)\right]}{(s^2+1)^4}\right]_{s\to s+2}$$

$$= (s^2+1)\left[\frac{\left[(s^2+1)(-2s)-4s(1-s^2)\right]}{(s^2+1)^4}\right]_{s\to s+2}$$

$$= \left[\frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}\right]_{s\to s+2}$$

$$= \left[\frac{2s^3-6s}{(s^2+1)^3}\right]_{s\to s+2}$$

$$\therefore \quad L[t^2e^{-2t}cost] = \frac{2(s+2)^3-6(s+2)}{((s+2)^2+1)^3}$$

(ii) $L[t^2e^{4t}sin3t] = L[t^2sin3t]_{s\to s-4} = \frac{d^2}{ds^2}L[sin3t]_{s\to s-4}$

$$= \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)_{s\to s-4}$$

$$= \frac{d}{ds}\left[\frac{(s^2+9)0-3(2s)}{(s^2+9)^2}\right]_{s\to s-4}$$

$$= \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right]_{s\to s-4} = -6\frac{d}{ds}\left[\frac{s}{(s^2+9)^2}\right]_{s\to s-4}$$

$$= -6\left[\frac{\left[(s^2+9)^2(1)-(s)2(s^2+9)(2s)\right]}{(s^2+9)^4}\right]_{s\to s-4}$$

$$= -6(s^2+9)\left[\frac{\left[(s^2+9)-4s^2\right]}{(s^2+9)^4}\right]_{s\to s-4}$$

$$= -6\left[\frac{9-3s^2}{(s^2+9)^3}\right]_{s\to s-4}$$

$$= \left[\frac{18s^2-54}{(s^2+9)^3}\right]_{s\to s-4}$$

$$\therefore \quad L[t^2e^{4t}sin3t] = \frac{18(s-4)^2-54}{((s-4)^2+9)^3}$$

Problems using the formula

$$\boldsymbol{L\left[\frac{f(t)}{t}\right] = \int_s^\infty L[f(t)]ds}$$

This formula is valid if $\lim_{t\to 0}\frac{f(t)}{t}$ is finite.

The following formula is very useful in this section

$$\int \frac{ds}{s} = logs$$

$$\int \frac{ds}{s+a} = \log(s+a)$$

$$\int \frac{s\,ds}{s^2+a^2} = \frac{1}{2}\log(s^2+a^2)$$

$$\int \frac{a\,ds}{s^2+a^2} = \tan^{-1}\frac{s}{a}$$

Example: Find $\boldsymbol{L\left[\frac{cosat}{t}\right]}$

Solution:

$$\lim_{t\to 0}\frac{cosat}{t} = \frac{cosa(0)}{0} = \frac{1}{0} = \infty$$

∴ Laplace transform does not exists.

Example: Find $\boldsymbol{L\left[\frac{sin^3t}{t}\right]}$

Solution:

$$\boldsymbol{\frac{sin^3t}{t} = \frac{3sint-sin3t}{4t}}$$

$$\lim_{t\to 0}\boldsymbol{\frac{sin^3t}{t}} = \lim_{t\to 0}\boldsymbol{\frac{3sint-sin3t}{4t}}$$

$$= \frac{0-0}{0} = \frac{0}{0} \quad \text{(by applying L−Hospital rule)}$$

$$= \lim_{t\to 0}\frac{3sint-sin3t}{4t} = 0$$

Hence Laplace transform exists

$$L\left[\frac{sin^3t}{t}\right] = L\left[\frac{3sint-sin3t}{4t}\right]$$

$$= \frac{1}{4}\int_s^\infty L[(3sint - sin3t)]ds$$

$$= \frac{1}{4}\int_s^\infty \left(3\frac{1}{s^2+1} - \frac{3}{s^2+9}\right)ds$$

$$= \frac{1}{4}\left[3\tan^{-1}s - \tan^{-1}\frac{s}{3}\right]_s^\infty$$

$$= \frac{1}{4}\left[3(\tan^{-1}\infty - \tan^{-1}s) - \left(\tan^{-1}\infty - \tan^{-1}\frac{s}{3}\right)\right]$$

$$= \frac{1}{4}\left[(\frac{\pi}{2} - \tan^{-1}s) - \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right)\ \right]$$

$$= \frac{1}{4}\left[\cot^{-1}s - cot^{-1}\frac{s}{3}\right]$$

Example: Find $L\left[e^{-2t}\frac{sin2tcos3t}{t}\right]$

Solution:

$$L\left[e^{-2t}\frac{sin2tcos3t}{t}\right] = L\left[\frac{sin2tcos3t}{t}\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty L(\sin(3t+2t) - \sin(3t-2t))ds\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty L((\sin 5t) - L(\sin t))ds\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty \left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right]\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left[\tan^{-1}\frac{s}{5} - \tan^{-1}s\right]_s^\infty\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left[\left(\tan^{-1}\infty - \tan^{-1}\frac{s}{5}\right) - (\tan^{-1}\infty - \tan^{-1}s)\right]\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{5}\right) - \left(\frac{\pi}{2} - \tan^{-1}s\right)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\cot^{-1}\frac{s}{5} - cot^{-1}s\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\cot^{-1}\frac{(s+2)}{5} - cot^{-1}(s+2)\right]$$

Problems using $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$

Example: Find the Laplace transform for (i) $\int_0^t e^{-2t}dt$ **(ii)** $\int_0^t cos2tdt$ **(iii)** $\int_0^t tsin3tdt$ **(iv)** $t\int_0^t costdt$

Solution:

(i) $L\left[\int_0^t e^{-2t}dt\right] = \frac{1}{s}L[e^{-2t}] = \frac{1}{s}\left(\frac{1}{s+2}\right)$

$$\therefore L\left[\int_0^t e^{-2t}dt\right] = \frac{1}{s(s+2)}$$

(ii) $L\left[\int_0^t cos2tdt\right] = \frac{1}{s}L[cos2t] = \frac{1}{s}\left(\frac{s}{s^2+4}\right)$

$$\therefore L\left[\int_0^t cos2tdt\right] = \frac{1}{s^2+4}$$

(iii) $L\left[\int_0^t tsin3tdt\right] = \frac{1}{s}L[tsin3t]$

$$= \frac{1}{s}\left[\frac{-d}{ds}[L[sin3t]]\right]$$

$$= \frac{-1}{s}\left[\frac{d}{ds}\left[\frac{3}{s^2+9}\right]\right]$$

$$= \frac{-1}{s}\left[\frac{-6s}{(s^2+9)^2}\right]$$

$$\therefore L\left[\int_0^t tsin3tdt\right] = \frac{6}{(s^2+9)^2}$$

(iv) $L\left[t\int_0^t costdt\right] = \frac{-d}{ds}L\left[\int_0^t costdt\right]$

$$= \frac{-d}{ds}\left[\frac{1}{s}\left(\frac{s}{s^2+1}\right)\right]$$

$$= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= -\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$\therefore L\left[\int_0^t tsin3tdt\right] = \frac{2s}{(s^2+1)^2}$$

Example: Find the Laplace transform for $e^{-t}\int_0^t tcos4tdt$

Solution:

$$L\left[e^{-t}\int_0^t tcos4tdt\right] = L\left[\int_0^t tcos4tdt\right]_{s\to s+1} = \left[\frac{-1}{s}\frac{d}{ds}L(cos4t)\right]_{s\to s+1}$$

$$= -\left(\frac{1}{s}\frac{d}{ds}\frac{s}{s^2+16}\right)_{s\to s+1}$$

$$= \left[\frac{-1}{s}\ \frac{(s^2+16)1-s(2s)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{(s^2+16-2s^2)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{-1}{s}\frac{(-s^2+16)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\frac{(s^2-16)}{(s^2+16)^2}\right]_{s\to s+1}$$

$$\therefore L\left[e^{-t}\int_0^t tcos4tdt\right] = \frac{1}{s+1}\left[\frac{(s+1)^2-16}{((s+1)^2+16\ )^2}\right]$$

Example: Find the Laplace transform of $e^{-t}\int_0^t \frac{sint}{t}dt$

Solution:

$$L\left[e^{-t}\int_0^t \frac{sint}{t}dt\right] = L\left[\int_0^t \frac{sint}{t}dt\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}L\left(\frac{sint}{t}\right)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\int_s^\infty L(sint)ds\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\int_s^\infty \frac{1}{s^2+1}\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}[\tan^{-1}s\,]_s^\infty\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}(\tan^{-1}\infty - \tan^{-1}s)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}(\frac{\pi}{2} - \tan^{-1}s)\right]_{s\to s+1}$$

$$= \left[\frac{1}{s}\cot^{-1} s\right]_{s\to s+1}$$

$$\therefore L\left[e^{-t}\int_0^t \frac{sint}{t}dt\right] = \frac{1}{s+1}cot^{-1}(s+1)$$

Evaluation of integrals using Laplace transform

Note: (i) $\int_0^\infty f(t)e^{-st}dt = L[f(t)]$

(ii) $\int_0^\infty f(t)e^{-at}dt = \left[L[f(t)]\right]_{s=a}$

(iii) $\int_0^\infty f(t)dt = \left[L[f(t)]\right]_{s=0}$

Example: Find the values of the following integrals using Laplace transforms:

(i) $\int_0^\infty te^{-2t}cos2tdt$ **(ii)** $\int_0^\infty t^2e^{-t}sintdt$ **(iii)** $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right)dt$

(iv) $\int_0^\infty \left(\frac{1-cost}{t}\right)e^{-t}dt$ **(v)** $\int_0^\infty \left(\frac{e^{-at}-cosbt}{t}\right)dt$

Solution:

(i) $\int_0^\infty te^{-2t}cos2tdt = L[tcos2t]_{s=2} = \left[\frac{-d}{ds}L(cos2t)\right]_{s=2}$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+4}\right)_{s=2}$$

$$= -\left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s=2}$$

$$= -\left[\frac{(4-s^2)}{(s^2+4)^2}\right]_{s=2}$$

$$= -\frac{(4-4)}{(4+4)^2} = 0$$

(ii) $\int_0^\infty t^2e^{-t}sintdt = L[t^2sint]_{s=1} = \frac{d^2}{ds^2}L[sint]_{s=1}$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)_{s=1}$$

$$= \frac{d}{ds}\left[\frac{-1(2s)}{(s^2+1)^2}\right]_{s=1}$$

$$= -2\frac{d}{ds}\left[\frac{s}{(s^2+1)^2}\right]_{s=1}$$

$$= -2\left[\frac{\left[(s^2+1)^2(1)-s.2(s^2+1)(2s)\right)]}{(s^2+1)^4}\right]_{s=1}$$

$$= -2\left[\frac{\left[(s^2+1)[(s^2+1)-4s^2)\right]]}{(s^2+1)^4}\right]_{s=1}$$

$$= -2\left[\frac{(1-3s^2)}{(s^2+1)^3}\right]_{s=1}$$

$$= \left[\frac{6s^3-2}{(s^2+1)^3}\right]_{s=1} = \frac{4}{8} = \frac{1}{2}$$

(iii) $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right) dt = L\left[\frac{e^{-t}-e^{-2t}}{t}\right]_{s=0} = \int_s^\infty [L[e^{-t} - e^{-2t}]ds]_{s=0}$

$= \int_s^\infty \left[[L(e^{-t}) - L(e^{-2t})]\square s\right]_{s=0}$

$= \int_s^\infty \left[\left(\frac{1}{s+1} - \frac{1}{s+2}\right) ds\right]_{s=0}$

$= \{[\log(s+1) - \log(s+2)]_s^\infty\}_{s=0}$

$= \left\{\left[log\,\frac{s+1}{s+2}\right]_s^\infty\right\}_{s=0}$

$= \left\{log\,\frac{s\left(1+\frac{1}{s}\right)}{s\left(1+\frac{2}{s}\right)}\right\}_{s}^{\infty}{}_{s=0}$

$= \left[0 - log\,\frac{s+1}{s+2}\right]_{s=0}$ $\quad\quad \because log1 = 0$

$= \left[log\frac{s+2}{s+1}\right]_{s=0} = log2$

(iv) $\int_0^\infty \left(\frac{1-cost}{t}\right) e^{-t} dt$

$\int_0^\infty \left(\frac{1-cost}{t}\right) e^{-t} dt = L\left[\frac{1-cost}{t}\right]_{S=1} = \int_s^\infty [L[(1-cost)]ds]_{S=1}$

$= \int_s^\infty \left[[L(1) - L(cost)]ds\right]_{S=1}$

$= \int_s^\infty \left[\left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds\right]_{S=1}$

$= \left\{\left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty\right\}_{S=1}$

$= \left\{\left[\log s - \log\sqrt{s^2+1}\right]_s^\infty\right\}_{S=1}$

$= \left\{\left[log\,\frac{s}{\sqrt{s^2+1}}\right]_s^\infty\right\}_{S=1}$

$= \left[0 - log\,\frac{s}{\sqrt{s^2+1}}\right]_{S=1}$

$= \left[log\,\frac{\sqrt{s^2+1}}{s}\right]_{S=1}$

$= log\sqrt{2}$

(v) $\int_0^\infty \left(\frac{e^{-at}-cosbt}{t}\right) dt$

$\int_0^\infty \left(\frac{e^{-at}-cosbt}{t}\right) dt = L\left[\frac{e^{-at}-cosbt}{t}\right]_{S=0} = \int_s^\infty [L[(e^{-at} - cosbt)]ds]_{S=0}$

$= \int_s^\infty \left[[L(e^{-at}) - L(cosbt)]ds\right]_{S=0}$

$= \int_s^\infty \left[\left(\frac{1}{s+a} - \frac{s}{s^2+b^2}\right) ds\right]_{S=0}$

$= \left\{\left[\log(s+a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty\right\}_{S=0}$

$$= \left\{\left[\log(s+a) - \log\sqrt{s^2+b^2}\right]_s^{\infty}\right\}_{S=0}$$

$$= \left\{\left[log\frac{s+a}{\sqrt{s^2+b^2}}\right]_s^{\infty}\right\}_{S=0}$$

$$= \left[0 - log\frac{s+a}{\sqrt{s^2+b^2}}\right]_{s=0}$$

$$= \left[log\frac{\sqrt{s^2+b^2}}{s+a}\right]_{s=0}$$

$$= log\frac{\sqrt{b^2}}{a}$$

$$= log\frac{b}{a}$$