STRAIN ENERGY METHOD
(Introduction)

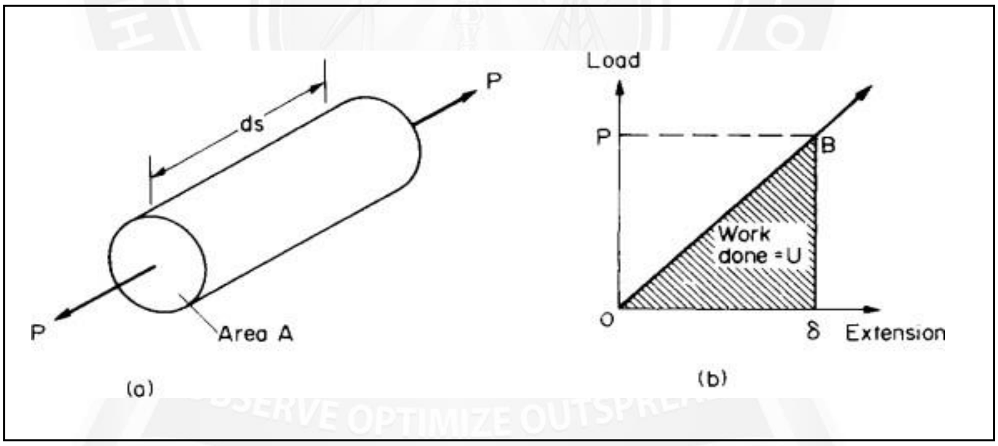
Strain Energy

Strain energy is as the energy which is stored within a material when work has been done on the material. Here it is assumed that the material remains elastic whilst work is done on it so that all the energy is recoverable and no permanent deformation occurs due to yielding of the material,

Strain energy U = work done

Thus for a gradually applied load the work done in straining the material will be given by the shaded area under the load-extension graph of Fig.7.1

$$U = P\delta$$



Work done by a gradually applied load.

The unshaded area above the line OB of Fig. 7.1 is called the complementary energy, a quantity which is utilized in some advanced energy methods of solution and is not considered within the terms of reference of this text.

Strain Energy (Tension or Compression)

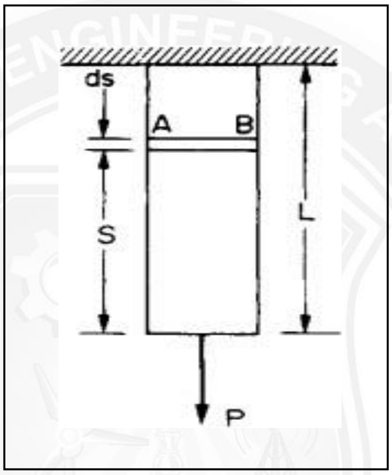
(a) Neglecting the weight of the bar: -

Consider a small element of a bar, length ds, shown in Fig. 7.1. If a graph is drawn of load against elastic extension the shaded area under the graph gives the work done and hence the strain energy,

Strain energy; **$U=P^2L / 2AE$**

(b) Including the weight of the bar: -

Consider now a bar of length L mounted vertically, as shown in Fig. 7.2. At any section A B the total load on the section will be the external load P together with the weight of the bar material below AB.

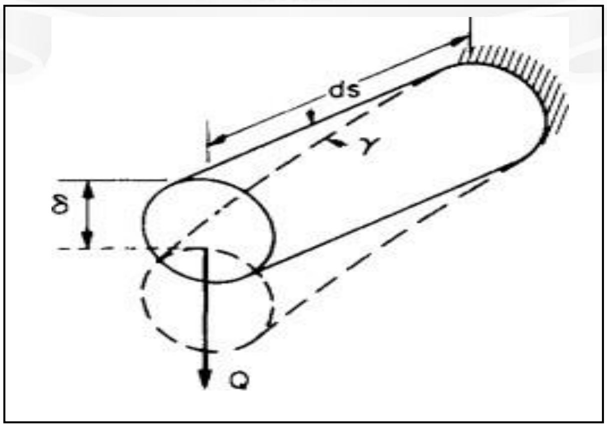


Direct load - tension or compression.

Strain energy; **$U=P^2L/2AE + P\rho gL^2/2E + (\rho g)^2 AL^3/6E$**

Strain Energy : Shear

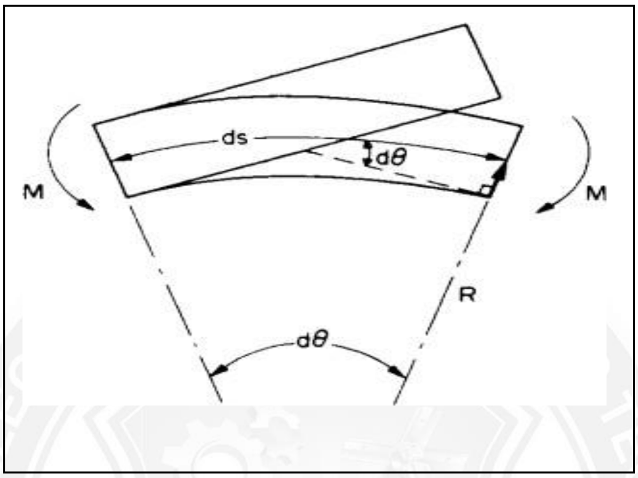
Consider the elemental bar now subjected to a shear load Q at one end causing deformation through the angle y (the shear strain) and a shear deflection 6, as shown in Fig.



Strain energy; **$U=Q^2L / 2AG$**

Strain Energy – Bending

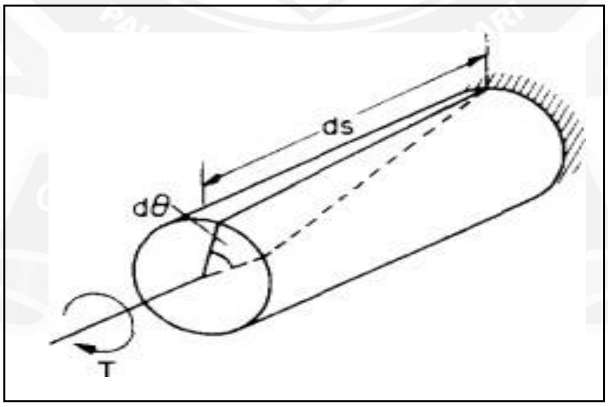
Let the element now be subjected to a constant bending moment M causing it to bend into an arc of radius R and subtending an angle $d\Theta$ at the center (Fig. 7.4). The beam will also have moved through an angle $d\Theta$.



Strain energy; $\mathbf{U= M^2 L / 2EI}$

Strain Energy - Torsion

The element is now considered subjected to a torque T as shown in Fig. producing an angle of twist $d\Theta$ radians.



Strain energy; $\mathbf{U=T^2 L / 2GJ}$

Note: - It should be noted that in the four types of loading case considered above the strain energy expressions are all identical in form,

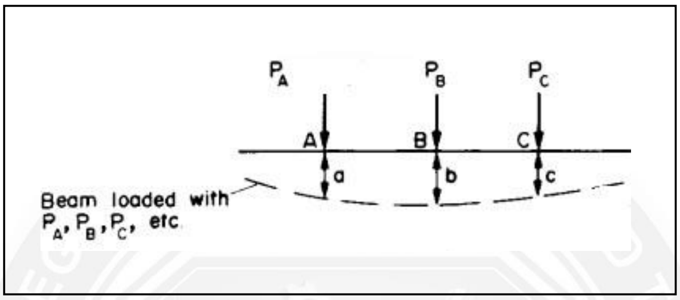
U = (unit applied load)^{2}x L / 2(product of two related constants)

Castigliano's first theorem assumption for deflection: -

If the total strain energy of a body or framework is expressed in terms of the external loads and is partially differentiated with respect to one of the loads the result is the deflection of the point of application of that load and in the direction of that load,

i.e, a=ΔU/ΔPa , b=ΔU/ΔPb and c=ΔU/Δpc

Where a,b and c are deflections of a beam under loads P_a, P_b and P_c etc. as shown in fig 7.6.



In most beam applications the strain energy, and hence the deflection, resulting from end loads and shear forces are taken to be negligible in comparison with the strain energy resulting from bending (torsion not normally being present),

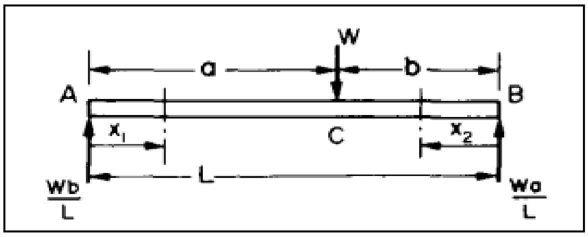
Therefore deflection; **Δ = ρv / ρp**

Application of Castigliano's theorem to angular movements:

If the total strain energy, expressed in terms of the external moments, be partially differentiated with respect to one of the moments, the result is the angular deflection (in radians) of the point of application of that moment and in its direction.

Example 1: -

Using Castigliano's first theorem, obtain the expressions for (a) the deflection under a single concentrated load applied to a simply supported beam as shown in Figure below, (b) the deflection at the center of a simply supported beam carrying a uniformly distributed load.



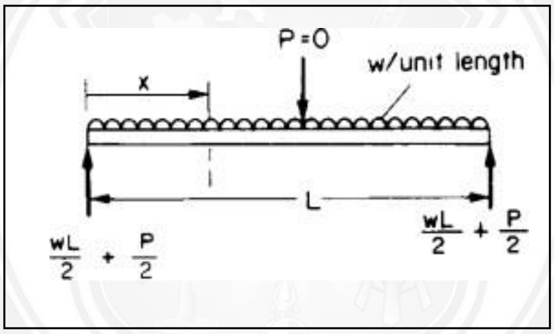
$$\delta = \int_B^A \frac{M}{EI}\frac{\partial M}{\partial W}\,ds$$

$$= \int_A^C \frac{M}{EI}\frac{\partial M}{\partial W}\,ds + \int_C^B \frac{M}{EI}\frac{\partial M}{\partial W}\,ds$$

$$= \frac{1}{EI}\int_0^a \frac{Wbx_1}{L} \times \frac{bx_1}{L} \times dx_1 + \frac{1}{EI}\int_0^b \frac{Wax_2}{L} \times \frac{ax_2}{L} \times dx_2$$

$$= \frac{Wb^2}{L^2EI}\int_0^a x_1^2\,dx_1 + \frac{Wa^2}{L^2EI}\int_0^b x_2^2\,dx_2$$

$$= \frac{Wb^2a^3}{3L^2EI} + \frac{Wa^2b^3}{3L^2EI} = \frac{Wa^2b^2}{3L^2EI}(a+b) = \mathbf{\frac{Wa^2b^2}{3LEI}}$$



$$\delta = \int_0^L \frac{Mm}{EI}\,ds = 2\int_0^{L/2} \frac{Mm}{EI}\,ds$$

$$M = \frac{wL}{2} - \frac{wx^2}{2} \quad \text{and} \quad m = \frac{x}{2}$$

$$\delta = \frac{2}{EI}\int_0^{L/2} \left(\frac{wLx}{2} - \frac{wx^2}{2}\right)\frac{x}{2}\,dx$$

$$= \frac{1}{2EI}\int_0^{L/2} (wLx^2 - wx^3)\,dx$$

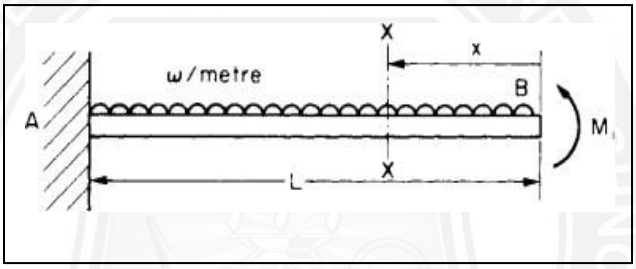
$$\delta = \frac{w}{2EI}\left[\frac{Lx^3}{3} - \frac{x^4}{4}\right]_0^{L/2}$$

$$= \frac{wL^4}{2EI}\left[\frac{1}{24} - \frac{1}{64}\right]$$

$$= \frac{wL^4}{2EI}\left[\frac{8-3}{192}\right] = \frac{5WL^4}{384EI}$$

Example 2: -

Derive the equation for the slope at the free end of a cantilever carrying a uniformly distributed load over its full length.



$$M = M_i - \frac{wx^2}{2}$$

$$\frac{\partial M}{\partial M_i} = 1$$

$$\theta = \int_0^L \frac{M}{EI}\cdot\frac{\partial M}{\partial M_i}\cdot dx$$

$$= \frac{1}{EI}\int_0^L \left(M_i - \frac{wx^2}{2}\right)(1)\,dx$$

$$\theta = \frac{-w}{2EI}\int_0^L x^2 . dx = \frac{wL^3}{6EI} \textbf{ radian}$$