UNIT–II

FOURIER SERIES

PROBLEMS BASED ON HALF RANGE COSINE AND SINE SERIES

HALF RANGE SERIES

It is often necessary to obtain a Fourier expansion of a function for the range$(0, \pi)$ which is half the period of the Fourier series, the Fourier expansion of such a function consists a cosine or sine terms only.

(i) Half Range Cosine Series

The Fourier cosine series for f(x) in the interval $(0,\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

2 π

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$ and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$$

(ii) Half Range Sine Series

The Fourier sine series for f(x) in the interval $(0,\pi)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

Example 1

If c is the constant in ($0 < x < \pi$) then show that

$$c = (4c / \pi) \ \{ \sin x + (\sin 3x /3) + \sin 5x / 5) + ...\}$$

Given $f(x) = c$ in $(0,\pi)$.

$$\text{Let } f(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} c \sin nx \, dx$$

$$= \frac{2c}{\pi}\left[\frac{- \cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2c}{\pi}\left\{\frac{-(-1)^n}{n} + \frac{1}{n}\right\}$$

$$b_n = (2c/n\pi) [1 - (-1)^n]$$

∞

$$\therefore f(x) = \sum_{n=1} (2c / n\pi)(1-(-1)^n) \sin nx$$

$$\text{i.e, } c = \frac{4c}{\pi} \left\{ \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \dots \dots \right\}$$

Example 2

Find the Fourier Half Range Sine Series and Cosine Series for f(x) = x in the interval$(0,\pi)$.

Sine Series

$$\text{Let} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx \text{ ------------------(1)}$$

$$\text{Here} \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\,dx = \frac{2}{\pi} \int_0^{\pi} x \, d(-\cos nx / n)$$

$$= \frac{2}{\pi} \left[(x) \left(\frac{-\cos nx}{n} \right) - (1) \left(\frac{\quad}{\quad} \right) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left(\frac{-\pi(-1)^n}{n} \right)$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$$

Cosine Series

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ ----------------------- (2)}$$

$$\text{Here} \quad a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x \, d(\sin nx / n)$$

$$= \frac{1}{\pi} \left[(x) \frac{\sin nx}{n} - (1) \frac{-\cos nx}{n^2} \right]_0^{\pi}$$

$$a_n = \frac{2}{n^2\pi} \left[(-1)^n - 1 \right]$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi} [(-1)^n - 1] \cos nx$$

$$\Rightarrow x = \frac{\pi}{2} + \frac{4}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \ldots\ldots \right]$$

Example 3

Find the sine and cosine half-range series for the function function .f(x) =

x , $0 < x \le \pi/2$

$= \pi - x$, $\pi/2 \le x < \pi$

Sine series

$$\text{Let } f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$b_n = (2/\pi)\int_0^{\pi} f(x)\sin nx\,dx$$

$$= (2/\pi)\left[\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx\right]$$

$$= (2/\pi)\left[\int_0^{\pi/2} x\,d\left(\frac{-\cos nx}{n}\right) + \int_{\pi/2}^{\pi} (\pi - x)\,d\left(\frac{-\cos nx}{n}\right)\right]$$

$$= (2/\pi)\left\{\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= (2/\pi)\left\{\left[\frac{-(\pi/2)\cos n(\pi/2)}{n} + \frac{\sin n(\pi/2)}{n^2}\right] - \left[\frac{-(\pi/2)\cos n(\pi/2)}{n} - \frac{\sin n(\pi/2)}{n^2}\right]\right\}$$

$$= (2/\pi)\left\{\frac{2\sin n(\pi/2)}{n^2}\right\}$$

$$= \frac{4}{n^2\pi}\sin(n\pi/2)$$

Therefore, $f(x) = (4/\pi)\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n^2}\sin nx$

ie, $f(x) = (4/\pi)\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \text{--------}\right]$

3^2 5^2

Cosine series

.Let $f(x) = (a_o/2) + \sum_{n=1}^{\infty} a_n \cos nx$., where n=1

$$a_o = (2/\pi) \int_0^{\pi} f(x)\, dx$$

$$= (2/\pi) \left\{ \int_0^{\pi/2} x\, dx + \int_{\pi/2}^{\pi} (\pi - x)\, dx \right\}$$

$$= (2/\pi) \left\{ (x^2/2)\Big|_0^{\pi/2} + (\pi x - x^2/2)\Big|_{\pi/2}^{\pi} \right\} = \pi/2$$

$$a_n = (2/\pi) \int_0^{\pi} f(x) \cos nx\, dx$$

$$= (2/\pi) \left\{ \int_0^{\pi/2} x \cos nx\, dx + \int_{\pi/2}^{\pi} (\pi - x) \cos nx\, dx \right\}$$

$$= (2/\pi) \left\{ \int_0^{\pi/2} x\, d\left(\frac{\sin nx}{n}\right) + \int_{\pi/2}^{\pi} (\pi - x)\, d\left(\frac{\sin nx}{n}\right) \right\}$$

$$= (2/\pi) \left\{ \left\{ x \left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right) \right\}_0^{\pi/2} + (\pi - x)\left(\frac{\sin nx}{n}\right) - (-1) - \left(\frac{\cos nx}{n^2}\right) \Big|^{\pi} \right\}$$

$$= (2/\pi) \left(\frac{(\pi/2) \sin n(\pi/2)}{} + \frac{\cos n(\pi/2)}{} - \frac{1}{} \right)$$

n n^2 n^2

$$+\left(-\frac{\cos nx}{n^2} - \frac{(\pi/2)\sin n(\pi/2)}{n} + \frac{\cos n(\pi/2)}{n^2}\right)\Bigg\}$$

$$=(2/\pi)\left\{\frac{2\cos n(\pi/2) - \{1+(-1)^n\}}{n^2}\right\}$$

$$\text{Therefore, } f(x) = (\pi/4)+(2/\pi)\sum_{n=1}^{\infty}\frac{2\cos n(\pi/2) - \{1+(-1)^n\}}{n^2}\cos nx\,.$$

$$= (\pi/4)-(2/\pi)\ \cos 2x + \frac{\cos 6x}{3^2} + \text{-------------}$$

Exercises

1.Obtain cosine and sine series for f(x) = x in the interval 0< x < π. Hence show that $1/1^2 + 1/3^2 + 1/5^2 + \ldots = \pi^2/8$.

2.Find the half range cosine and sine series for $f(x) = x^2$ in the range $0 \le x \le \pi$

3.Obtain the half-range cosine series for the function f(x) = xsinx in $(0,\pi)$..

4.Obtain cosine and sine series for $f(x) = x(\pi-x)$ in $0< x < \pi$

5.Find the half-range cosine series for the function6.f(x) =

$(\pi x) / 4$, $0<x< (\pi/2)$

$= (\pi/4)(\pi-x)$, $\pi/2 < x < \pi$.

7.Find half range sine series and cosine series

forf(x) = x in $0<x< (\pi/2)$

= 0 in $\pi/2 < x < \pi$.

8.Find half range sine series and cosine series for the function f(x) == π - x in the interval $0 < x < \pi$.

9.Find the half range sine series of f(x) = x cosx in $(0,\pi)$

10.Obtain cosine series for

$f(x) = \cos x$, $0<x< (\pi/2)$

$= 0, \ \pi/2 < x < \pi.$