

# UNIT-II

## FOURIER SERIES

PROBLEMS BASED ON HALF RANGE COSINE AND SINE SERIES

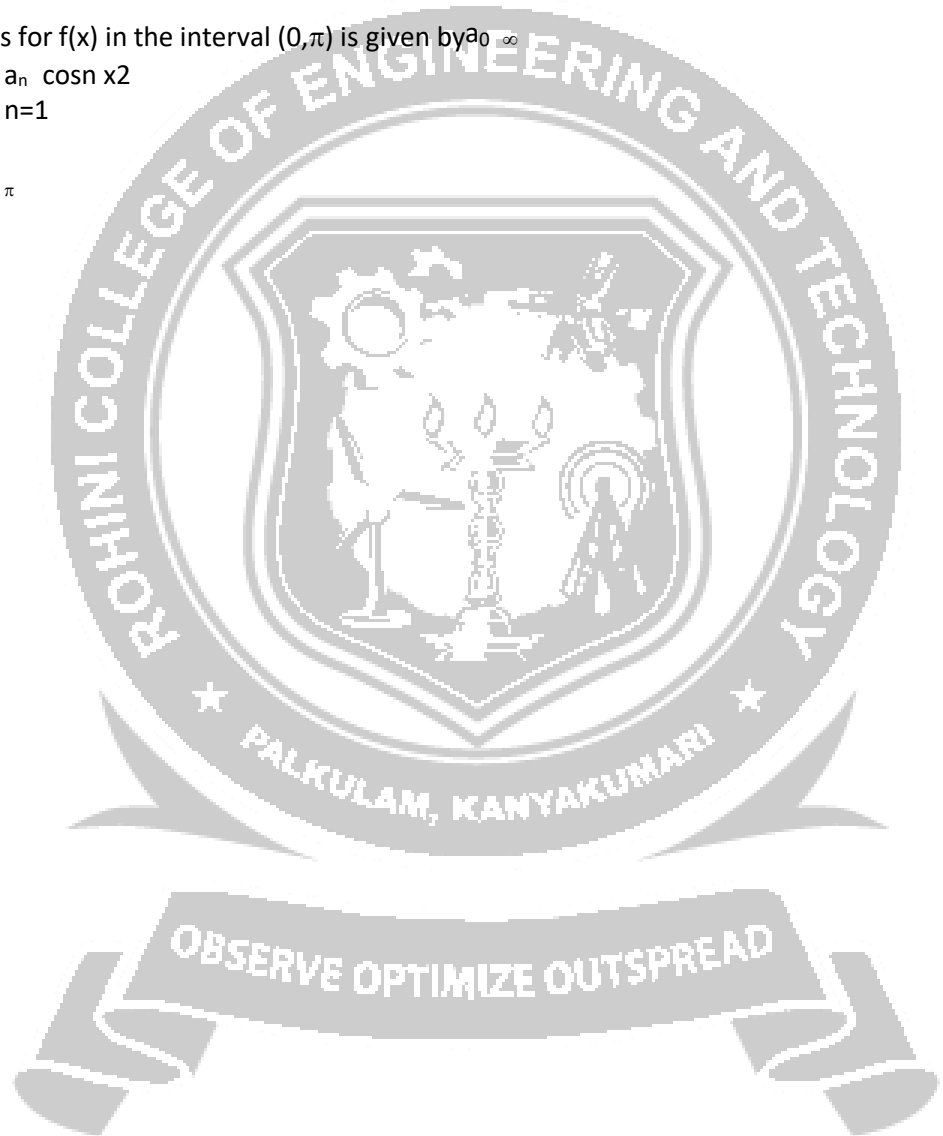
### HALF RANGE SERIES

It is often necessary to obtain a Fourier expansion of a function for the range  $(0, \pi)$  which is half the period of the Fourier series, the Fourier expansion of such a function consists a cosine or sine terms only.

#### (i) Half Range Cosine Series

The Fourier cosine series for  $f(x)$  in the interval  $(0, \pi)$  is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$



where  $a_0 = \frac{1}{\pi} \int_0^{\pi} f(x) dx$  and

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

**(ii) Half Range Sine Series**

The Fourier sine series for  $f(x)$  in the interval  $(0, \pi)$  is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where  $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$

**Example 1**

If  $c$  is the constant in  $(0 < x < \pi)$  then show that

$$c = (4c / \pi) \{ \sin x + (\sin 3x / 3) + \sin 5x / 5) + \dots \}$$

Given  $f(x) = c$  in  $(0, \pi)$ .

Let  $f(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$

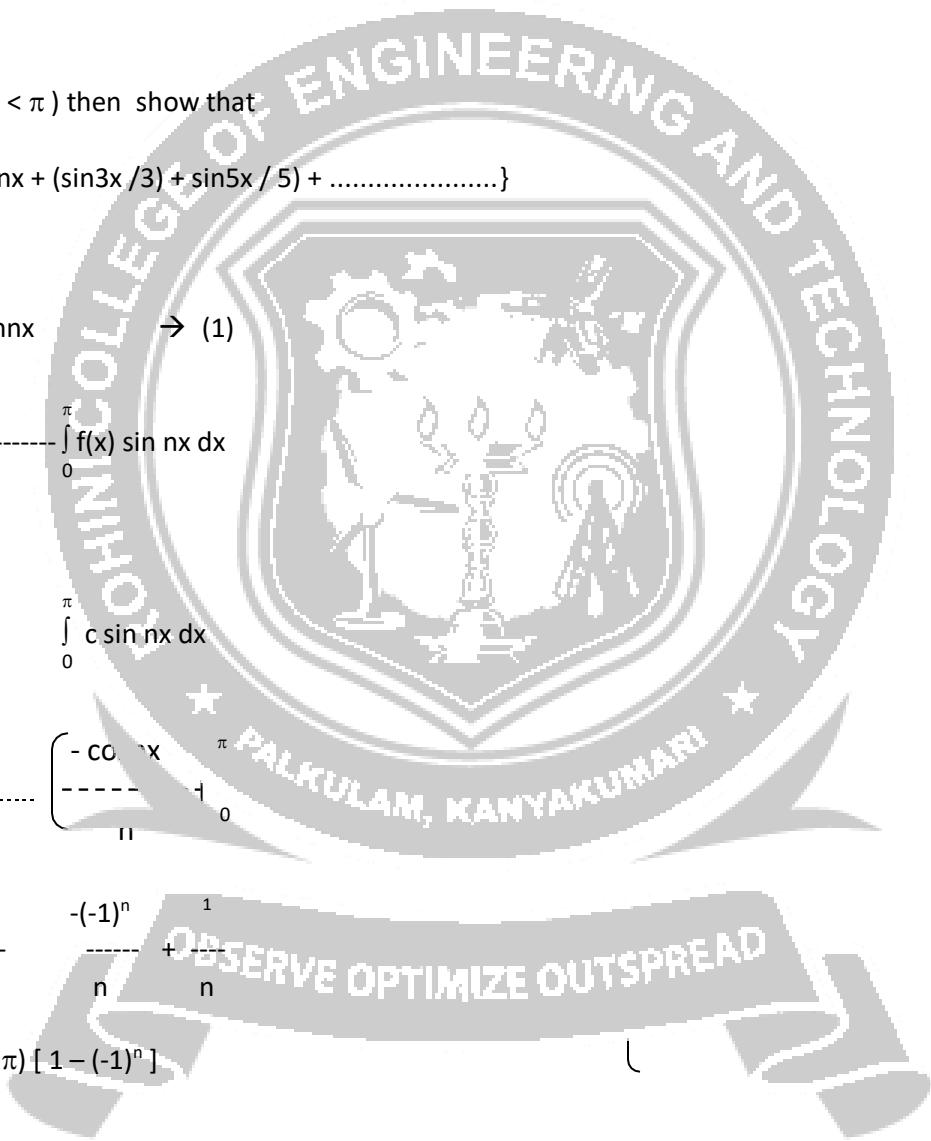
$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} c \sin nx dx$$

$$= \frac{2c}{\pi} \left[ -\frac{\cos nx}{n} \right]_0^{\pi}$$

$$= \frac{2c}{\pi} \left[ \frac{-(-1)^n}{n} + \frac{1}{n} \right]$$

$$b_n = (2c/n\pi) [1 - (-1)^n]$$



$$\therefore f(x) = \sum_{n=1}^{\infty} (2c / n\pi) (1 - (-1)^n) \sin nx$$

$$\text{i.e, } c = \frac{4c}{\pi} \left\{ \sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \dots \dots \right\}$$

**Example 2**

Find the Fourier Half Range Sine Series and Cosine Series for  $f(x) = x$  in the interval  $(0, \pi)$ .

**Sine Series**

Let  $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$  -----(1)

Here  $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} x d(-\cos nx / n)$

$$= \frac{2}{\pi} \left( x \left[ \frac{-\cos nx}{n} \right] - \int_0^{\pi} \left[ \frac{-\cos nx}{n} \right] dx \right)$$

$$= \frac{2}{\pi} \left( \frac{-\pi (-1)^n}{n} \right)$$

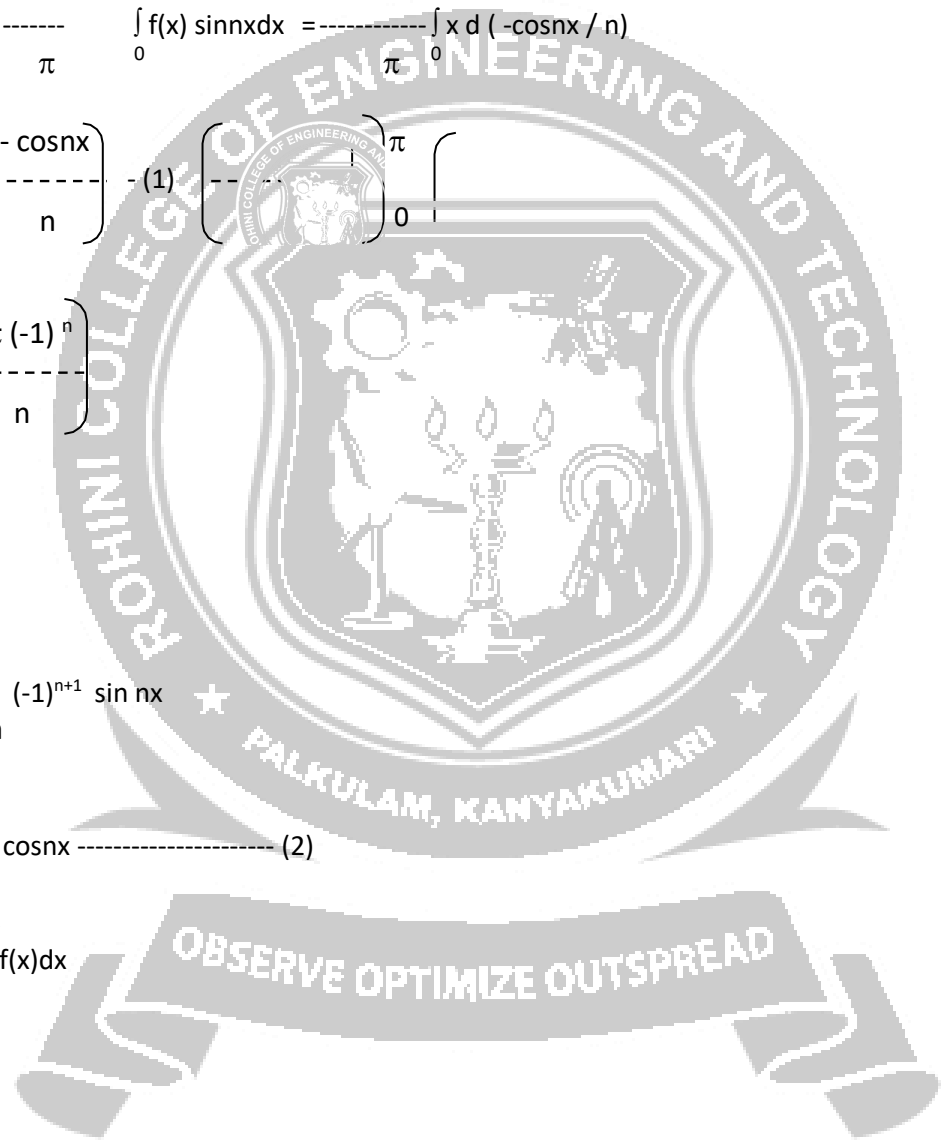
$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin nx$$

**Cosine Series**

Let  $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$  ----- (2)

Here  $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$



$$= \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left[ \frac{x^2}{2} \right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} x d(\sin nx / n)$$

$$= \frac{1}{\pi} \left[ x \frac{\sin nx}{n} - \int_0^{\pi} \sin nx dx \right]_0^{\pi}$$

$$a_n = \frac{2}{n^2 \pi} [(-1)^n - 1]$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi} [(-1)^n - 1] \cos nx$$

$$\Rightarrow x = \frac{\pi}{2} + \frac{4}{\pi} \left[ \frac{\cos x}{1^2} - \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} - \dots \right]$$

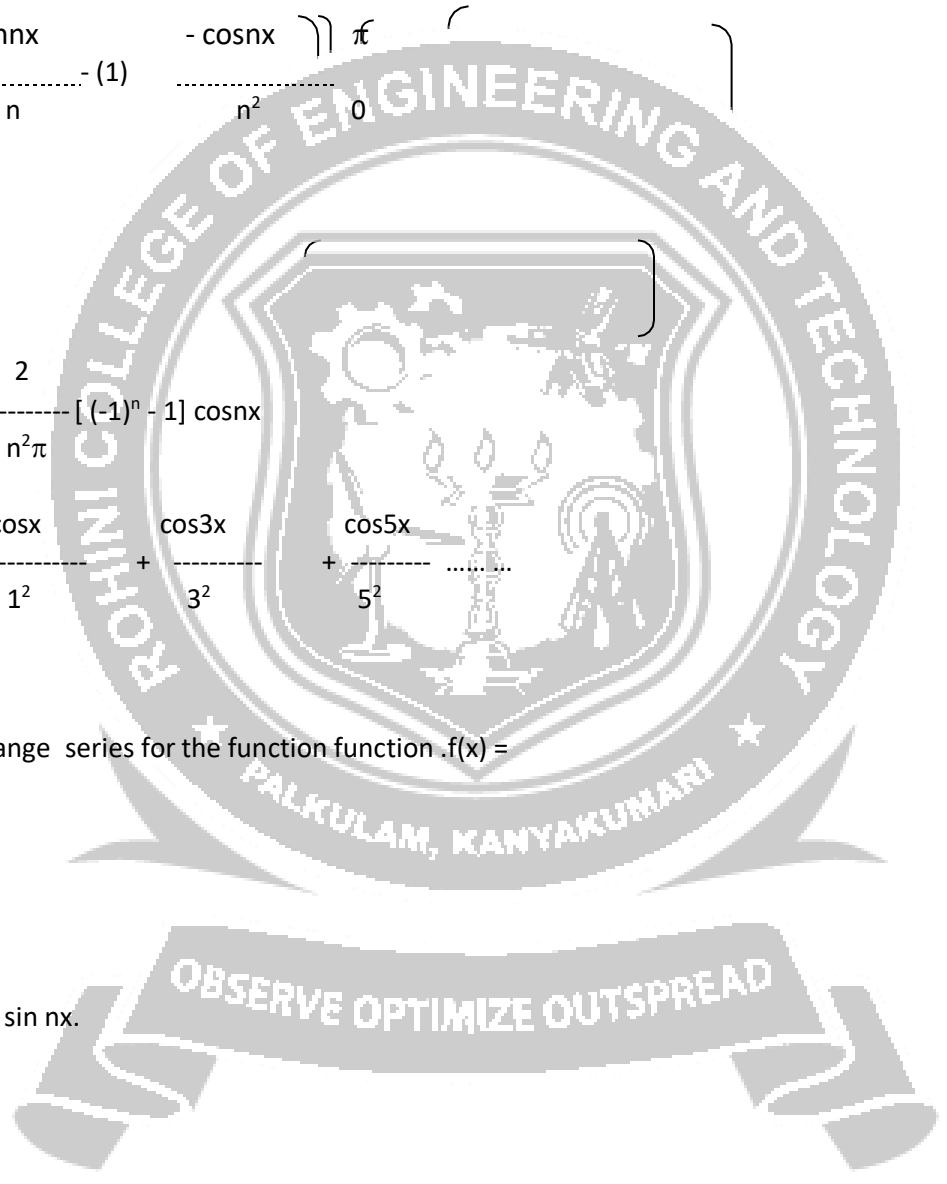
**Example 3**

Find the sine and cosine half-range series for the function  $f(x) = x$ ,  $0 < x \leq \pi/2$

$$= \pi - x, \pi/2 \leq x < \pi$$

**Sine series**

$$\text{Let } f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$



$$b_n = (2/\pi) \int_0^{\pi} f(x) \sin nx \, dx$$

$$= (2/\pi) \int_{\pi/2}^{\pi/2} x \sin nx \, dx + \int_0^{\pi} (\pi-x) \sin nx \, dx$$

$$= (2/\pi) \left[ x \cdot \frac{-\cos nx}{n} + \int_{\pi/2}^{\pi} (\pi-x) \, dx \cdot \frac{-\cos nx}{n} \right]$$

$$= (2/\pi) \left\{ x \left[ \frac{-\cos nx}{n} \right] - (1) \left[ \frac{-\sin nx}{n^2} \right] \right\}_{\pi/2}^{\pi}$$

$$+ (\pi-x) \frac{\cos nx}{n} - (-1) \frac{\sin nx}{n^2}$$

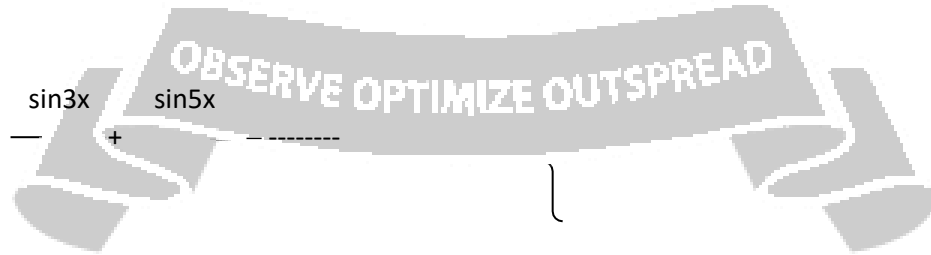
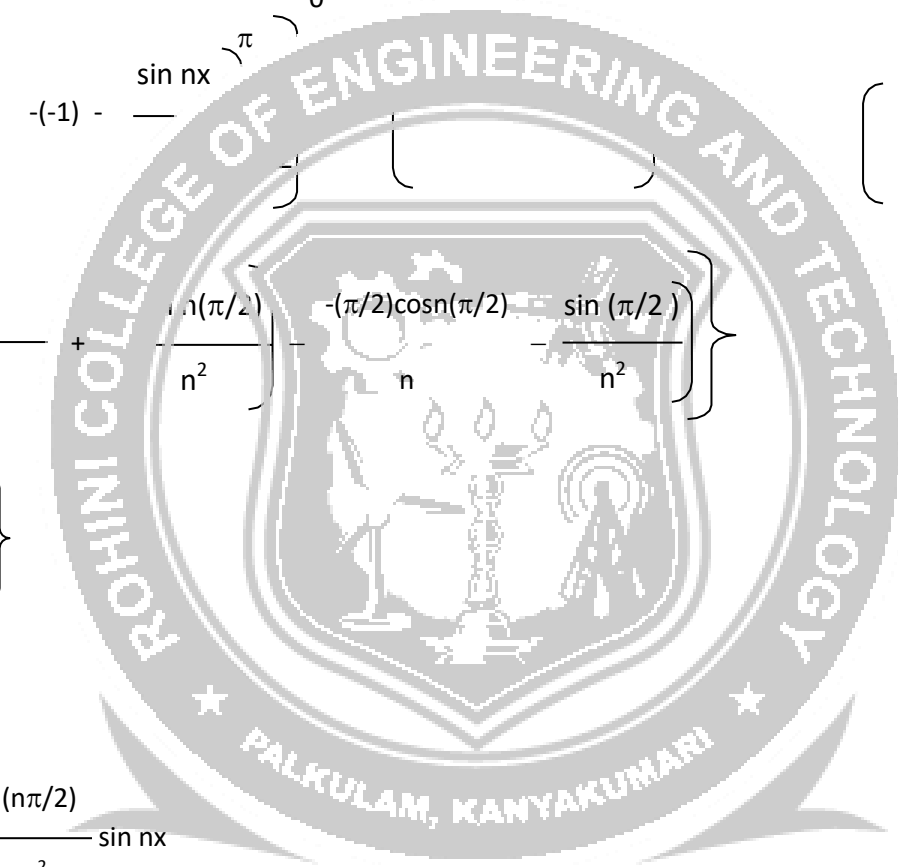
$$= (2/\pi) \left\{ \left[ \frac{(\pi/2) \cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2} \right] - \left[ \frac{-(\pi/2) \cos(n\pi/2)}{n} - \frac{\sin(n\pi/2)}{n^2} \right] \right\}$$

$$= (2/\pi) \left\{ \frac{2 \sin(n\pi/2)}{n^2} \right\}$$

$$= \frac{4}{n^2 \pi} \sin(n\pi/2)$$

Therefore,  $f(x) = (4/\pi) \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n^2} \sin nx$

ie,  $f(x) = (4/\pi) \sin x - \frac{\sin 3x}{8} + \frac{\sin 5x}{8} - \dots$



$$3^2 \quad 5^2$$

### Cosine series

Let  $f(x) = (a_0/2) + \sum_{n=1}^{\infty} a_n \cos nx$ , where  $n=1$

$$a_0 = (2/\pi) \int_0^{\pi} f(x) dx$$

$$= (2/\pi) \left[ \int_0^{\pi/2} x dx + \int_{\pi/2}^{\pi} (\pi-x) dx \right]$$

$$= (2/\pi) \left[ \left( \frac{x^2}{2} \right)_0^{\pi/2} + \left( \pi x - \frac{x^2}{2} \right)_{\pi/2}^{\pi} \right] = \pi/2$$

$$a_n = (2/\pi) \int_0^{\pi} f(x) \cos nx dx$$

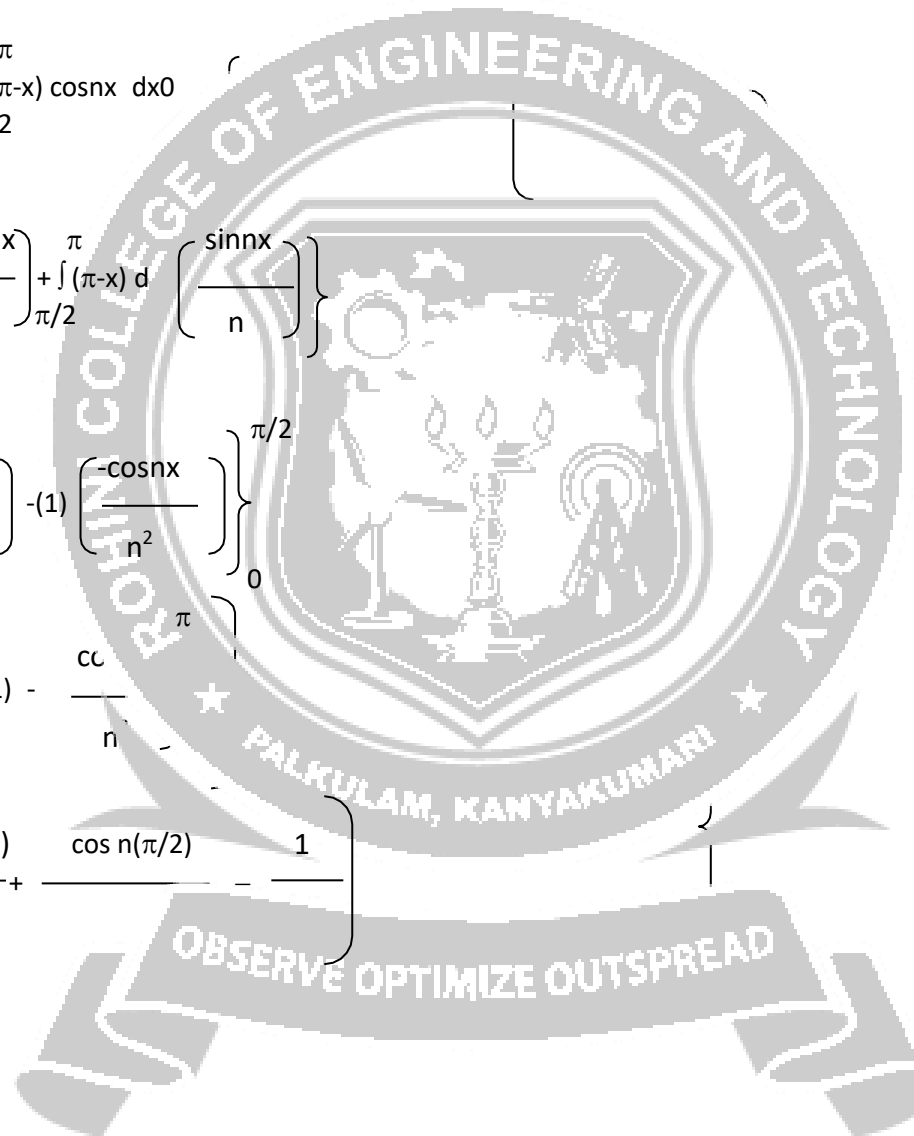
$$= (2/\pi) \left[ \int_0^{\pi/2} x \cos nx dx + \int_{\pi/2}^{\pi} (\pi-x) \cos nx dx \right]$$

$$= (2/\pi) \left[ \int_0^{\pi/2} x d \left( \frac{\sin nx}{n} \right) + \int_{\pi/2}^{\pi} (\pi-x) d \left( \frac{\sin nx}{n} \right) \right]$$

$$= (2/\pi) \left[ \left\{ x \left( \frac{\sin nx}{n} \right) - (1) \left( \frac{-\cos nx}{n^2} \right) \right\}_0^{\pi/2} \right]$$

$$+ (\pi-x) \left[ \frac{\sin nx}{n} - (-1) - \frac{\cos nx}{n} \right]_{\pi/2}^{\pi}$$

$$= (2/\pi) \left[ \frac{(\pi/2) \sin(\pi/2) + \cos n(\pi/2)}{n^2} - \frac{1}{n^2} \right]$$



$$+ \left\{ \frac{\cos nx}{n^2} - \frac{(\pi/2) \sin n(\pi/2)}{n} + \frac{\cos n(\pi/2)}{n^2} \right\}$$

$$= (2/\pi) \left\{ \frac{2 \cos n(\pi/2) - \{1+(-1)^n\}}{n^2} \right\}$$

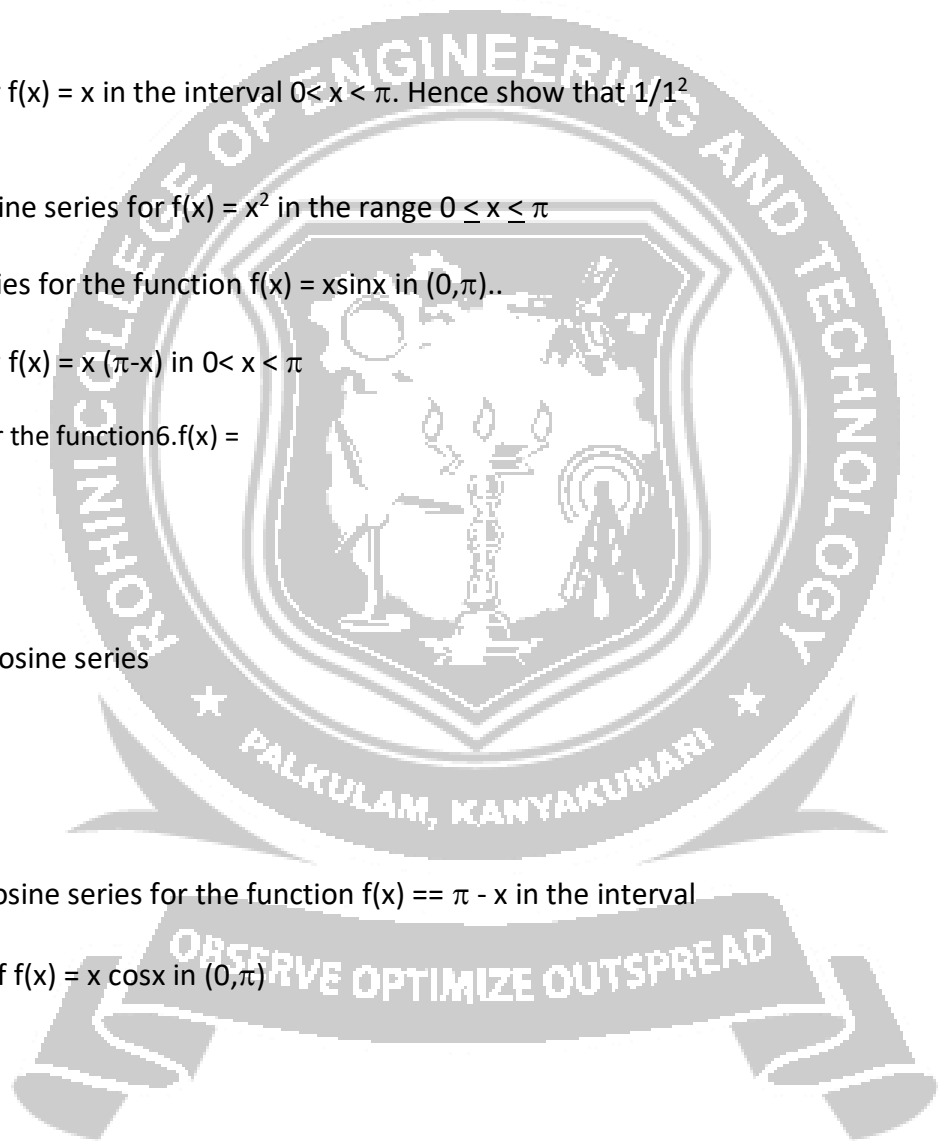
Therefore,  $f(x) = (\pi/4) + (2/\pi) \sum_{n=1}^{\infty} \frac{2 \cos n(\pi/2) - \{1+(-1)^n\}}{n^2} \cos nx.$

$$= (\pi/4) - (2/\pi) \cos 2x + \frac{\cos 6x}{3^2} + \dots$$

**Exercises**

1. Obtain cosine and sine series for  $f(x) = x$  in the interval  $0 < x < \pi$ . Hence show that  $1/1^2 + 1/3^2 + 1/5^2 + \dots = \pi^2/8$ .
2. Find the half range cosine and sine series for  $f(x) = x^2$  in the range  $0 \leq x \leq \pi$
3. Obtain the half-range cosine series for the function  $f(x) = x \sin x$  in  $(0, \pi)$ .
4. Obtain cosine and sine series for  $f(x) = x(\pi-x)$  in  $0 < x < \pi$
5. Find the half-range cosine series for the function  $f(x) = (\pi x) / 4, 0 < x < (\pi/2)$   
 $= (\pi/4)(\pi-x), \pi/2 < x < \pi.$
7. Find half range sine series and cosine series  
for  $f(x) = x$  in  $0 < x < (\pi/2)$   
 $= 0$  in  $\pi/2 < x < \pi.$
8. Find half range sine series and cosine series for the function  $f(x) = \pi - x$  in the interval  $0 < x < \pi.$
9. Find the half range sine series of  $f(x) = x \cos x$  in  $(0, \pi)$
10. Obtain cosine series for

$$f(x) = \cos x, \quad 0 < x < (\pi/2)$$



$$= 0, \pi/2 < x < \pi.$$