UNIT IV – TWO DIMENSIONAL VECTOR VARIABLE PROBLEMS

PART - A

1. **What is meant by axisymmetric field problem? Give example.*(April/May 2010)***

   In some of the three dimensional solids like flywheel, turbine, discs etc, the material is symmetric with respect to their axes. Hence the stress developed is also symmetric. Such solids are known as axisymmetric solids. Due to this condition, three dimensional solids can be treated as two dimensional elements.

2. **List the required conditions for a problem assumed to be axisymmetric. *(April/May 2011)***

   The condition to be axisymmetric is as follows:

- Problem domain must be symmetric about the axis of revolution.
- All boundary conditions must be symmetric about the axis of revolution.
- All loading conditions must be symmetric about the axis of revolution.

3. **What is Plane stress and Plane strain condition? *(April/May 2015), (May/June 2013)***

   Plane stress - A state of plane stress is said to exist when the elastic body is very thin and there is no load applied in the coordinate direction parallel to the thickness.

   Example: A ring press-fitted on a shaft in a plane stress problem. In plane stress problem σ_z, τ_{yz}, τ_{zx} are zero.

   Plane strain – A state of plane strain is said to exist when the strain at the plane perpendicular to the plane of application of load is constant.

4. **What are the forces acting on shell elements? Give its applications**

   The two forces in which the shell element is subjected to are:

   Bending moments

   Membrane forces

   Shell elements can be employed in the analysis of the following structures,

   Example:

- Sea shell, egg shell (the wonder of the nature);
- Containers, pipes, tanks;
- Car bodies;
- Roofs, buildings (the Superdome), etc.

5. **Write the constitutive relations for axisymmetric problems.**

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\mu)(1-2\mu)} \begin{pmatrix} 1-\mu & \mu & \mu & 0 \\ & 1-\mu & \mu & 0 \\ & & 1-\mu & 0 \\ & & & \frac{1-2\mu}{2} \end{pmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$

6. **Define body force.**

   A body force is distributed force acting on every elemental volume of the body.
   Unit: force per unit volume.

7. **Write the governing equation for 2D bending of plates.**

$$D\nabla^4 w = q(x,y),$$

where

$$\nabla^4 \equiv \left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right),$$

$$D = \frac{Et^3}{12(1-\nu^2)} \quad \text{(the bending rigidity of the plate)},$$

q = lateral distributed load (force/area).

8. Write the stress strain relationship for plane stress problems.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} \right)$$

9. Differentiate material non linearity and geometric non linearity. ***(Nov/Dec 2012)***

Material Non linearity	Geometric non linearity
(i) The stress – strain relation for the material may not be linear.	(i) The Strain – Displacement relations are not linear.
(ii) Basic non-linear relations are time dependent complex constitutive relations	(ii) Need consideration of actual strain displacement relations rather than linear strain displacement.

10. Write the equilibrium equations for two dimensional elements. ***(Nov/Dec 2012)***

In elasticity theory, the stresses in the structure must satisfythe following equilibrium equations,

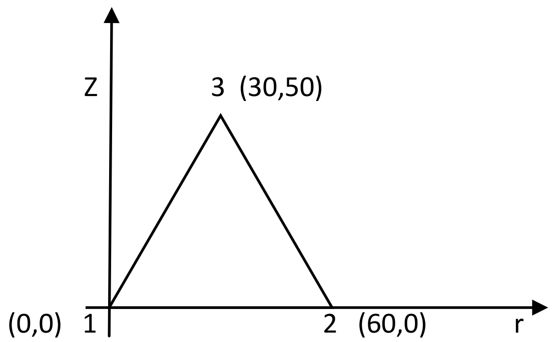
$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

wherefx and fy are body forces (such as gravity forces) per unit volume.

PART - B

1. For the axe symmetric element shown in fig .Determine the element stresses. Take E= 2.1 x 10^5 N/mm^2 ν = 0.25. The co-ordinates shown in fig are in mm. The nodal displacements are u_1=0.05 mm, u_2=0.02 mm, u_3=0.0 mm, $\omega_1 = 0.03\,mm, \omega_2 = 0.02\,mm, \omega_3 = 0.0\,mm$.



Given data:

r_1 = 0 mm z_1=0 mm u_1=0.05 mm $\omega_1 = 0.03\, mm$

r_2 = 60 mm z_2=0 mm u_2=0.02 mm $\omega_2 = 0.02\, mm$

r_3 = 30 mm z_3=50 mm u_3=0.0 mm $\omega_3 = 0.0\, mm$

E= 2.1 x 10^5 N/mm^2, ν = 0.25

To find

i. Radial stress σ_r
ii. Circumferential stress σ_θ
iii. Longitudinal stress σ_z
iv. Shear stress τ_{rz}

Formula used

$$\{\sigma\} = [D]\,[B]\,\{u\}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = [D]\,[B]\,\begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

Solution:

$\{\sigma\}$ = [D] [B] {u}

D = Stress - Strain relationship matrix

$$D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$= \frac{2.1 \text{ x } 10^5}{(1+0.25)(1-2\times 0.25)} \begin{bmatrix} 1-0.25 & 0.25 & 0.25 & 0 \\ 0.25 & 1-0.25 & 0.25 & 0 \\ 0.25 & 0.25 & 1-0.25 & 0 \\ 0 & 0 & 0 & \frac{1-2\times 0.25}{2} \end{bmatrix}$$

$$[D] = 336\times 10^3 \times 0.25 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 84\times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[B] =Strain displacement relationship matrix or gradient matrix

$$[B] = \frac{1}{2A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta 1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (60 \times 50) - (30 \times 0) = 3000 mm^2$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (30 \times 0) - (0 \times 50) = 0$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (0 \times 0) - (60 \times 0) = 0$$

$\beta_1 = z_2 - z_3 = 0 - 50 = -50$; $\beta_2 = y_3 - y_1 = 50 - 0 = 50$; $\beta_3 = y_1 - y_2 = 0 - 0 = 0$
$\gamma_1 = r_3 - r_2 = 30 - 60 = -30$; $\gamma_2 = r_1 - r_3 = 0 - 30 = -30$; $\gamma_3 = r_2 - r_1 = 60 - 0 = 60$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{0 + 60 + 30}{3} = 30\, mm$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{0 + 0 + 50}{3} = 16.67\, mm$$

$$\frac{\alpha_1}{r} + \beta^1 + \frac{\gamma_1 z}{r} = \frac{3000}{30} + (-50) + \frac{(-30 \times 16.67)}{30} = 33.33\, mm$$

$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = 0 + 50 + \frac{(-30 \times 16.67)}{30} = 33.33\, mm$$

$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = 0 + 0 + \frac{60 \times 16.67}{30} = 33.33\, mm$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 60 & 0 \\ 1 & 30 & 50 \end{vmatrix}$$

$$= \frac{1}{2}\, [1(3000 - 0) - 0(50 - 0) + 0(30 - 60)] = 1500\, mm^2$$

$$[B] = \frac{1}{2 \times 1500}\begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 33.33 & 0 & 33.33 & 0 & 33.33 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix}$$

$$[D]\,[B] = 84 \times 10^3 \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times 3.34 \times 10^{-4}$$

$$\begin{bmatrix} -50 & 0 & 50 & 0 & 0 & 0 \\ 33.33 & 0 & 33.33 & 0 & 33.33 & 0 \\ 0 & -30 & 0 & -30 & 0 & 60 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix}$$

$$= 28 \begin{bmatrix} -116.67 & -30 & 183.33 & -30 & 33.33 & 60 \\ 49.99 & -30 & 149.99 & -30 & 99.99 & 60 \\ -16.67 & -90 & 83.33 & -90 & 33.33 & 180 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = 28 \begin{bmatrix} -116.67 & -30 & 183.33 & -30 & 33.33 & 60 \\ 49.99 & -30 & 149.99 & -30 & 99.99 & 60 \\ -16.67 & -90 & 83.33 & -90 & 33.33 & 180 \\ -30 & -50 & -30 & 50 & 60 & 0 \end{bmatrix} \begin{Bmatrix} 0.05 \\ 0.03 \\ 0.02 \\ 0.02 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix}\sigma_r\\ \sigma_\theta\\ \sigma_z\\ \tau_{rz}\end{Bmatrix} = 28\begin{Bmatrix}-3.66\\ 4\\ -3.66\\ -2.6\end{Bmatrix} = \begin{Bmatrix}-102.65\\ 112\\ -102.65\\ -72.8\end{Bmatrix}$$

Results

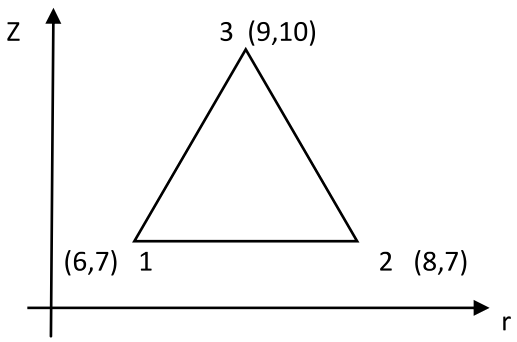
Radial stress σ_r = -102.65 N/mm^2

Circumferential stress σ_θ = 112 N/mm^2

Longitudinal stress σ_z = -102.65 N/mm^2

Shear stress τ_{rz} = -72.8 N/mm^2

2. **Calculate the element stiffness matrix and the thermal force vector for the axisymmetric triangular element shown in figure. The element experiences a 15^{0}c increase in temperature. The co-ordinates are in mm. Take α= 10 x 10^{-6}/^{0}c ; E= 2x 10^5 N/mm^2, ν = 0.25**



Given data:

r_1 = 6 mm z_1=7 mm

r_2 = 8 mm z_2=7 mm

r_3 = 9 mm z_3=10 mm

E= 2 × 10^5 N/mm^2, ν = 0.25, α= 10 × 10^{-6}/^{0}c

To find

Thermal force vector $\{F\}_t$

Formula used

$[K]=[B]^T[D][B]\ 2\pi r\ A$

$\{F\}= [B]^T[D]\{e_t\}2\pi r\ A$

Solution:

[B] =Strain displacement relationship matrix or gradient matrix

$$[B] = \frac{1}{2A}\begin{bmatrix}\beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0\\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} & 0\\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3\\ \gamma_1 & \beta 1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3\end{bmatrix}$$

$$\alpha_1 = r_2 z_3 - r_3 z_2 = (8 \times 10) - (9 \times 7) = 17mm^2$$

$$\alpha_2 = r_3 z_1 - r_1 z_3 = (9 \times 7) - (6 \times 10) = 3mm^2$$

$$\alpha_3 = r_1 z_2 - r_2 z_1 = (6 \times 7) - (8 \times 7) = 13mm^2$$

$\beta_1 = z_2 - z_3 = 7 - 10 = -3mm$; $\beta_2 = y_3 - y_1 = 10 - 7 = 3$; $\beta_3 = y_1 - y_2 = 7 - 7 = 0$

$\gamma_1 = r_3 - r_2 = 9 - 8 = 1mm$; $\gamma_2 = r_1 - r_3 = 6 - 9 = -3$; $\gamma_3 = r_2 - r_1 = 8 - 6 = 2$

$$r = \frac{r_1 + r_2 + r_3}{3} = \frac{6+8+9}{3} = 7.67\ mm$$

$$z = \frac{z_1 + z_2 + z_3}{3} = \frac{7+7+10}{3} = 8\ mm$$

$$\frac{\alpha_1}{r} + \beta^1 + \frac{\gamma_1 z}{r} = \frac{17}{7.67} + (-3) + \frac{(1 \times 8)}{7.67} = 0.26\ mm$$

$$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} = \frac{3}{7.67} + 3 + \frac{(-3 \times 8)}{7.67} = 0.26\ mm$$

$$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} = \frac{-14}{7.67} + 0 + \frac{2 \times 8}{7.67} = 0.26\ mm$$

$$A = \frac{1}{2}\begin{vmatrix} 1 & r_1 & z_1 \\ 1 & r_2 & z_2 \\ 1 & r_3 & z_3 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 1 & 6 & 7 \\ 1 & 8 & 7 \\ 1 & 9 & 10 \end{vmatrix} = \frac{1}{2}\ [1(80 \times 63) - 6(10-7) + 7(9-8) = 3\ mm^2$$

$$[\mathrm{B}] = \frac{1}{2 \times 3}\begin{bmatrix} -3 & 0 & 3 & 0 & 0 & 0 \\ 0.26 & 0 & 0.26 & 0 & 0.26 & 0 \\ 0 & 1 & 0 & -3 & 0 & 2 \\ 1 & -3 & -3 & 3 & 2 & 0 \end{bmatrix}; \quad [\mathrm{B}]^{\mathrm{T}} = 0.167\begin{bmatrix} -3 & 0.26 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 3 & 0.26 & 0 & -3 \\ 0 & 0 & -3 & 3 \\ 0 & 0.26 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$[\mathrm{D}] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$= \frac{2 \times 10^5}{(1+0.25)(1-2\times 0.25)}\begin{bmatrix} 1-0.25 & 0.25 & 0.25 & 0 \\ 0.25 & 1-0.25 & 0.25 & 0 \\ 0.25 & 0.25 & 1-0.25 & 0 \\ 0 & 0 & 0 & \frac{1-2\times 0.25}{2} \end{bmatrix}$$

$$= 320 \times 10^5 \times 0.25\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\mathrm{B}]^{\mathrm{T}}[\mathrm{D}] = 0.167\begin{bmatrix} -3 & 0.26 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 3 & 0.26 & 0 & -3 \\ 0 & 0 & -3 & 3 \\ 0 & 0.26 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \times 8\times 10^4\begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= 13.36\times 10^3\begin{bmatrix} -8.7 & -2.2 & -2.7 & 1 \\ 1 & 1 & 3 & -3 \\ 9.26 & 3.78 & 3.26 & -3 \\ -3 & -3 & -9 & 3 \\ 0.26 & 0.78 & 0.26 & 2 \\ 2 & 2 & 6 & 0 \end{bmatrix}$$

$$[B]^T[D][B] = 13.36\times10^3\begin{bmatrix} -8.7 & -2.2 & -2.7 & 1 \\ 1 & 1 & 3 & -3 \\ 9.26 & 3.78 & 3.26 & -3 \\ -3 & -3 & -9 & 3 \\ 0.26 & 0.78 & 0.26 & 2 \\ 2 & 2 & 6 & 0 \end{bmatrix} \times 0.167 \begin{bmatrix} -3 & 0.26 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 3 & 0.26 & 0 & -3 \\ 0 & 0 & -3 & 3 \\ 0 & 0.26 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$[K] = 321.27 \times 10^3 \begin{bmatrix} 26.63 & -5.7 & -29.79 & 11.21 & 1.42 & -5.4 \\ -5.7 & 12 & 12.26 & -18 & -5.7 & 6 \\ -29.79 & 12.26 & 37.76 & -18.78 & -5.01 & 6.5 \\ 11.21 & -18 & -18.78 & 36 & 5.2 & -18 \\ 1.42 & -5.7 & -5.01 & 5.2 & 4.2 & 0.52 \\ -5.4 & 6 & 6.5 & -18 & 0.52 & 12 \end{bmatrix}$$

Thermal force vector $\{F\} = [B]^T[D]\{e_t\}2\pi r A$

$$\{e_t\} = \begin{Bmatrix} \alpha\Delta t \\ \alpha\Delta t \\ 0 \\ \alpha\Delta t \end{Bmatrix} = \begin{Bmatrix} 10 \times 10^{-6} \times 15 \\ 10 \times 10^{-6} \times 15 \\ 0 \\ 10 \times 10^{-6} \times 15 \end{Bmatrix} = 10^{-6} \begin{Bmatrix} 150 \\ 150 \\ 0 \\ 150 \end{Bmatrix}$$

$$\{F\} = [B]^T[D]\{e_t\}\, 2\pi r A \quad = 13.36\times10^3 \begin{bmatrix} -8.7 & -2.2 & -2.7 & 1 \\ 1 & 1 & 3 & -3 \\ 9.26 & 3.78 & 3.26 & -3 \\ -3 & -3 & -9 & 3 \\ 0.26 & 0.78 & 0.26 & 2 \\ 2 & 2 & 6 & 0 \end{bmatrix} \times 10^{-6} \begin{Bmatrix} 150 \\ 150 \\ 0 \\ 150 \end{Bmatrix} \times 2\pi \times 7.67 \times 3$$

$$= 1.927 \begin{Bmatrix} -1493.46 \\ -150 \\ 1506.54 \\ -450 \\ 456.54 \\ 600 \end{Bmatrix}$$

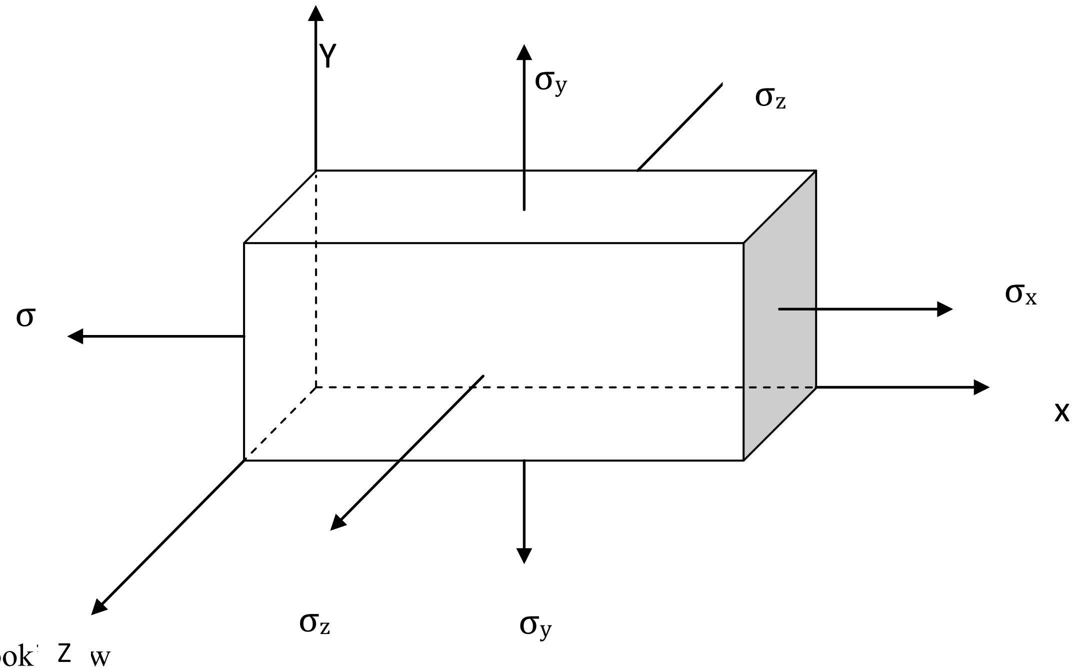
$$\text{Thermal force vector } \{F\} = \begin{Bmatrix} F_{1u} \\ F_{1w} \\ F_{2u} \\ F_{2w} \\ F_{3u} \\ F_{3w} \end{Bmatrix} = \begin{Bmatrix} -2878.25 \\ -289.08 \\ 2903.45 \\ -867.25 \\ 879.86 \\ 1156.34 \end{Bmatrix}$$

3. DERIVE THE EXPRESSION FOR STRESS – STRAIN RELATIONSHIP FOR A 2D- ELEMENT?

EQUATION OF ELASTICITY

1. Stress – strain relationship matrix for a two dimensional element

Consider a three dimensional body as shown in fig. which is subjected to a stress σ_x σ_y and σ_z



Hook' w

$$\sigma = Ee$$

$$e = \frac{\sigma}{E}$$

The stress in the x direction produces a positive strain in x direction as shown in fig.

$$e_x = \frac{\sigma_x}{E}$$

The positive stress in the y direction produces a negative strain in the x direction

$$e_y = \frac{-\nu\sigma_y}{E}$$

The positive stress in the z direction produces a negative strain in the x direction

$$e_z = \frac{-\nu\sigma_z}{E}$$

$$e_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$e_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$e_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

Solving 3 equations

$$\sigma_x = \frac{E}{(1+v)(1-2v)} \left[e_{x(1-v)+v\, e_y + V\, e_2} \right]$$

$$\sigma_J = \frac{E}{(1+v)(1-2v)} \left[v\, e_{x(1-v) -\, e_y + V\, e_2} \right]$$

$$\sigma_2 = \frac{E}{(1+v)(1-2v)} \left[v\, e_{x+v\, e_y + (1-v)\, e_2} \right]$$

The shear stress and shear strain relationship

$\tau = G\gamma$ where, τ - Shear Stress

γ – Shear Strain

G – Modular of rigidity

$\tau_{xy} = G\gamma_{xy}$

$\tau_{yz} = G\gamma_{yz}$

$\tau_{zx} = G\gamma_{zx}$

G $\longrightarrow$ Modular of rigidity = $\frac{E}{2(1+v)}$

$\tau_{xy} = \frac{E}{2(1+v)} \gamma_{xy}$; $\tau_{xy} = \frac{E}{2(1+v)(1-2v)^{\gamma y 2}} \frac{(1-2v)}{2}$

$\tau_{yz} = \frac{E}{(1+v)(1-2v)} \frac{(1-2v)}{2} \gamma_{xz}$;$\tau_{yz} = \frac{E}{(1+v)(1-2v)} \frac{(1-2v)}{(2)} \gamma_{yz}$

$\tau_{zx} \frac{E}{(1+v)(1-2v)} = \frac{(1-2v)}{(2)} \gamma_{zx}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$\{\sigma\} = \{D\} \{e\}$$

D- in a stress strain relation ship matrix

$$\{D\} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

Where E – Yours Modules

V – Poisson Ratio

(i) PLANE STRESS CONDITION:-

Plane stress is defined to be a state of stress in which the normal stress (σ) and shear stress (τ) cleared perpendicular to the plane are assumed to be zero.

Normal stress σ_z = 0; Shear Stress $\tau_{xz} + \tau_{yz} = 0$

$\therefore \sigma_z = \tau_{xz} = \tau_{yz} = 0$

$\therefore\ e_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E}$; $e_y = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E}$

$$e_x = \frac{\sigma_x}{E} - v\frac{\sigma_y}{E}$$

$$ve_y = -v^2\ \frac{\sigma_x}{E} + v\frac{\sigma_y}{E}$$

$$e_x + ve_y = \frac{\sigma_x}{E} - \frac{v^2\sigma_x}{E}$$

$$e_x + ve_y = \frac{\sigma_x}{E} - (1 - v^2)$$

$$\sigma_x = \frac{E}{(1-v^2)}\left(e_x + v\,e_y\right)$$

$$v\,e_x = v\frac{\sigma_x}{E}\ -V^2\ \frac{\sigma_y}{E}$$

$$e_y = -v\frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$

$$v\,e_x + e_y = -V^2\frac{\sigma_y}{E} + \frac{\sigma_y}{E}$$

$$v\,e_x + e_y = \frac{\sigma_y}{E}(1 - v^2)$$

$$\sigma_y = \frac{E}{(1-v^2)}\left(ve_x + e_y\right)$$

Share Stress τ_{xz} = G γ_{xz}

Where G ⟶ Modular of rigidity = $\frac{E}{2(1+v)}$

γ_{xy} ⟶ Share Strain

V – Poisson ratio

$$\tau_{xy} = \frac{E}{2(1+v)}\ \gamma_{xy}$$

$$\tau_{xy} = \frac{E}{(1+v)(1-v)} \times \frac{(1-v)}{2}\ \gamma_{xy}$$

$$\tau_{xy} = \frac{E}{(1-v)^2} \times \frac{(1-v)}{2} \times \gamma_{xy}$$

Above equation matrix form

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1-v)} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ \tau_{xy} \end{Bmatrix}$$

Two dimensional stress strain relationship matrix for phase stress location.

$$\{D\} = \frac{E}{1-v} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

(ii) PLANE STRAIN CONDITION

Plane strain is defined to be a state of strain in which the strain normal to the xy plane and the shear strain are assumed to be zero.

Normal strain $e_z = 0$

Shear Stress $\gamma_{xz} = 0 = \gamma_{yz}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-\mathrm{v} & \mathrm{v} & \mathrm{v} & 0 & 0 & 0 \\ \mathrm{v} & 1-\mathrm{v} & \mathrm{v} & 0 & 0 & 0 \\ \mathrm{v} & \mathrm{v} & 1-\mathrm{v} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ e_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

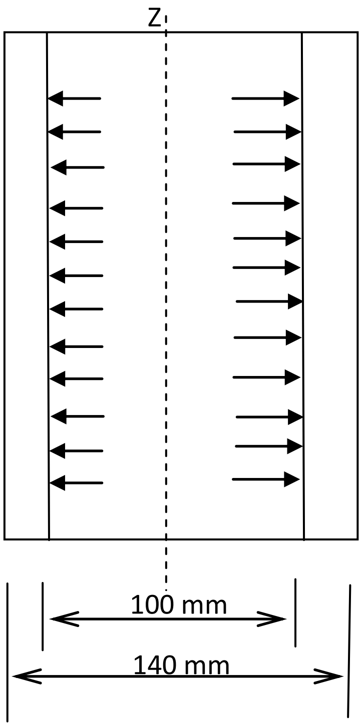
$e_z = 0$; $\gamma_{x0} = \gamma_{yz} = 0$ Sub in above matrix.

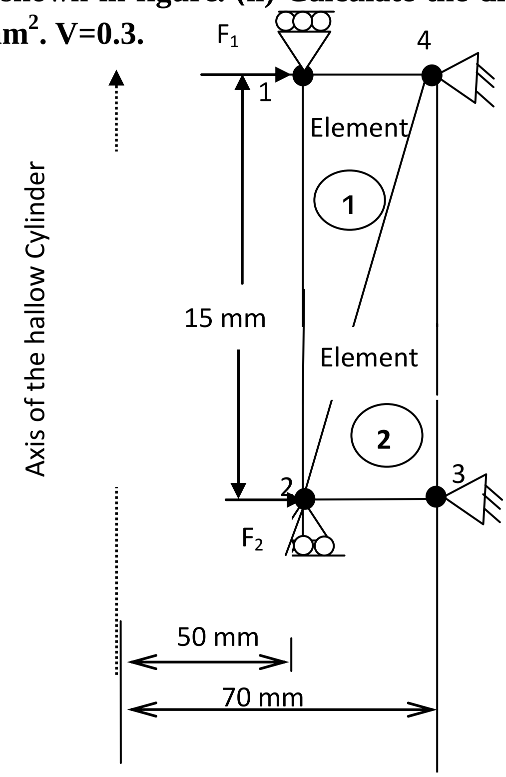
$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \gamma_{xy} \end{Bmatrix} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix} \begin{Bmatrix} e_x \\ e_y \\ \gamma_{xy} \end{Bmatrix}$$

Stress Strain relationship matrix for phase strain condition.

$$\{D\} = \frac{E}{(1+v)(1-2v)} \begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}$$

4. **A long hollow cylinder of inside diameter 100 mm and outside diameter 140 mm is subjected to an internal pressure of 4 N/mm^2 as shown in figure.(i) By using two elements on the 15 mm length shown in figure. (ii) Calculate the displacements at the inner radiusTake E=2×10^5 N/mm^2. V=0.3.**





Given data:

Inner diameter, d_e= 100mm

Inner radius r_e= 50 mm

Outer diameter D_e=140 mm

Outer radius R_e=70mm

Internal pressure P=4N/mm^2

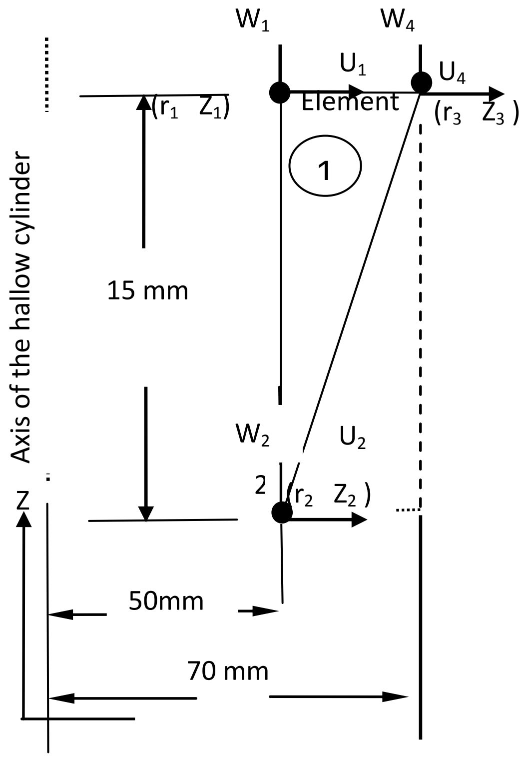
Length l_e=15mm

Young's modulus E=2×10^5 N/mm^2

Poison's ratio v= 0.3

To Find

u_1, w1, u2, w2, u3, w3, u4, w4



Formula used

$\{F\}=[K]\ \{U\}$

Solution

For element (1)

(Nodal displacements u_1, w1, u2, w2, u4, w4)

Co ordinates

At node 1

r_1=50mm

z_1=15mm

At node 2

r_1=50mm

z_1=0mm

At node 3

r_1=70mm

z_1=15mm

We know that, $where \quad r = \frac{r_1+r_2+r_3}{3} \quad = \frac{50+50+70}{3}$

r = 56.6667mm

$z = \frac{z_1+z_2+z_3}{3} = \frac{15+0+15}{3}$; z= 10 mm

Area of the triangle element = $\frac{1}{2} \times \ Breadth \ \times \ Height$

$$=\frac{1}{2}\times\ 20\ \times\ 15\ ; \qquad A = 150 \text{ mm}$$

We know that,

Stiffness matrix for axisymmetric triangular element (1),

$[K]_1 = 2\,\pi$ rA $[B]^T[D][B]$

Stress strain relationship matrix $[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$

Stress strain relationship matrix $[D] = \frac{2X10^5}{(1+0.3)(1-(2\times 0.3))}\begin{bmatrix} 1-0.3 & 0.3 & 0.3 & 0 \\ 0.3 & 1-0.3 & 0.3 & 0 \\ \nu & \nu & 1-0.3 & 0 \\ 0 & 0 & 0 & \frac{1-(2\times0.3)}{2} \end{bmatrix}$

$$= \frac{2\times10^5}{0.5}\begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ \nu & \nu & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

$$= 384.6153\times10^3\begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ \nu & \nu & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

We know that , strain-Displacement matrix

$$[B]=\frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta 1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\alpha_1 = r_2 z_3 - r_3 z_2 \qquad \alpha_2 = r_3 z_1 - r_1 z_3 \qquad \alpha_3 = r_1 z_2 - r_2 z_1$$

$$\alpha_1 = (50\times15) - (70\times0) \qquad \alpha_2 = (70\times15) - (50\times15) \qquad \alpha_3 = (50\times0) - (50\times15)$$

$$\alpha_1 = 750\ mm^2 \qquad \alpha_2 = 300 mm^2 \qquad \alpha_3 = -750 mm^2$$

$$\beta_1 = z_2 - z_3 \qquad \beta_2 = y_3 - y_1 \qquad \beta_3 = y_1 - y_2$$
$$\gamma_1 = r_3 - r_2 \qquad \gamma_2 = r_1 - r_3 \qquad \gamma_3 = r_2 - r_1$$

$$\beta_1 = 0 - 15 \qquad \beta_2 = 15 - 15 \qquad \beta_3 = 15 - 0$$
$$\gamma_1 = 70 - 50 \qquad \gamma_2 = 50 - 70 \qquad \gamma_3 = 50 - 50$$

$$\beta_1 = -15mm \qquad \beta_2 = 0 \qquad \beta_3 = 15mm$$
$$\gamma_1 = 20mm \qquad \gamma_2 = -20mm \qquad \gamma_3 = 0$$

$$\frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} = \frac{750}{56.6667} + (-15) + \frac{20\times10}{56.6667} \qquad = 1.7647 \text{ mm}$$

$$\frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} = \frac{300}{56.6667} + 0 + \frac{(-20\times10)}{56.6667} \qquad = 1.7647 \text{ mm}$$

$$\frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} = \frac{-750}{56.6667} + 15 + 0 \qquad = 1.7647 \text{mm}$$

Substitute$\beta_1,\beta_2,\beta_3,\frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r},\frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r},\ \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r},\gamma_1,\gamma_2,\gamma_3$ and A values in equations no 5, we get,

$$[B]=\frac{1}{2\times150}\begin{bmatrix}-15 & 0 & 0 & 0 & 15 & 0\\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0\\ 0 & 20 & 0 & -20 & 0 & 0\\ 20 & -15 & -20 & 0 & 0 & 15\end{bmatrix}$$

$$[B]=3.333\times10^{-3}\begin{bmatrix}-15 & 0 & 0 & 0 & 15 & 0\\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0\\ 0 & 20 & 0 & -20 & 0 & 0\\ 20 & -15 & -20 & 0 & 0 & 15\end{bmatrix}$$

$$[B]^T=3.333\times10^{-3}\begin{bmatrix}-15 & 1.7647 & 0 & 20\\ 0 & 0 & 20 & -15\\ 0 & 1.7647 & 0 & -20\\ 0 & 0 & -20 & 0\\ 15 & 1.7647 & 0 & 0\\ 0 & 0 & 0 & 15\end{bmatrix}$$

$$[D][B] = 384.6153\times10^3\begin{bmatrix}0.7 & 0.3 & 0.3 & 0\\ 0.3 & 0.7 & 0.3 & 0\\ 0.3 & 0.3 & 0.7 & 0\\ 0 & 0 & 0 & 0.2\end{bmatrix}\times$$

$$3.33310^{-3}\begin{bmatrix}-15 & 0 & 0 & 0 & 15 & 0\\ 1.7647 & 0 & 1.7647 & 0 & 1.7647 & 0\\ 0 & 20 & 0 & -20 & 0 & 0\\ 20 & -15 & -20 & 0 & 0 & 15\end{bmatrix}$$

$$[D][B] = 1.282\times10^3\begin{bmatrix}-9.9706 & 6 & 0.5294 & -6 & 11.0294 & 0\\ 0.3 & 0.7 & 0.3 & -6 & 5.7353 & 0\\ 0.3 & 0.3 & 0.7 & -14 & 5.0294 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$[D][B][B]^T =1.282\times10^3\begin{bmatrix}-9.9706 & 6 & 0.5294 & -6 & 11.0294 & 0\\ 0.3 & 0.7 & 0.3 & -6 & 5.7353 & 0\\ \nu & \nu & 0.7 & -14 & 5.0294 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\mathrm{X}3.33\ 10^{-3}$$

$$\begin{bmatrix}-15 & 1.7647 & 0 & 20\\ 0 & 0 & 20 & -15\\ 0 & 1.7647 & 0 & -20\\ 0 & 0 & -20 & 0\\ 15 & 1.7647 & 0 & 0\\ 0 & 0 & 0 & 15\end{bmatrix}$$

$$[D][B][B]^T= 4.2733\begin{bmatrix}223.798 & -139.4118 & -85.7611 & 79.4118 & -155.32 & 60\\ -139.412 & 325 & 70.588 & -280 & 100.588 & -45\\ -85.7612 & 70.588 & 82.18 & -10.588 & 10.1211 & -60\\ 79.412 & -280 & -10.588 & 280 & -100.588 & 0\\ -155.3202 & 100.5882 & 10.1210 & -100.588 & 175.5621 & 0\\ 60 & -45 & -60 & 0 & 0 & 45\end{bmatrix}$$

Substitute $[D][B][B]^T$ value in equ no 4

$$[K]_1 = 2\pi \times 56.6667 \times 150 \times 4.2733$$

$$\times \begin{bmatrix} 223.798 & -139.4118 & -85.7611 & 79.4118 & -155.32 & 60 \\ -139.412 & 325 & 70.588 & -280 & 100.588 & -45 \\ -85.7612 & 70.588 & 82.18 & -10.588 & 10.1211 & -60 \\ 79.412 & -280 & -10.588 & 280 & -100.588 & 0 \\ -155.3202 & 100.5882 & 10.1210 & -100.588 & 175.5621 & 0 \\ 60 & -45 & -60 & 0 & 0 & 45 \end{bmatrix}$$

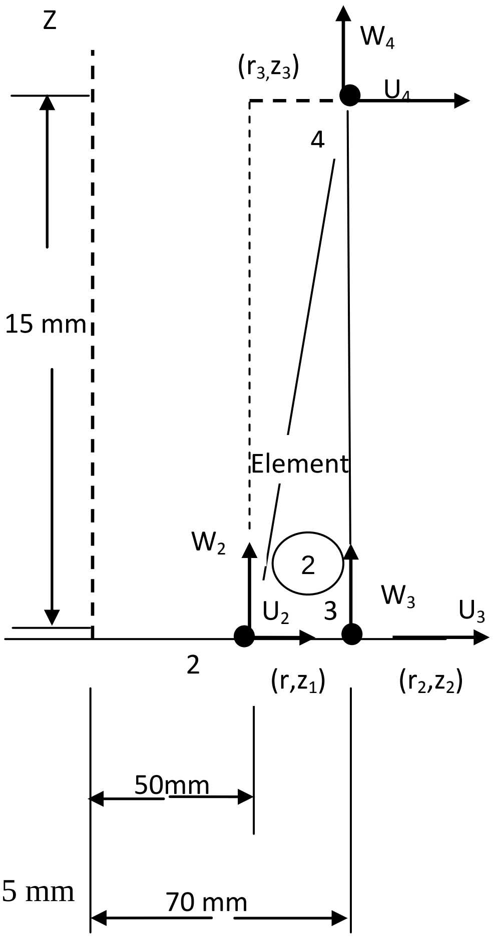
$$[K]_1 = 228224.6 \times \begin{bmatrix} 223.798 & -139.4118 & -85.7611 & 79.4118 & -155.32 & 60 \\ -139.412 & 325 & 70.588 & -280 & 100.588 & -45 \\ -85.7612 & 70.588 & 82.18 & -10.588 & 10.1211 & -60 \\ 79.412 & -280 & -10.588 & 280 & -100.588 & 0 \\ -155.3202 & 100.5882 & 10.1210 & -100.588 & 175.5621 & 0 \\ 60 & -45 & -60 & 0 & 0 & 45 \end{bmatrix}$$

u_1 w_1 u_2 w_2 u_4 w_4

$$[K]_1 = \begin{bmatrix} 51.076 & -31.817 & -19.573 & 18.124 & -35.448 & 13.693 \\ -31.817 & 74.173 & 16.110 & -63.903 & 22.597 & -10.270 \\ -19.573 & 16.110 & 18.755 & -2.416 & 2.310 & -13.693 \\ 18.124 & -63.903 & -2.416 & 63.903 & -22.597 & 0 \\ -35.448 & 22.597 & 2.310 & -22.597 & 40.068 & 0 \\ 13.693 & 10.270 & -13.693 & 0 & 0 & 10.270 \end{bmatrix}$$

For element (2) (Nodal displacements, u2, w2, u_3, w3, u4, w4)

Co ordinates



At node 2

r_1=50mm

z_1=0mm

At node 3

r_1=70mm

z_1=0mm

At node 4

r_1=70mm

z_1=15mm

We know that, $where \quad r = \frac{r_1+r_2+r_3}{3}$

$= \frac{50+70+70}{3}$

r = 63.3333mm,

$z = \frac{z_1+z_2+z_3}{3} = \frac{0+0+15}{3}$;z= 5 mm

Area of the triangle element = $\frac{1}{2} \times Breadth \times Height$

$$= \frac{1}{2} \times 20 \times 15$$

A = 150 mm

We know that,

Stiffness matrix for axisymmetric triangular element (2),

$[K]_2$ =2 π rA $[B]^T[D]$[B]

Stress strain relationship matrix$[D]$= $\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$

Stress strain relationship matrix$[D]$= $\frac{2X10^5}{(1+0.3)(1-(2\times0.3))}\begin{bmatrix} 1-0.3 & 0.3 & 0.3 & 0 \\ 0.3 & 1-0.3 & 0.3 & 0 \\ \nu & \nu & 1-0.3 & 0 \\ 0 & 0 & 0 & \frac{1-(2\times0.3)}{2} \end{bmatrix}$

$$=\frac{2\times10^5}{0.5}\begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ \nu & \nu & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

$$=384.6153\times10^3\begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ \nu & \nu & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

$$=384.6153\times10^3\begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ \nu & \nu & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}$$

We know that, strain-Displacement matrix

$$[B]=\frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ \frac{\alpha_1}{r}+\beta_1+\frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r}+\beta_2+\frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r}+\beta_3+\frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta 1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\alpha_1 = r_2 z_3 - r_3 z_2 \qquad \alpha_2 = r_3 z_1 - r_1 z_3 \qquad \alpha_3 = r_1 z_2 - r_2 z_1$$

$$\alpha_1 = (70 \times 15) - (70 \times 0) \qquad \alpha_2 = (70 \times 0) - (50 \times 15) \qquad \alpha_3 = (50 \times 0) - (70 \times 0)$$

$$\alpha_1 = 1050\, mm^2 \qquad \alpha_2 = -750 mm^2 \qquad \alpha_3 = 0$$

$$\beta_1 = z_2 - z_3 \qquad \beta_2 = y_3 - y_1 \qquad \beta_3 = y_1 - y_2$$
$$\gamma_1 = r_3 - r_2 \qquad \gamma_2 = r_1 - r_3 \qquad \gamma_3 = r_2 - r_1$$

$$\beta_1 = 0 - 15 \qquad \beta_2 = 15 - 0 \qquad \beta_3 = 0 - 0$$
$$\gamma_1 = 70 - 70 \qquad \gamma_2 = 50 - 70 \qquad \gamma_3 = 70 - 50$$

$$\beta_1 = -15mm \qquad \beta_2 = 15mm \qquad \beta_3 = 0$$
$$\gamma_1 = 0 \qquad \gamma_2 = -20mm \qquad \gamma_3 = -20mm$$

$\frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} \quad = \quad \frac{1050}{63.333} + (-15) + 0 \quad$ =1.579 mm

$\frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} \quad = \quad \frac{-750}{63.333} + 15 + \frac{(-20\times5)}{63.333} \quad$ = 1.579 mm

$\frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} \quad = 0 + 0 + \frac{(20\times5)}{63.333} \quad$ =1.579mm

Substitute $\beta_1, \beta_2, \beta_3, \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r}, \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r}, \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r}, \gamma_1, \gamma_2, \gamma_3$ and A values in equations no 10, we get,

$$[B]=\frac{1}{2\times150}\begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 1.579 & 0 & 1.579 & 0 & 1.579 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -15 & -20 & 15 & 20 & 0 \end{bmatrix}$$

$$[B]=3.333 \times 10^{-3}\begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 1.579 & 0 & 1.579 & 0 & 1.579 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -15 & -20 & 15 & 20 & 0 \end{bmatrix}$$

$$[D][B] = 384.6153\times10^{3}$$
$$\begin{bmatrix} 0.7 & 0.3 & 0.3 & 0 \\ 0.3 & 0.7 & 0.3 & 0 \\ 0.3 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} \times 3.333 \times 10^{-3} \begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 1.579 & 0 & 1.579 & 0 & 1.579 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -15 & -20 & 15 & 20 & 0 \end{bmatrix}$$

$$[D][B] = 1.282\times10^{3}\begin{bmatrix} -10.0263 & 0 & 10.9737 & -6 & 0.4737 & 6 \\ -3.3947 & 0 & 5.6053 & -6 & 1.1053 & 6 \\ -4.0263 & 0 & 4.9737 & -14 & 0.4737 & 14 \\ 0 & -3 & -4 & 3 & 4 & 0 \end{bmatrix}$$

We know that

$$[B]=3.333 \times 10^{-3}\begin{bmatrix} -15 & 0 & 15 & 0 & 0 & 0 \\ 1.579 & 0 & 1.579 & 0 & 1.579 & 0 \\ 0 & 0 & 0 & -20 & 0 & 20 \\ 0 & -15 & -20 & 15 & 20 & 0 \end{bmatrix}$$

$$[B]^{T} = 3.333 \times 10^{-3}\begin{bmatrix} -15 & 1.579 & 0 & 0 \\ 0 & 0 & 0 & -15 \\ 15 & 1.579 & 0 & -20 \\ 0 & 0 & -20 & 15 \\ 0 & 1.579 & 0 & 20 \\ 0 & 0 & 20 & 0 \end{bmatrix}$$

$$[D][B][B]^T = 1.282\times10^{3}\begin{bmatrix} -10.0263 & 0 & 10.9737 & -6 & 0.4737 & 6 \\ -3.3947 & 0 & 5.6053 & -6 & 1.1053 & 6 \\ -4.0263 & 0 & 4.9737 & -14 & 0.4737 & 14 \\ 0 & -3 & -4 & 3 & 4 & 0 \end{bmatrix} 3.333\times10^{-3} \begin{bmatrix} -15 & 1.579 & 0 & 0 \\ 0 & 0 & 0 & -15 \\ 15 & 1.579 & 0 & -20 \\ 0 & 0 & -20 & 15 \\ 0 & 1.579 & 0 & 20 \\ 0 & 0 & 20 & 0 \end{bmatrix}$$

$$[D][B][B]^T = 4.2733\begin{bmatrix} 145.034 & 0 & -155.755 & 80.526 & -5.360 & -80.526 \\ 0 & 45 & 60 & -45 & -60 & 0 \\ -155.755 & 60 & 253.456 & -159.474 & -71.149 & 99.474 \\ 80.526 & -45 & -159.474 & 325 & 50.256 & -280 \\ -5.360 & -60 & -71.149 & 50.526 & 81.745 & 9.474 \\ -80.526 & 0 & 99.474 & -280 & 9.474 & 280 \end{bmatrix}$$

Substitute $[D][B][B]^T$ value in equ no 8

$$[K]_2 = 2\pi\times63.333\times150\times4.2733\times\begin{bmatrix} 145.034 & 0 & -155.755 & 80.526 & -5.360 & -80.526 \\ 0 & 45 & 60 & -45 & -60 & 0 \\ -155.755 & 60 & 253.456 & -159.474 & -71.149 & 99.474 \\ 80.526 & -45 & -159.474 & 325 & 50.256 & -280 \\ -5.360 & -60 & -71.149 & 50.526 & 81.745 & 9.474 \\ -80.526 & 0 & 99.474 & -280 & 9.474 & 280 \end{bmatrix}$$

$$[K]_2 = 255.074\text{X}10^{3}\begin{bmatrix} 145.034 & 0 & -155.755 & 80.526 & -5.360 & -80.526 \\ 0 & 45 & 60 & -45 & -60 & 0 \\ -155.755 & 60 & 253.456 & -159.474 & -71.149 & 99.474 \\ 80.526 & -45 & -159.474 & 325 & 50.256 & -280 \\ -5.360 & -60 & -71.149 & 50.526 & 81.745 & 9.474 \\ -80.526 & 0 & 99.474 & -280 & 9.474 & 280 \end{bmatrix}$$

$$[K]_2 = 10^{6}\begin{bmatrix} 36.994 & 0 & -39.729 & 20.540 & -1.367 & -20.540 \\ 0 & 11.478 & 15.304 & -11.478 & -15.304 & 0 \\ -39.729 & 15.304 & 64.650 & -40.678 & -18.148 & 25.373 \\ 20.540 & -11.478 & -40.678 & 82.899 & 12.877 & -71.421 \\ -1.367 & -15.304 & -18.148 & 12.877 & 20.851 & 2.417 \\ -20.540 & 0 & 25.373 & -71.421 & 2.417 & 71.421 \end{bmatrix}$$

Assemble the equations.

Global stiffness matrix, [K] =

51.076 +0	-31.817 +0	-19.573 +0	18.124+0	0	0	-35.448+0	13.693+0
-31.817 +0	74.173 +0	16.110 +0	- 63.903+0	0	0	22.957+0	-10.270+0
-19.573 +0	16.110 +0	18.755+ 36.994	-2.416 + 0	-39.729 +0	20.540 +0	2.310- 1.367	-13.693 -20.540
18.124 +0	-63.903 +0	-2.416 +0	63.903 +11.478	0+15.304	0 -11.478	-22.957 -15.304	0+0
0	0	0+ (-39.729)	0+ 15.304	0+64.650	0+ (-40.678)	0+ (-18.148)	0+25.373
0	0	20.540+0	- 11.478+0	- 40.678+0	82.899+0	12.887+0	-71.421+0
-35.448 +0	22.597+0	2.310- 1.367	-22.597 -15.304	0 -18.148	0+12.887	40.068+20 .851	0+2.417
13.693+ 0	-10.270+0	-13.693 -20.540	0+0	0+25.373	0-71.421	0+2.417	10.270 +71.421

Global stiffness matrix, [K] =

51.076	-31.817	-19.573	18.124	0	0	-35.448	13.693
-31.817	74.173	16.110	-63.903	0	0	22.957	-10.270
-19.573	16.110	55.749	-2.416	-39.729	20.540	0.943	-34.233
18.124	-63.903	-2.416	75.381	15.304	-11.478	-38.261	0
0	0	(-39.729)	15.304	64.650	-40.678	-18.148	25.373
0	0	20.540	-11.478	-40.678	82.899	12.887	-71.421
-35.448	22.597	0.943	-38.261	18.148	12.887	60.919	2.417
13.693	-10.270	-34.233	0	25.373	71.421	2.417	81.691

We know that

$$\{F\}=[K]\,\{U\}$$

$$\begin{Bmatrix} F_{1u} \\ F_{2u} \\ F_{3u} \\ F_{4u} \\ F_{5u} \\ F_{6u} \end{Bmatrix} = 10^{6} \begin{bmatrix} 51.076 & -31.817 & -19.573 & 18.124 & 0 & 0 & -35.448 & 13.693 \\ -31.817 & 74.173 & 16.110 & -63.903 & 0 & 0 & 22.957 & -10.270 \\ -19.5573 & 16.110 & 55.759 & -2.416 & -39.729 & 20.540 & 0.943 & -34.233 \\ 18.124 & -63.903 & -2.416 & 75.381 & 15.304 & -11.478 & -38.261 & 0 \\ 0 & 0 & -39.729 & 15.304 & 64.650 & -40.678 & -18.148 & 25.373 \\ 0 & 0 & 20.540 & -11.478 & -40.678 & 82.899 & 12.887 & -71.421 \\ -35.448 & 22.957 & 0.943 & -38.261 & -18.148 & 12.887 & 60.919 & 2.417 \\ 13.693 & -10.270 & -34.233 & 0 & 25.373 & -71.421 & 2.417 & 81.691 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \end{Bmatrix}$$

Forces we know that

$$F_{1u} = F_{2u} = \frac{2\,P\pi r_{el_e}}{2} = \frac{2\times\pi\times50\times15\times4}{2} = 9424.77 \text{ N}$$

The remaining forces are zero F_{1w}, F_{2w}, F_{3u}, F_{3w}, $F4_{w}$, are zero.

Displacements

1. Node 1 is moving in r direction. $u_1 \neq 0$ but $w_1 = 0$
2. Node 2 is moving in r direction. $u_2 \neq 0$ but $w_2 = 0$
3. Node 3 & 4 are fixed. So u_3, w_3 u_4 and w_4 are zero.

Substitute nodal force and nodal displacements values in eqn 12

$$\begin{Bmatrix} 9424.77 \\ 0 \\ 9424.77 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = 10^6 \begin{bmatrix} 51.076 & -31.817 & -19.573 & 18.124 & 0 & 0 & -35.448 & 13.693 \\ -31.817 & 74.173 & 16.110 & -63.903 & 0 & 0 & 22.957 & -10.270 \\ -19.5573 & 16.110 & 55.759 & -2.416 & -39.729 & 20.540 & 0.943 & -34.233 \\ 18.124 & -63.903 & -2.416 & 75.381 & 15.304 & -11.478 & -38.261 & 0 \\ 0 & 0 & -39.729 & 15.304 & 64.650 & -40.678 & -18.148 & 25.373 \\ 0 & 0 & 20.540 & -11.478 & -40.678 & 82.899 & 12.887 & -71.421 \\ -35.448 & 22.957 & 0.943 & -38.261 & -18.148 & 12.887 & 60.919 & 2.417 \\ 13.693 & -10.270 & -34.233 & 0 & 25.373 & -71.421 & 2.417 & 81.691 \end{bmatrix} \times \begin{Bmatrix} u_1 \\ 0 \\ u_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Delete second row, second column, fourth row, fourth column, fifth row, fifth column, sixth row, sixth column, seventh row, seventh column, and eighth row and eight column of the above matrix. Hence the Equation reduces to

$$\begin{Bmatrix} 9424.77 \\ 9424.77 \end{Bmatrix} = 10^6 \begin{bmatrix} 51.706 & -19.5573 \\ -19.5573 & 55.759 \end{bmatrix} \text{ x } \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$9424.77 = 10^6\ (51.706u_1 - 19.573u_2)$$

$$9424.77 = 10^6\ (-19.573u_1 - 55.749u_2)$$

Above equations we solving and we get

$$u_1 = 2.88\times10^{-4}\text{mm} \qquad u_2 = 2.70\times10^{-4}\text{mm}$$

RESULTS

DISPLACEMENTS

$u_1 = 2.88\times10^{-4}$mm	$w_1=0$
$u_2 = 2.70\times10^{-4}$mm	$w_2=0$
$u_3 = 0$	$w_3=0$
$u_4 = 0$	$w_4=0$

5. **DERIVE THE EXPRESSION FOR STRAIN-DISPLACEMENT RELATIONSHIP FOR AXISYMMETRIC ELEMENT.**

Shape function are given below

$$U = N_1u_1+N_2u_2+N_3u_3 \quad \text{--------------------------- } 1$$

$$W = N_1w_1+N_2w_2 +N_3w_3 \text{ --------------------- } 2$$

Radial strain $e_r = \frac{\partial u}{\partial r}$

Eqn 1 d.w.r to "r "

$$e_r = \frac{\partial u}{\partial r} = \frac{\partial N_1}{\partial r}u_1+\frac{\partial N_2}{\partial r}u_2 +\frac{\partial N_3}{\partial r}u_3 \text{ ------------------- } 3$$

Circumferential strain $e_\Theta = \frac{u}{r}$

$$e_\Theta = \frac{N_1}{r}u_1 +\frac{N_2}{r}u_2 + \frac{N_3}{r}u_3 \text{ --------- } 4$$

Longitudinal strain $e_z = \frac{\partial w}{\partial z}$

$$e_z = \frac{\partial N_1}{\partial z}w_1 + \frac{\partial N_2}{\partial z}w_2 + \frac{\partial N_3}{\partial z}w_3 \text{ ---------- } 5$$

Shear strain $\Upsilon_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$

$$\Upsilon_{rz} = \frac{\partial N_1}{\partial z}u_1 + \frac{\partial N_2}{\partial z}u_2 + \frac{\partial N_3}{\partial z}u_3 + \frac{\partial N_1}{\partial r}w_1+\frac{\partial N_2}{\partial r}w_2 +\frac{\partial N_3}{\partial r}w_3 \text{ ------ } 6$$

Arranging equation 3, 4, 5 & 6 in matrix form

$$\begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix} \text{ ------------- } 7$$

Shape function

$$N_1 = \frac{1}{2A}[\alpha_1 + \beta_1 r + \gamma_1 z] \quad ;$$

$$N_2 = \frac{1}{2A}[\alpha_2 + \beta_2 r + \gamma_2 z] \quad ;$$

$$N_3 = \frac{1}{2A}[\alpha_3 + \beta_3 r + \gamma_3 z];$$

$$\frac{\partial N_1}{\partial r} = \frac{\beta_1}{2A}$$

$$\frac{N_1}{r} = \frac{1}{2A}\left[\frac{\propto_1}{r} + \beta_1 + \frac{\gamma_1 z}{r}\right]$$

$$\frac{\partial N_1}{\partial z} = \frac{\gamma_1}{2A}$$

$$\frac{\partial N_2}{\partial r} = \frac{\beta_2}{2A}$$

$$\frac{N_2}{r} = \frac{1}{2A}\left[\frac{\propto_2}{r} + \beta_2 + \frac{\gamma_2 z}{r}\right]$$

$$\frac{\partial N_2}{\partial z} = \frac{\gamma_2}{2A}$$

$$\frac{\partial N_3}{\partial r} = \frac{\beta_3}{2A}$$

$$\frac{N_3}{r} = \frac{1}{2A}\left[\frac{\propto_3}{r} + \beta_3 + \frac{\gamma_3 z}{r}\right]$$

$$\frac{\partial N_3}{\partial z} = \frac{\gamma_3}{2A}$$

Above values substitute in eqn 7

$$\begin{Bmatrix} e_r \\ e_\theta \\ e_z \\ \gamma_{rz} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & b_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta 1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \end{Bmatrix}$$

{e} = [B]{u}

$$\beta_1 = z_2 - z_3 \qquad \beta_2 = z_3 - z_1 \qquad \beta_3 = z_1 - z_2$$
$$\gamma_1 = r_3 - r_2 \qquad \gamma_2 = r_1 - r_3 \qquad \gamma_3 = r_2 - r_1$$
$$\alpha_1 = r_2 z_3 - r_3 z_2 \qquad \alpha_2 = r_3 z_1 - r_1 z_3 \qquad \alpha_3 = r_1 z_2 - r_2 z_1$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & b_3 & 0 \\ \frac{\alpha_1}{r} + \beta_1 + \frac{\gamma_1 z}{r} & 0 & \frac{\alpha_2}{r} + \beta_2 + \frac{\gamma_2 z}{r} & 0 & \frac{\alpha_3}{r} + \beta_3 + \frac{\gamma_3 z}{r} & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta 1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}$$