Wave equation and their solution

From equation 4.24 we can write the Maxwell's equations in the differential form as

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$
$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\nabla\cdot\vec{D} = \vec{\rho}$$
$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} \qquad (4.29(a))$$
$$\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \qquad (4.29(b))$$
$$\nabla\cdot\vec{E} = 0 \qquad (4.29(c))$$
$$\nabla\cdot\vec{H} = 0 \qquad (4.29(d))$$

Let us consider a source free uniform medium having dielectric constant , magnetic permeability μ and conductivity . The above set of equations can be written as

Using the vector identity ,

$$\nabla\times\nabla\times\vec{A} = \nabla\cdot(\nabla\cdot\vec{A}) - \nabla^2 A$$

We can write from 4.29(b)

$$\nabla\times\nabla\times\vec{E} = \nabla\cdot(\nabla\cdot\vec{E}) - \nabla^2\vec{E}$$
$$= -\nabla\times\left(\mu\frac{\partial\vec{H}}{\partial t}\right)$$

or $$\nabla\cdot(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}(\nabla\times\vec{H})$$

Substituting $\nabla\times\vec{H}$ from 4.29(a)

$$\nabla\cdot\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}=-\mu\frac{\partial}{\partial t}\left(\sigma\vec{E}+\varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

But in source free medium $\nabla\cdot\vec{E}=0$ (eqn 4.29(c))

$$\nabla^2\vec{E}=\mu\sigma\frac{\partial\vec{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad (4.30)$$

In the same manner for equation eqn 4.29(a)

$$\nabla\times\nabla\times\vec{H}=\nabla\cdot\left(\nabla\cdot\vec{H}\right)-\nabla^2\vec{H}$$

$$=\sigma\left(\nabla\times\vec{E}\right)+\varepsilon\frac{\partial}{\partial t}\left(\nabla\times\vec{E}\right)$$

$$=\sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)+\varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Since $\nabla\cdot\vec{H}=0$ from eqn 4.29(d), we can write

$$\nabla^2\vec{H}=\mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right)+\mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right) \quad (4.31)$$

These two equations

$$\nabla^2\vec{E}=\mu\sigma\frac{\partial\vec{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla^2\vec{H}=\mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right)+\mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right)$$

are known as wave equations.

It may be noted that the field components are functions of both space and time. Forexample, if we consider a Cartesian co ordinate system, $\vec{E}$ and $\vec{H}$ essentially represents $\vec{E}(x,y,z,t)$ and $\vec{H}(x,y,z,t)$. For simplicity, we consider propagation in free space $\sigma=0$, $\mu=\mu_0$, $\varepsilon=\varepsilon_0$

The wave eqn in equations 4.30 and 4.31 reduces to

$$\nabla^2 \vec{E} = \mu_0 \varepsilon_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right) \qquad (4.32(a))$$

$$\nabla^2 \vec{H} = \mu_0 \varepsilon_0 \left(\frac{\partial^2 \vec{H}}{\partial t^2} \right) \qquad (4.32(b))$$

Further simplifications can be made if we consider in Cartesian co ordinate system aspecial case where $\vec{E}$ and $\vec{H}$ are considered to be independent in two dimensions, say $\vec{E}$ and $\vec{H}$ are assumed to be independent of *y* and *z*. Such waves are called plane waves.

From eqn (4.32 (a)) we can write

$$\frac{\partial^2 \vec{E}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E}}{\partial t^2} \right)$$

The vector wave equation is equivalent to the three scalarequations

$$\frac{\partial^2 \vec{E_x}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_x}}{\partial t^2} \right) \qquad (4.33(a))$$

$$\frac{\partial^2 \vec{E_y}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_y}}{\partial t^2} \right) \qquad (4.33(b))$$

$$\frac{\partial^2 \vec{E_z}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \vec{E_z}}{\partial t^2} \right) \qquad (4.33(c))$$

Since we have ,

$$\nabla \cdot \vec{E} = 0$$

$$\therefore \frac{\partial \vec{E_x}}{\partial x} + \frac{\partial \vec{E_y}}{\partial y} + \frac{\partial \vec{E_z}}{\partial z} = 0 \qquad (4.34)$$

As we have assumed that the field components are independent of y and z eqn(4.34)reduces to $\frac{\partial E_x}{\partial x} = 0$

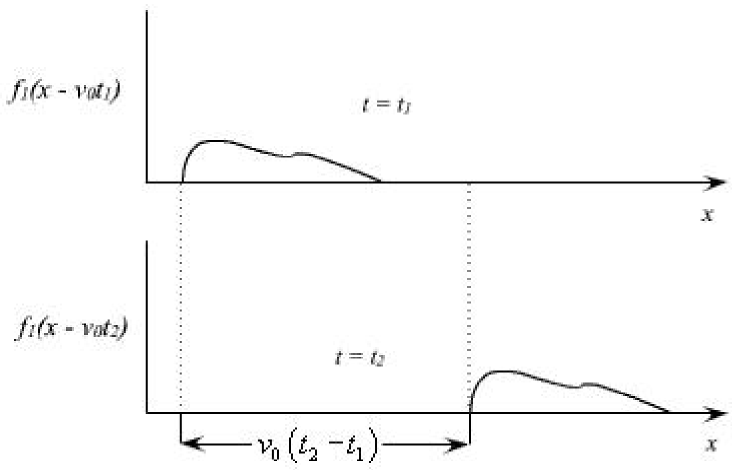

Fig 4.1 : Traveling wave in the + x direction

(www.brainkart.com/subject/Electromagnetic-Theory_206/)

A field component satisfying either of the last two conditions (i.e (ii) and (iii))is not a part of a plane wave motion and hence E_X is taken to be equal to zero. Therefore, a uniform plane wave propagating in x direction does not have a field component (*E* or *H*) acting along x as shown in figure 4.1.

Without loss of generality let us now consider a plane wave having E_y component only(Identical results can be obtained for E_Z component) .

The equation involving such wave propagation is given by

$$\frac{\partial^2 \overrightarrow{E_y}}{\partial x^2} = \varepsilon_0 \mu_0 \left(\frac{\partial^2 \overrightarrow{E_y}}{\partial t^2} \right) \qquad (4.36)$$

The above equation has a solution of the form

$$E_y = f_1(x - v_0 t) + f_2(x + v_0 t) \qquad (4.37)$$

where $v_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

Thus equation (4.37) satisfies wave eqn (4.36) can be verified by substitution.

$f_1(x - v_0 t)$ corresponds to the wave traveling in the + x direction while $f_2(x + v_0 t)$

corresponds to a wave traveling in the -x direction. The general solution of the wave eqn thus consists of two waves, one traveling away from the source and other traveling back towards the source. In the absence of any reflection, the second form of the eqn (4.37) is zero and the solution can be written as

$$E_y = f_1(x - v_0 t) \quad (4.38)$$

Such a wave motion is graphically shown in fig 4.4 at two instances of time t_1 and t_2.

Let us now consider the relationship between E and H components for the forward traveling wave.

Since $\vec{E} = \hat{a}_y E_y = \hat{a}_y f_1(x - v_0 t)$ and there is no variation along y and z.

$$\nabla \times \vec{E} = \hat{a}_z \frac{\partial E_y}{\partial x}$$

Since only z component of $\nabla \times \vec{E}$ exists, from (4.29(b))

$$\frac{\partial E_y}{\partial x} = -\mu_0 \frac{\partial H_z}{\partial t} \quad (4.39)$$

and from (4.29(a)) with $\vec{J} = 0$, only H_z component of magnetic field being present

$$\nabla \times \vec{H} = -\hat{a}_y \frac{\partial H_z}{\partial x}$$

$$\therefore -\frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{\partial E_y}{\partial t}$$

Substituting Eq (4.38)

$$\therefore H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} \cdot \int f_1'(x - v_0 t) dx + c$$

$$\frac{\partial H_z}{\partial x} = -\varepsilon_0 \frac{\partial E_y}{\partial t} = \varepsilon_0 v_0 f_1'(x - v_0 t)$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} \int \frac{\partial}{\partial x} f_1 dx + c$$

$$\therefore \frac{\partial H_z}{\partial x} = \varepsilon_0 \frac{1}{\sqrt{\mu_0 \varepsilon_0}} f_1'(x - v_0 t)$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} f_1 + c \quad (4.40)$$

The constant of integration means that a field independent of x may also exist. However,this field will not be a part of the wave motion.

Hence $$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y \qquad (4.41)$$

$$H_z = \sqrt{\frac{\varepsilon_0}{\mu_0}} E_y + c$$

which relates the *E* and *H* components of the traveling wave.

$$z_0 = \frac{E_y}{H_z} = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi \ \text{or} \ 377\Omega$$

$z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$ is called the characteristic or intrinsic impedance of the free space

Harmonic fields

In the previous section we introduced the equations pertaining to wave propagation and discussed how the wave equations are modified for time harmonic case. In this section we discuss in detail a particular form of electromagnetic wave propagation called'plane waves'.

The Helmhotz Equation:

In source free linear isotropic medium, Maxwell equations in phasor form are,

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad \nabla \times \vec{E} = 0$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} \qquad \nabla \times \vec{H} = 0$$

or,

$$\therefore \nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \times \vec{E}\right) - \nabla^2 \vec{E} = -j\omega\mu\nabla \times \vec{H}$$

$$-\nabla^2 \vec{E} = -j\omega\mu(j\omega\varepsilon\vec{E})$$

or,

$$\nabla^2 \vec{E} + \omega^2 \mu\varepsilon\vec{E} = 0$$

or, $$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$ where $$k = \omega\sqrt{k\varepsilon}$$

An identical equation can be derived for $\overrightarrow{H}$.

i.e., $\nabla^2 \overrightarrow{H} + k^2 \overrightarrow{H} = 0$

These equations

$$\left.\begin{aligned} &\nabla^2 \overrightarrow{E} + k^2 \overrightarrow{E} = 0 \ldots\ldots\ldots\ldots\ldots (a) \\ \& \quad &\nabla^2 \overrightarrow{H} + k^2 \overrightarrow{H} = 0 \ldots\ldots\ldots\ldots\ldots (b) \end{aligned}\right\} \ldots\ldots \quad (4.42)$$

are called homogeneous vector Helmholtz's equation.

$k = \omega\sqrt{\mu\varepsilon}$ is called the wave number or propagation constant of the medium.