3.2 Methods of Dimensions Analysis

If the number of variables involved in a physical phenomenon are known, then the relation among the variables can be determined by the following two methods.

- Reyleigh's method
- Buckingham's Pi-theorem

Reyleigh's method

This method is used for determining the expression for a variable which depends upon maximum three or four variables only. If the number of independent variables is more than five then it becomes difficult to find expression for dependent variable.

Let $X_1, X_2, X_3 --------X_n$ are the variables involved in a physical problem. Let X_1 be the dependent variable and $X_2, X_3 --------X_n$ are independent variable upon which X_1 depends.

$$X_1 = f(X_2, X_3 --------X_n)$$

i.e $f_1(X_1, X_2, X_3 --------X_n) = 0$ -------------------------- (i)

Where K is a constant and a,b,c are the arbitrary powers

Buckingham's Pi-theorem

If there are n variables (independent and dependent) in a physical phenomenon and these variables contain m fundamental dimensions (M,L,T) then the variables are arranged into (n-m) dimensionless terms. Each term is called π term.

Let X_1, X_2,X_3 – – – – – – – – –X_n , , are the variables involved in a physical problem. Let X1 be the dependent variable and X_2, X_3 – – – – – – – – –X_n , are independent variable upon which X_1 depends.

$$X_1 = f(X_2, X_3 --------X_n)$$

i.e $f_1(X_1, X_2, X_3 --------X_n) = 0$ -------------------------- (i)

Equation (i) is dimensionally homogeneous equation. It contains n variables. If there are m fundamental dimensions then according to Buckingham' π – Theorem, eqn.(i) can be written in terms of number of dimensionless groups or π – terms in which number of π – terms is equal to (n-m). Hence eqn.(i) becomes

$f(\pi_1, \pi_2, \pi_3 --------\pi_{n-m}) = 0$ ------------------------------------ (ii)

Each π term is dimensionless and independent of the system. Division or multiplication by a constant does not change the character of the π – term. Each π – term contains m+1 variables, where m is number of fundamental dimensions and is also called repeating variables. Let in the above case X_2, X_3 ,X_4 are repeating variables if fundamental dimension m (M, L, T) = 3 then each π – term is written as

$$\pi_1 = X_2^{a_1} X_3^{b_1} X_4^{c_1} X_1$$
$$\pi_2 = X_2^{a_2} X_3^{b_2} X_4^{c_2} X_1$$

$$\pi_{n-m} = X_2^{a_{n-m}} X_3^{b_{n-m}} X_4^{c_{n-m}} X_1 \text{------------------------- (iii)}$$

Each term is solved by the principle of dimensional homogeneity and values of a_1 ,b_1 ,c_1 etc are obtained. These values are substituted in the eqn. (iii) and values of π_1 π_2 π_3 – – – – – – – π_{n-m} , , are obtained. These values are substituted in eqn. (ii). The final equation for the phenomenon is obtained by expressing any one of the π – terms as a function of others as

$$\pi_1 = \phi(\pi_2, \pi_3 \text{--------} \pi_{n-m}) = 0$$
$$\pi_2 = \phi(\pi_1, \pi_3 \text{--------} \pi_{n-m}) = 0$$

Method of selecting repeating variable:

1. As far as possible dependent variable should not be selected as repeating variable.

2. Repeating variables should be selected in such a way that one variable contains geometric property (such as length l , diameter d, height H etc), other variable contains flow properties (such as velocity, acceleration etc.) and the third variable contains fluid properties (such as viscosity, density etc)

3. Selected repeating variable should not form dimensionless group.

4. Repeating variables together must have same number of fundamental dimensions.

5. No two repeating variables should have the same dimension. For most of the fluid mechanics problems the choice for the repeating variable may be
(i) d,γ ,ρ (ii) l,γ ,ρ (iii) l,γ ,μ (iv) d,γ ,μ

PROBLEM 1: A partially submerged body is towed in water. The resistance R to its motion depends on the density ρ , viscosity μ of water, length L of the body, velocity V of the body and acceleration g due to gravity. Show that the resistance to the motion can be expressed in the form of

$$R = \rho L^2 V^2 \phi\left[\left(\frac{\mu}{\rho V L}\right) \cdot \left(\frac{lg}{V^2}\right)\right].$$

Soln. The resistance R depends on ρ,μ,L,V, g

$$R = K\rho^a \cdot \mu^b \cdot l^c \cdot V^d \cdot g^e \quad \text{...}(i)$$

Substituting the dimensions on both sides of the equation (*i*),

$$MLT^{-2} = K(ML^{-3})^a \cdot (ML^{-1}T^{-1})^b \cdot L^c \cdot (LT^{-1})^d \cdot (LT^{-2})^e$$

Equating the powers of *M*, *L*, *T* on both sides

Power of *M*, $1 = a + b$

Power of *L*, $1 = -3a - b + c + d + e$

Power of *T*, $-2 = -b - d - 2e.$

There are 5 unknowns and 3 equations. Expressing the three unknowns in terms of two unknowns (μ and g) . Hence express a, c and d in terms of b and e. Solving we get

$$a = 1 - b$$
$$d = 2 - b - 2e$$
$$c = 1 + 3a + b - d - e = 1 + 3(1 - b) + b - (2 - b - 2e) - e$$
$$= 1 + 3 - 3b + b - 2 + b + 2e - e = 2 - b + e.$$

Substituting these values in equation (*i*), we get

$$R = K\rho^{1-b} . \mu^{b} . l^{2-b+e} . V^{2-b-2e} . g^{e}$$
$$= K\rho l^{2} . V^{2} . (\rho^{-b} \mu^{b} l^{-b} V^{-b}) . (l^{e} . V^{-2e} . g^{e})$$
$$= K\rho l^{2} V^{2} . \left(\frac{\mu}{\rho V l}\right)^{b} . \left(\frac{lg}{V^{2}}\right)^{e}$$
$$= \mathbf{K\rho l^{2} V^{2} \phi\left[\left(\frac{\mu}{\rho V l}\right) . \left(\frac{lg}{V^{2}}\right)\right].} \quad \textbf{Ans.}$$

PROBLEM 2: The resisting force R of a supersonic plane during flight can be considered as dependent upon the length of the aircraft L, velocity V, air viscosity μ, air density , and bulk modulus of air k. Express the functional relationship between the variables and the resisting force.

Solution. The resisting force R depends upon

(*i*) density, l, (*ii*) velocity, V,
(*iii*) viscosity, μ, (*iv*) density, ρ,
(*v*) Bulk modulus, K.

$$\therefore \quad R = A l^{a} . V^{b} . \mu^{c} . \rho^{d} . K^{e} \quad ...(i)$$

Substituting the dimensions on both sides of the equation (*i*),

$$MLT^{-2} = AL^{a} . (LT^{-1})^{b} . (ML^{-1}T^{-1})^{c} . (ML^{-3})^{d} . (ML^{-1}T^{-2})^{e}$$

Equating the powers of M, L, T on both sides,

Power of M, $1 = c + d + e$
Power of L, $1 = a + b - c - 3d - e$
Power of T, $-2 = -b - c - 2e.$

There are five unknowns but equations are only three. Expressing the three unknowns in terms of two unknowns (μ and K).

∴ Express the values of a, b and d in terms of c and e.

Solving,
$$d = 1 - c - e$$
$$b = 2 - c - 2e$$
$$a = 1 - b + c + 3d + e = 1 - (2 - c - 2e) + c + 3(1 - c - e) + e$$
$$= 1 - 2 + c + 2e + c + 3 - 3c - 3e + e = 2 - c.$$

Substituting these values in (*i*), we get

$$R = A\, l^{2-c} . V^{2-c-2e} . \mu^{c} . \rho^{1-c-e} . K^{e}$$

$$= A\, l^2 . V^2 . \rho(l^{-c} V^{-c} \mu^c \rho^{-c}) . (V^{-2e} . \rho^{-e} . K^e)$$

$$= A\, l^2 V^2 \rho \left(\frac{\mu}{\rho V L}\right)^c \cdot \left(\frac{K}{\rho V^2}\right)^e$$

PROBLEM 3:Using Buckingham' π – Theorem show that velocity through circular orifice is given by

$$V = \sqrt{2gH}\, \phi\left(\frac{D}{H}, \frac{\mu}{\rho V H}\right),$$

where H is head causing flow, D is diameter of the orifice, μ is coefficient viscosity, ρ is mass density and g is acceleration due to gravity

Solution. Given :

V is a function of H, D, μ, ρ and g

$\therefore \quad V = f(H, D, \mu, \rho, g)$ or $f_1(V, H, D, \mu, \rho, g) = 0$

$\therefore$ Total number of variable, $n = 6$ ...(*i*)

Writing dimension of each variable, we have

$$V = LT^{-1}, H = L, D = L, \mu = ML^{-1}T^{-1}, \rho = ML^{-3}, g = LT^{-2}.$$

Thus number of fundamental dimensions, $m = 3$

$\therefore$ Number of π-terms $= n - m = 6 - 3 = 3.$

Equation (*i*) can be written as $f_1(\pi_1, \pi_2, \pi_3) = 0$ ...(*ii*)

Each π-term contains $m + 1$ variables, where $m = 3$ and is also equal to repeating variables. Here V is a dependent variable and hence should not be selected as repeating variable. Choosing H, g, ρ as repeating variable, we get three π-terms as

$$\pi_1 = H^{a_1} . g^{b_1} . \rho^{c_1} . V$$

$$\pi_2 = H^{a_2} . g^{b_2} . \rho^{c_2} . D$$

$$\pi_3 = H^{a_3} . g^{b_3} . \rho^{c_3} . \mu$$

First π-term $\quad \pi_1 = H^{a_1} . g^{b_1} . \rho^{c_1} . V$

Substituting dimensions on both sides

$$M^0L^0T^0 = L^{a_1} . (LT^{-2})^{b_1} . (MT^{-3})^{c_1} . (LT^{-1})$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c, \quad \therefore \quad c_1 = 0$

Power of L, $\quad 0 = a_1 + b_1 - 3c_1 + 1, \quad \therefore \quad a_1 = -b_1 + 3c_1 - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$

Power of T, $\quad 0 = -2b_1 - 1, \quad \therefore \quad b_1 = -\frac{1}{2}$

Substituting the values of a_1, b_1 and c_1 in π_1,

$$\pi_1 = H^{-\frac{1}{2}} . g^{-\frac{1}{2}} . \rho^0 . V = \frac{V}{\sqrt{gH}}.$$

Second π-term $\quad \pi_2 = H^{a_2} . g^{b_2} . \rho^{c_2} . D$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_2} . (LT^{-2})^{b_2} . (ML^{-3})^{c_2} . L$$

Equating the powers of M, L, T,

Power of M, $0 = c_2$ $\therefore$ $c_2 = 0$

Power of L, $0 = a_2 + b_2 - 3c_2 + 1$, $a_2 = -b_2 + 3c_2 - 1 = -1$

Power of T, $0 = -2b_2$, $\therefore$ $b_2 = 0$

Substituting the values of a_2, b_2, c_2 in π_2,

$$\pi_2 = H^{-1} \cdot g^0 \rho^0 \cdot D = \frac{D}{H}.$$

Third π-term $\pi_3 = H^{a_3} \cdot g^{b_3} \cdot \rho^{c_3} \cdot \mu$

Substituting the dimensions on both sides

$$M^0L^0T^0 = L^{a_3} \cdot (LT^{-2})^{b_3} \cdot (ML^{-3})^{c_3} \cdot ML^{-1}T^{-1}$$

Equating the powers of M, L, T on both sides

Power of M, $0 = c_3 + 1$, $\therefore$ $c_3 = -1$

Power of L, $0 = a_3 + b_3 - 3c_3 - 1$, $\therefore$ $a_3 = -b_3 + 3c_3 + 1 = \frac{1}{2} - 3 + 1 = -\frac{3}{2}$

Power of T, $0 = -2b_3 - 1$, $\therefore$ $b_3 = -\frac{1}{2}$

Substituting the values of a_3, b_3 and c_3 in π_3,

$$\pi_3 = H^{-3/2} \cdot g^{-1/2} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{H^{3/2}\rho\sqrt{g}}$$

$$= \frac{\mu}{H\rho\sqrt{gH}} = \frac{\mu V}{H\rho V\sqrt{gH}}$$ [Multiply and Divide by V]

$$= \frac{\mu}{H\rho V} \cdot \pi_1$$ $\left\{\because \frac{V}{\sqrt{gH}} = \pi_1\right\}$

Substituting the values of π_1, π_2 and π_3 in equation (*ii*),

$$f_1\left(\frac{V}{\sqrt{gH}}, \frac{D}{H}, \pi_1 \frac{\mu}{H\rho V}\right) = 0 \text{ or } \frac{V}{\sqrt{gH}} = \phi\left[\frac{D}{H}, \pi_1 \frac{\mu}{H\rho V}\right]$$

or $$V = \sqrt{2gH}\ \phi\left[\frac{D}{H}, \frac{\mu}{\rho VH}\right].$$ **Ans.**

Multiplying by a constant does not change the character of π-terms.