1.6.1. PROBLEMS UNDER BASIS

Let V be a vector space with $\dim(V) = n$. Then any basis of V contains n elements.

Let β be a set with cardinality(number of elements) $|\beta|$.

- If $|\beta| < n$ or $|\beta| > n$, then S does not form a basis of V.
- If β is a linearly independent set in V with $|\beta| = n$, then β forms a basis in V.

Example. Determine whether $(1,1,1), (1,0,1)$ forms a basis of R^3

Sol: Since $\dim(R^3) = 3$, any basis of R^3 contains three elements. Let $\beta = \{(1,1,1), (1,0,1)\}$. Since β contains two elements, β does not form a basis of R^3.

Example 80. Show that the sets of vectors

$\{(1,2,1), (3,1,5), (-1,0,1), (1,-1,2)\}$ do not form a basis for $V_3(R)$.

Sol: Since $\dim(V_3(R)) = 3$, any basis of $V_3(R)$ contains three elements.

Let $\beta = \{(1,2,1), (3,1,5), (-1,0,1), (1,-1,2)\}$. Since β contains four elements, does not form a basis of $V_3(R)$.

Example Verify the vectors $(1,-1,2), (1,-2,1), (1,1,4)$ in R° forms a basis of R^3.

Sol: Let $\beta = \{(1,-1,2), (1,-2,1), (1,1,4)\}$

$\dim(R^3) = 3$, which is finite.

In R^3, any independent set with three elements is a basis of R^3.

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & 4 \end{vmatrix}$$

$= 1(-8-1) + 1(4-1) + 2(1+2) = 0$

$\therefore \beta$ is a linearly dependent set in R^3.

$\therefore \beta$ does not form a basis of R^3.

Example. Verify the vectors $(1,2,0), (2,3,0), (8,13,0)$ of R^3 is a basis of $\boldsymbol{R}^3$

Sol: Let $\beta = \{(1,2,0), (2,3,0), (8,13,0)\}$

$\dim(R^3) = 3$, which is finite.

In R^3, any independent set with three elements is a basis of $\mathbb{R}^3$.

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 8 & 13 & 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 8 & 13 & 0 \end{vmatrix} = 0$$

$\therefore \beta$ is a linearly dependent set in R^3.

$\therefore \beta$ is not a basis of R^3

Example Verify the vectors $(2,1,0), (-3,-3,1), (-2,1,-1)$ in R^3 basis of R^3

Sol: Let $\beta = \{(2,1,0), (-3,-3,1), (-2,1,-1)\}$.

$\dim(R^3) = 3$, which is finite.

In R^3, any independent set with three elements is a basis of R^3.

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -3 & 1 \\ -2 & 1 & -1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ -3 & -3 & 1 \\ -2 & 1 & -1 \end{vmatrix} = -1 \neq 0$$

$\therefore \beta$ is a linearly independent set in R^3.

$\therefore \beta$ is a basis of R^3.

Example. Check whether the following are basis for the space R^3

(a) $\{(1,1,-1),(2,3,4),(4,1,-1),(0,1,-1)\}$

(b) $\{(1,1,-1),(0,3,4),(0,0,-1)\}$

(C) $\{(1,2,0),(0,1,-1)\}$

Sol:

$\dim(R^3) = 3$, which is finite.

In R^3, any independent set with three elements is a basis for R^3.

(a) $\beta = \{(1,1,-1),(2,3,4),(4,1,-1),(0,1,-1)\}$

Since β is contains four elements, it is not a basis for R^3.

(b) $\beta = \{(1,1,-1),(0,3,4),(0,0,-1)\}$

The set contains three elements

Let $v_1 = (1,1,-1), v_2 = (0,3,4), v_3 = (0,0,-1)$

To prove S is a basis we have to prove S is a linearly independent.

Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{bmatrix}$

$|A| = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{bmatrix} = -3 \neq 0$

$\therefore \beta$ is linearly independent in R^3

$\Rightarrow \beta$ is a basis in R^3

(C) $\beta = \{(1,2,0),(0,1,-1)\}$

Since the set contains two elements, it does not form a basis in R^3.

Example 85. Determine $\{1+2x+x^2, 3+x^2, x+x^2\}$ is a basis for $P_2(R)$. Sol:

$\dim P_2(R) = 3$, which is finite. In $P_2(R)$, any independent set with three elements is a basis.

Given $v_1 = 1 + 2x + x^2, v_2 = 3 + x^2, v_3 = x + x^2$

The vector equation is

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\alpha_1(1 + 2x + x^2) + \alpha_2(3 + x^2) + \alpha_3(x + x^2) = 0$$

$$(\alpha_1 + 3\alpha_2) + (2\alpha_1 + \alpha_3)x + (\alpha_1 + \alpha_2 + \alpha_3)x^2 = 0$$

Equating the like terms, we get

$$\alpha_1 + 3\alpha_2 = 0$$

$$2\alpha_1 + \alpha_3 = 0$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 0$$

Let A be the coefficients matrix,

$$\therefore A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = -4 \neq 0$$

the system of homogenous equations have only the trivial solution

$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$

$\therefore v_1, v_2, v_3$ are linearly independent

Hence v_1, v_2, v_3 is a basis of $P_2(R)$

Therefore $\{1 + 2x + x^2, 3 + x^2, x + x^2\}$ is a basis over R.

Example 86. Let $V = P_2(R)$ and. $\beta = \{1, 1 + x, 1 + x + x^2\}$. Check whether S forms a basis in V.

Sol: $\dim P_2(R) = 3$, which is finite.

In $P_2(R)$, any independent set with three elements is a basis.

Given $v_1 = 1, v_2 = 1 + x, v_3 = 1 + x + x^2$

The vector equation is

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\alpha_1(1) + \alpha_2(1 + x) + \alpha_3(1 + x + x^2) = 0$$

$$\alpha_3 + \alpha_1 + \alpha_2 + \alpha_2 x + \alpha_3 x + \alpha_3 x^2 = 0x^2 + 0x + 0$$

$$(\alpha_3 + \alpha_1 + \alpha_2) + (\alpha_2 + \alpha_3)x + \alpha_3 x^2 = 0x^2 + 0x + 0$$

Equating the like terms, we get

$\alpha_3 + \alpha_1 + \alpha_2 = 0$ ... (1)

$\alpha_2 + \alpha_3 = 0$ ... (2)

$\alpha_3 = 0$

(2) $\Rightarrow \alpha_2 = 0$

(1) $\Rightarrow \alpha_1 = 0$

$\therefore$ β is linearly independent set in $P_2(R)$,

Therefore β is a basis in $P_2(R)$,

Example 87. If the vectors $\{u, v, w\}$ form a basis for R^3, show that the vectors $\{u, u - w, u + v - 2w\}$ also forms a basis for R^3.

Sol: $\dim(R^3) = 3$, which is finite.

In R^3, any independent set with three elements is a basis for R^3.

Let $\beta = \{u, v, w\}$ and $\beta_1 = \{uu - w, u + v - 2w\}$

Given β forms a basic for R^3.

$\therefore$ β is a linearly independent set in R^3.

In a finite dimensional vector space, any two bases has same number of elements.

Also in a finite dimensional vector space, any independent set with number elements $\dim(V)$ is a basis.

To prove β_1 is a basis for R^3, it is enough to prove β_1 is a linearly independent set. The vector equation is

$\alpha_1 u + \alpha_2(u - w) + \alpha_3(u + v - 2w) = 0$

$\alpha_1 u + \alpha_2 u - \alpha_2 w + \alpha_3 u + \alpha_3 v - 2\alpha_3 w = 0$

$(\alpha_1 + \alpha_2 + \alpha_3)u + \alpha_3 v + (-\alpha_2 - 2\alpha_3)w = 0$

Since u, v and w are linearly independent,

$\alpha_1 + \alpha_2 + \alpha_3 = 0 \ldots\ldots\ldots\ldots$ (1)

$\alpha_3 = 0$

$\alpha_2 - 2\alpha_3 = 0 \ldots\ldots\ldots\ldots$ (2)

(2) $\Rightarrow -\alpha_2 - 2(0) = 0$

$\alpha_2 = 0$

(1)$\Rightarrow \alpha_1 = 0$

$\therefore \alpha_1 u + \alpha_2(u - w) + \alpha_3(u + v - 2w) = 0 \Rightarrow \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$

$\therefore \beta_1$ is a linearly independent set.

Hence β_1 is a basis of R^3.

$$= \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}$$

Equating the like terms, we get

$\alpha_1 = 2$

$\alpha_2 = 3$

$\alpha_3 = 4$

$\alpha_4 = -7$

The coordinate of A relative to the usual basis is $(2,3,4,-7)$.

1.6.2. PROBLEMS UNDER BASIS AND DIMENSION OF A SUBSPACE

Let W be a subspace of a vector space V over F. To find the basis dimension of :

- From W, find linear span of W. Let it be β.
- Check β is linearly independent or not.
- If β is linearly independent set, then β forms a basis in W.
- $\dim(W) = |\beta|$

Example 91. Find the dimension of the subspace W of the vector space R^3 over R if $W = \{(a,0,0)/a \in R\}$

Sol: Let $v \in W$. Then

$v = (a,0,0) = a(1,0,0)$

$\therefore \beta = \{(1,0,0)\}$ spans $\underline{W}$.

Any set with one element is linearly independent

$\therefore B$ is a linearly independent set in W.

$\therefore B = \{(1,0,0)\}$ is a basis of W.

$\therefore \dim(W) = 1$

Example 92. Find the dimension of the subspace W of the vector space R^3 over

R, if $W = \{(a_1, a_2, a_3)/(2a_1 - 7a_2 + a_3 = 0)\}$

Sol: $W = \{(a_1, a_2, a_3)/(2a_1 - 7a_2 + a_3 = 0)\}$

Given $2a_1 - 7a_2 + a_3 = 0$

$$\Rightarrow a_3 = -2a_1 + 7a_2$$

Let $v \in W$. Then

$v = (a_1, a_2, a_3)$

$(a_1, a_2, a_3) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$

$= a_1(1,0,0) + a_2(0,1,0) + (-2a_1 + 7a_2)(0,0,1)$

$= a_1(1,0,0) + a_2(0,1,0) - 2a_1(0,0,1) + 7a_2(0,0,1)$

$= (a_1, 0,0) + (0, a_2, 0) + (0,0, -2a_1) + (0,0,7a_2)$

$= (a_1, 0, -2a_1) + (0, a_2, 7a_2)$

$= a_1(1,0,-2) + a_2(0,1,7)$

$\therefore \beta = \{(1,0,-2), (0,1,7)\}$ spans W i.e., $L(\beta) = W$

Next we prove that B is a linearly independent set in W.

Consider the vector equation

$a_1v_1 + a_2v_2 = 0$

$a_1(1,0,-2) + a_2(0,1,7) = 0$

$(a_1, a_2, -2a_1 + 7a_2) = 0$

$\Rightarrow a_1 = a_2 = 0$

$\therefore \beta$ is a linearly independent set in W.

$\therefore \beta = \{(1,0,-2),(0,1,7)\}$ is a basis of W

Since the basis contains two elements, $\dim(W) = 2$

Example 93. Find the dimension of the subspace W of the vector space $F^!$ over F, if $W = \{(a_1, a_2, a_3, a_4, a_5)/a_1 - a_3 + a_4 = 0\}$

Sol: $W = \{(a_1, a_2, a_3, a_4, a_5)/a_1 - a_3 + a_4 = 0\}$

Given $a_1 - a_3 + a_4 = 0$

$$\Rightarrow a_4 = a_3 - a_1$$

Let $v \in W$. Then

$v = (a_1, a_2, a_3, a_4, a_5)$

$(a_1, a_2, a_3, a_4, a_5)$

$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0) + a_4(0,0,0,1,0)$

$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0) + (a_3 - a_1)(0,0,0,1,0) + a_5(0,0,0,0,1)$

$$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0) + a_3(0,0,0,1,0) - a_1(0,0,0,1,0)$$

$$+a_5(0,0,0,0,1)$$

$= a_1(1,0,0,-1,0) + a_2(0,1,0,0,0) + a_3(0,0,1,1,0) + a_5(0,0,0,0,1)$

$\therefore \beta = a_1(1,0,0,-1,0), (0,0,-1,0), (0,1,0,0,0), (0,0,1,1,0), (0,0,0,0,1)\}$ spans W

i.e., $L(\beta) = W$

Next we prove that β is a linearly independent set in W.

Consider the vector equation

$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$

$a_1(1,0,0,-1,0) + a_2(0,1,0,0,0) + a_3(0,0,1,1,0) + a_4(0,0,0,0,1) = 0$

$(a_1, a_2, a_3, -a_1 + a_3, a_4) = 0$

$\Rightarrow a_1 = a_2 = a_3 = a_4 = 0$

$\therefore \beta$ is a linearly independent set in W.

$\therefore \beta = \{(1,0,0,-1,0), (0,1,0,0,0), (0,0,1,1,0), (0,0,0,0,1)\}$ is a basis of W.

Since the basis contains four elements, $\dim(W) = 4$.

Example 94. Find the dimension of the subspace W of the vector space F^5

over$R$, if $W = \{(a_1, a_2, a_3, a_4, a_5)/a_2 = a_3 = a_4, a_1 + a_5 = 0\}$

Sol: $W = \{(a_1, a_2, a_3, a_4, a_5)/a_2 = a_3 = a_4 = 0, a_1 + a_5 = 0\}$

Given $a_1 + a_5 = 0$

$\Rightarrow a_5 = -a_1$

Also given $a_2 = a_3 = a_4$

$\therefore a_3 = a_2$ and $a_4 = a_2$

Let $v \in W$. Then

$v = (a_1, a_2, a_3, a_4, a_5)$

$(a_1, a_2, a_3, a_4, a_5)$

$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0) + a_3(0,0,0,1,0) + a_5(0,0,0,0,1)$

$$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_2(0,0,1,0,0) + a_2(0,0,0,1,0) - a_1(0,0,0,0,1) = a_1(1,0,0,0,-1) + a_2(0,1,1,1,0)$$

$\beta = \{(1,0,0,0,-1),(0,1,1,1,0)\}$ spans W

i.e., $L(\beta) = W$

Next we prove that β is a linearly independent set in W.

Consider the vector equation

$$a_1 v_1 + a_2 v_2 = 0$$

$$a_1(1,0,0,0,-1) + a_2(0,1,1,1,0) = 0$$

$$(a_1, a_2, a_2, a_2, -a_1) = 0$$

$$\Rightarrow a_1 = a_2 = 0$$

$\therefore \beta$ is a linearly independent set in W.

$\therefore \beta = \{(1,0,0,0,-1),(0,1,1,1,0)\}$ is a basis of W. Since the basis contains two elements, $\dim(W) = 2$

Example 95. Find the dimension of the subspace W of the vector space R^3 over R, if $W = \{(a,b,c): 2a + 3b = c; 7c + 9b = a\}$

Sol:

$$W = \{(a,b,c): 2a + 3b = c; 7c + 9b = a\}$$

Given

$$2a + 3b = c$$

$$2a + 3b - c = 0$$

Also given

$$7c + 9b = a$$

$a - 9b - 7c = 0 \ldots (2)$

Solve (1) and (2)

$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -9 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$

Let $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -9 & -7 \end{bmatrix}$

$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & -21 & -12 \end{bmatrix} R_2 \to R_2 - R_1$

$\begin{vmatrix} 2 & 3 \\ 0 & -21 \end{vmatrix} = -42 \neq 0$

$R(A) = 2 <$ the number of unknowns $= 3$

Therefore the system has an infinite number of solutions.

From the last row, we get

$-21b - 12c = 0$

$-21b = 12c$

$b = -\frac{4}{7}c$

Let $c = k$

$\therefore b = -\frac{4}{7}k$

From the first equation, we get

$2a + 3b - c = 0$

$2a - \frac{12}{7}k - k = 0$

$2a = \frac{19}{7}k$

$a = \frac{19}{14}k$

where k is a parameter

$W = \left\{\left(\frac{19}{14}k, -\frac{4}{7}k, k\right)\right\} : k \in R\Big\}$

$= \left\{\left(\frac{19}{14}, -\frac{4}{7}, 1\right)k\right\} : k \in R\Big\}$

$\therefore \beta = \left\{\left(\frac{19}{14}, -\frac{4}{7}, 1\right)\right\}$ spans W.

i.e., $L(\beta) = W$

Any set with one non vector is linearly independent

$\therefore \beta$ is a linearly independent set in W. $\therefore \beta = \left\{\left(\frac{19}{14}, -\frac{4}{7}, 1\right)\right\}$ is a basis of W.

Since the basis contains one element, $\dim(W) = 1$

Example(96) Find the dimension of the subspace W of the vector space $M_{2\times 2}(R)$ over R, if $W = \left\{\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} : \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}\right\} = \mathbf{0}\Big\}$

Sol: $W = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c + d\right\}$

Given

$a + b + c + d = 0$

$d = -a - b - c \ldots (1)$

Let $v \in W$. Then

$v = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-a-b-c)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - a\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

$\therefore \beta = \left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\right\}$ spans W.

i.e., $L(\beta) = W$

Next we prove that β is a linearly independent set in W.

Consider the vector equation

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$$

$$a_1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + a_2\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + a_3\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = 0$$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & -a_1 - a_2 & -a_3 \end{bmatrix} = 0$$

$$\Rightarrow a_1 = a_2 = a_3 = 0$$

$\therefore \beta$ is a a linearly independent set in W.

$\therefore \beta = \left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\right\}$ is a basis of W.

Since the basis contains three elements, $\dim(W) = 3$