EIGEN VALUES AND EIGEN VECTORS

Definition

The values of λ obtained from the characteristic equation $|A - \lambda I| = 0$ are called Eigenvalues of 'A'.[or Latent values of A or characteristic values of A]

Definition

Let A be square matrix of order 3 and λ be scaler. The column matrix $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ which satisfies $(A - \lambda I)X = 0$ is called Eigen vector or Latent vector or characteristic vector.

Example: Find the Eigen values for the matrix $\begin{pmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 2 + 3 + 2 = 7$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 + 3 + 4 = 11$$

$$s_3 = |A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 2(6-2) - 2(2-1) + 1(2-3)$$

$$= 8 - 2 - 1 = 5$$

Characteristic equation is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$$\Rightarrow \lambda = 1 \ , \lambda^2 - 6\lambda + 5 = 0$$

$$\Rightarrow \lambda = 1, \ (\lambda - 1)(\lambda - 5) = 0$$

The Eigen values are $\lambda = 1, 1, 5$

Example: Determine the Eigen values for the matrix $\begin{pmatrix} \mathbf{-2} & \mathbf{2} & \mathbf{-3} \\ \mathbf{2} & \mathbf{1} & \mathbf{-6} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{0} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= -2 + 1 + 0 = -1$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -12 - 3 - 6 = -21$$

$$s_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = -2(0-12) - 2(0-6) - 3(-4+1)$$

$$= 24 + 12 + 9 = 45$$

Characteristic equation is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\Rightarrow \lambda = -3 \ , \quad \lambda^2 - 2\lambda - 15 = 0$$

$$\Rightarrow \lambda = -3, \ (\lambda+3)(\lambda-5) = 0$$

The Eigen values are $\lambda = -3, -3, 5$

Eigen values and Eigen vectors for Non – Symmetric matrix

Example: Find the Eigen values and Eigen vectors for the matrix $\begin{pmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$$= 8 + 7 + 3 = 18$$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 5 + 20 + 20 = 45$$

$$s_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 8(21-16) + 6(-18+8) + 2(24-14)$$

$$= 40 - 40 + 20 = 0$$

Characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\Rightarrow \lambda = 0, \ (\lambda - 15)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 0, 3, 15$$

To find the Eigen vectors:

Case(i) When $\lambda = 0$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0 \ldots (1)$$

$$-6x_1 + 7x_2 - 4x_3 = 0 \ldots (2)$$

$$2x_1 - 4x_2 + 3x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Case (ii) When $\lambda = 3$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0 \ldots (4)$$

$$-6x_1 + 4x_2 - 4x_3 = 0 \ldots (5)$$

$$2x_1 - 4x_2 + 0x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Case (iii) When $\lambda = 15$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$7x_1 - 6x_2 + 2x_3 = 0 \ldots (7)$$
$$-6x_1 - 8x_2 - 4x_3 = 0 \ldots (8)$$
$$2x_1 - 4x_2 - 12x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$
$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$
$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$
$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$; $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$; $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Example: Determine the Eigen values and Eigen vectors of the matrix $\begin{pmatrix} \mathbf{7} & \mathbf{-2} & \mathbf{0} \\ \mathbf{-2} & \mathbf{6} & \mathbf{-2} \\ \mathbf{0} & \mathbf{-2} & \mathbf{5} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = \text{sum of the main diagonal element}$$
$$= 7 + 6 + 5 = 18$$
$$s_2 = \text{sum of the minors of the main diagonalelement}$$
$$= \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix} = 26 + 35 + 38 = 99$$
$$s_3 = |A| = \begin{vmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix} = 182 - 20 + 0 = 162$$

Characteristic equation is $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

$$\Rightarrow \lambda = 3, (\lambda^2 - 15\lambda + 54) = 0$$
$$\Rightarrow \lambda = 3, \ (\lambda - 9)(\lambda - 6) = 0$$
$$\Rightarrow \lambda = 3, 6, 9$$

To find the Eigen vectors:

Case (i) When $\lambda = 3$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 7-3 & -2 & 0 \\ -2 & 6-3 & -2 \\ 0 & -2 & 5-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 + 0x_3 = 0 \ldots (1)$$

$$-2x_1 + 3x_2 - 2x_3 = 0 \ldots (2)$$

$$0x_1 - 2x_2 + 2x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{4-0} = \frac{x_2}{0+8} = \frac{x_3}{12-4}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{8}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Case (ii) When $\lambda = 6$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 7-6 & -2 & 0 \\ -2 & 6-6 & -2 \\ 0 & -2 & 5-6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 - 2x_2 + 0x_3 = 0 \ldots (4)$$

$$-2x_1 + 0x_2 - 2x_3 = 0 \ldots (5)$$

$$0x_1 - 2x_2 - x_3 = 0 \ldots (6)$$

From (4)and (5)

$$\frac{x_1}{4-0} = \frac{x_2}{0+2} = \frac{x_3}{0-4}$$

$$\frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{-4}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Case (iii) When $\lambda = 9$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 7-9 & -2 & 0 \\ -2 & 6-9 & -2 \\ 0 & -2 & 5-9 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 0x_3 = 0 \ldots (7)$$

$$-2x_1 - 3x_2 - 2x_3 = 0 \ldots (8)$$

$$0x_1 - 2x_2 - 4x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{4-0} = \frac{x_2}{0-4} = \frac{x_3}{6-4}$$

$$\frac{x_1}{4} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$; $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$; $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Example: Determine the Eigen values and Eigen vectors of the matrix $\begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = \text{sum of the main diagonal element}$$

$$= 3 + 2 + 5 = 10$$

$$s_2 = \text{sum of the minors of the main diagonalelement}$$

$$= \begin{vmatrix} 2 & 6 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 10 + 15 + 6 = 31$$

$$s_3 = |A| = \begin{vmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{vmatrix} = 30$$

Characteristic equation is $\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$

$$\Rightarrow \lambda = 2, (\lambda^2 - 8\lambda + 15) = 0$$

$$\Rightarrow \lambda = 2, \ (\lambda - 5)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 2,3,5$$

To find the Eigen vectors:

Case(i) When $\lambda = 2$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 + 4x_3 = 0 \ldots (1)$$

$$0x_1 + 0x_2 + 6x_3 = 0 \ldots (2)$$

$$0x_1 + 0x_2 + 3x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{6-0} = \frac{x_2}{0-6} = \frac{x_3}{0-0}$$

$$\frac{x_1}{6} = \frac{x_2}{-6} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Case (ii) When $\lambda = 3$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 + x_2 + 4x_3 = 0 \ldots (4)$$

$$0x_1 - x_2 + 6x_3 = 0 \ldots (5)$$

$$0x_1 + 0x_2 + 2x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{4+6} = \frac{x_2}{0-0} = \frac{x_3}{0-0}$$

$$\frac{x_1}{10} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Case (iii) When $\lambda = 5$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 + 4x_3 = 0 \ldots (7)$$

$$0x_1 - 3x_2 + 6x_3 = 0 \ldots (8)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{6+12} = \frac{x_2}{0+12} = \frac{x_3}{6-0}$$

$$\frac{x_1}{18} = \frac{x_2}{12} = \frac{x_3}{6}$$

$$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$; $X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $X_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Problems on Symmetric matrices with repeated Eigen values

Example: Determine the Eigen values and Eigen vectors of the matrix $\begin{pmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = \text{sum of the main diagonal element}$$

$$= 6 + 3 + 3 = 12$$

$$s_2 = \text{sum of the minors of the main diagonalelement}$$

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 8 + 14 + 14 = 36$$

$$s_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32$$

Characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$$\Rightarrow \lambda = 2, (\lambda^2 - 10\lambda + 16) = 0$$

$$\Rightarrow \lambda = 2, \ (\lambda - 2)(\lambda - 8) = 0$$

$$\Rightarrow \lambda = 2,2,8$$

To find the Eigen vectors:

Case (i) When $\lambda = 8$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \ldots (1)$$

$$-2x_1 - 5x_2 - x_3 = 0 \ldots (2)$$

$$2x_1 - x_2 - 5x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 2$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0 \ldots (4)$$

$$-2x_1 + x_2 - x_3 = 0 \ldots (5)$$

$$2x_1 - x_2 + x_3 = 0 \ldots (6)$$

In (1) put $x_1 = 0 \quad \Rightarrow -2x_2 = -2x_3$

$$\Rightarrow \frac{x_2}{1} = \frac{x_3}{1} \ \Rightarrow X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

In (1) put $x_2 = 0 \quad \Rightarrow 4x_1 + 2x_3 = 0$

$$\Rightarrow 4x_1 = -2x_3$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_3}{2} \ \Rightarrow X_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $\quad X_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

Example: Find the Eigen values and Eigen vectors of the matrix $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = \text{sum of the main diagonal element}$$

$$= 2 + 3 + 2 = 7$$

$$s_2 = \text{sum of the minors of the main diagonalelement}$$

$$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 + 3 + 4 = 1$$

$$s_3 = |A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 5$$

Characteristic equation is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$$\Rightarrow \lambda = 1, (\lambda^2 - 6\lambda + 5) = 0$$

$$\Rightarrow \lambda = 1, \ (\lambda - 1)(\lambda - 5) = 0$$

$$\Rightarrow \lambda = 1, 1, 5$$

To find the Eigen vectors:

Case (i) When $\lambda = 5$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x_1 + 2x_2 + x_3 = 0 \ldots (1)$$

$$x_1 - 2x_2 + x_3 = 0 \ldots (2)$$

$$x_1 + 2x_2 - 3x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{2+2} = \frac{x_2}{1+3} = \frac{x_3}{6-2}$$

$$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + 2x_2 + x_3 = 0 \ldots (4)$$

$$x_1 + 2x_2 + x_3 = 0 \ldots (5)$$

$$x_1 + 2x_2 + x_3 = 0 \ldots (6)$$

In (1) put $x_1 = 0 \quad \Rightarrow 2x_2 = -x_3$

$$\Rightarrow \frac{x_2}{-1} = \frac{x_3}{2} \Rightarrow X_2 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

In (1) put $x_2 = 0 \quad => x_1 = -x_3$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_3}{1} \Rightarrow X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$; $X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$