UNIT- II
INTERPOLATION AND APPROXIMATION

PROBLEMS BASED ON CUBIC SPLINES FORMULA

4.4 Cubic Splines

Interpolating with a cubic spline

The cubic spline interpolation formula is

$$S(x) = y(x) = y = \frac{1}{6h}[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i]$$
$$+ \frac{1}{h}(x_i - x)[y_{i-1} - \frac{h^2}{6}M_{i-1}] + \frac{1}{h}(x - x_{i-1})[y_i - \frac{h^2}{6}M_i]$$

where

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}[y_{i-1} - 2y_i + y_{i+1}]$$

n = number of data

i = number of intervals [i.e., $i = 1, 2, 3, , (n-1)$]

h = length of interval = interval length.

Note : If M_i and y_i'' values are not given, then assume $M_0 = M_n = 0$ $\left[\text{or } y_0'' = y_n'' = 0\right]$, and find $M_1, M_2, \cdots, M_{n-1}$ in 1^{st} interval, 2^{nd} interval, $\cdots$, $(n-1)^{th}$ interval value.

Note : Order of convergence of the cubic spline is 4.

Find the cubic spline approximation for the function f(x) given by the data:

x	0	1	2	3
$y = f(x)$	1	2	33	244

with $M_0 = 0 = M_3$. Hence estimate the value $f(0.5), f(1.5), f(2.5)$. {AU2010}

Solution: We know that cubic spline interpolation formula for $x_{i-1} \le x < x_i, i = 1,2,3$ is

$$S_i(x) = y(x) = y = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right]$$

$$+\frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right]$$

$$+\frac{1}{h}(x - x_{i-1})\left[y_i - \frac{h^2}{6}M_i\right] \quad (1)$$

$$\text{where } M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right] \quad (2)$$

n = number of data = 4

i = number of intervals = 3 i.e., $i = 1,2,3$.

h = length of inteval = 1

Solving (3)&(4), $(3) \Rightarrow 4M_1 + M_2 = 180$

$$4 \times (4) \Rightarrow 4M_1 + 16M_2 = 4320$$

$$\text{i.e., } (3) + 4 \times (4) \Rightarrow -15M_2 = 4140$$

$$\Rightarrow M_2 = 276$$

$$(3) \Rightarrow 4M_1 = 180 - 276$$

$$\Rightarrow M_1 = -24$$

<u>To find Cubic spline</u>

When $i = 1$,

Cubic spline in $x_{i-1} \le x \le x_i$

$$\text{i.e., } x_0 \le x \le x_1$$

$$\text{i.e., } 0 \le x \le 1$$

i.e., Cubic spline in $0 \le x \le 1$ is

$$y_1(x) = S_1(x) = \frac{1}{6(1)}\left[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1\right]$$

$$+ \frac{1}{1}(x_1 - x)\left[y_0 - \frac{1}{6}M_0\right] + \frac{1}{1}(x - x_0)\left[y_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}\left[(1 - x)^3(0) + (x - 0)^3(-24)\right]$$

$$+ (1 - x)[1 - 0] + (x - 0)[2 - (-24)]$$

$$= -4x^3 + (1 - x) + 6x$$

$$= -4x^3 + 5x + 1$$

When $i = 2$,

Cubic spline in $x_{i-1} \le x \le x_i$

i.e., $x_1 \le x \le x_2$

i.e., $1 \le x \le 2$

i.e., Cubic spline in $1 \le x \le 2$ is

$$y_2(x) = S_2(x) = \frac{1}{6(1)}\left[(x_2 - x)^3 M_1 + (x - x_1)^3 M_2\right]$$

$$+ \frac{1}{1}(x_2 - x)\left[y_1 - \frac{1}{6}M_1\right] + \frac{1}{1}(x - x_1)\left[y_2 - \frac{1}{6}M_2\right]$$

$$= \frac{1}{6}\left[(2 - x)^3(-24) + (x - 1)^3(276)\right]$$

$$+ (2 - x)\left[2 - \frac{1}{6}(-24)\right] + (x - 1)\left[33 - \frac{1}{6}(276)\right]$$

$$= -4(2 - x)^3 + 46(x - 1)^3 + 6(2 - x) - 13(x - 1)$$

$$= 50x^3 - 162x^2 + 162x - 53$$

When $i = 3$,

Cubic spline in $x_{i-1} \le x \le x_i$

i.e., $x_2 \le x \le x_3$

i.e., $2 \le x \le 3$

i.e., Cubic spline in $2 \le x \le 3$ is

$$y_3(x) = S_3(x) = \frac{1}{6(1)}\left[(x_3 - x)^3 M_2 + (x - x_2)^3 M_3\right]$$

$$+ \frac{1}{1}(x_3 - x)\left[y_2 - \frac{1}{6}M_2\right] + \frac{1}{1}(x - x_2)\left[y_3 - \frac{1}{6}M_3\right]$$

$$= \frac{1}{6}\left[(3 - x)^3(276) + 0\right]$$

$$+ (3 - x)\left[33 - \frac{1}{6}(276)\right] + (x - 2)[244 - 0]$$

$$= 46\left(27 - x^3 + 9x^2 - 27x\right) - 13(3 - x) + 244x - 488$$

$$= -46x^3 + 414x^2 - 985x + 715$$

$\therefore$ Cubic spline is

$$S(x) = \begin{cases} S_1(x) = y_1(x) = -4x^3 + 5x + 1, & 0 \leq x \leq 1 \\ S_2(x) = y_2(x) = 50x^3 - 162x^2 + 167x - 53, & 1 \leq x \leq 2 \\ S_3(x) = y_3(x) = -46x^3 + 414x^2 - 985x + 715, & 2 \leq x \leq 3 \end{cases}$$

When $x = 0.5, y_1(x = 0.5) = S_1(x = 0.5) = -4(0.5)^3 + 5(0.5)^2 + 1 = 3$

When $x = 1.5, y_2(x = 1.5) = S_2(x = 1.5) = 50(1.5)^3 - 162(1.5)^2 + 167(1.5) - 53 = 1.75$

When $x = 2.5, y_3(x = 2.5) = S_3(x = 2.5) = -46(2.5)^3 + 414(2.5)^2 - 985(2.5) + 715 = 121.25$

From the following table

x	1	2	3
$y = f(x)$	-8	-1	18

. Find cubic spline and compute $y(1.5), y'(1), y(2.5)$ and $y'(3)$.

Solution:

$$S(x) = \begin{cases} S_1(x) = y_1(x) = 3(x-1)^3 + 4x - 12, & 1 \le x \le 2 \\ S_2(x) = y_2(x) = 3(3-x)^3 + 22x - 48, & 2 \le x \le 3 \end{cases}$$

&

$$y(x = 1.5) = S_1(x = 1.5) = -\frac{45}{8}, y'(x = 1) = S_1'(x = 1) = 4$$

$$y(x = 2.5) = S_2(x = 2.5) = 7.375, y'(x = 3) = S_2'(x = 3) = 22.$$

Fit a natural cubic spline for the following data:

x	0	1	2	3
$y = f(x)$	1	4	0	−2

{AU 2008}

Solution: Assume $M_0 = 0 = M_3$.

$$S(x) = \begin{cases} S_1(x) = y_1(x) = -2x^3 + 5x + 1, & [0,1] \\ S_2(x) = y_2(x) = 3x^3 - 15x^2 + 20x - 4, & [1,2] \\ S_3(x) = y_3(x) = -x^3 + 9x^2 - 28x + 28, & [2,3] \end{cases}$$

4.4.3 Anna University Questions

1. If $f(0) = 1, f(1) = 2, f(2) = 33$ and $f(3) = 244$, find a cubic spline approximation, assuming $M(0) = M(3) = 0$. Also, find $f(2.5)$. (AM10)

Solution: Hint :

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = -4x^3 + 5x + 1, & x \in [0,1] \\ S_2(x) = y_2(x) = 50x^3 - 162x^2 + 1670x - 53, & x \in [1,2] \\ S_3(x) = y_3(x) = -46x^3 + 414x^2 - 985x + 715, & x \in [2,3] \end{cases}$$

$$f(x) = -46x^3 + 414x^2 - 985x + 715 \quad x \in [2,3]$$

$$f'(x) = -138x^2 + 828x - 985$$

$$f'(x = 2.5) = -138(2.5)^2 + 828(2.5) - 985$$

$$= 222.5$$

4. The following values of x and y are given:

x:	1	2	3	4
y:	1	2	5	11

Find the cubic splines and evaluate $y(1.5)$ and $y'(3)$ (MJ12)

Solution: Hint :

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = \frac{1}{3}\left(x^3 - 3x^2 + 5x\right), & x \in [1,2] \\ S_2(x) = y_2(x) = \frac{1}{3}\left(x^3 - 3x^2 + 5x\right), & x \in [2,3] \\ S_3(x) = y_3(x) = \frac{1}{3}\left(-2x^3 + 24x^2 - 76x + 81\right), & x \in [3,4] \end{cases}$$

$$y(x) = \frac{1}{3}\left(x^3 - 3x^2 + 5x\right) \Rightarrow y(1.5) = 1.375, \quad x \in [1,2]$$

$$y'(x) = \frac{1}{3}\left(3x^2 - 6x + 5\right) \Rightarrow y'(3) = 4.666666667, \quad x \in [2,3]$$

(or)

$$y'(x) = \frac{1}{3}\left(-6x^2 + 48x - 76\right) \Rightarrow y'(3) = 4.666666667, \quad x \in [3,4]$$

5. Obtain the cubic spline for the following data to find $y(0.5)$. (ND12)

x:	−1	0	1	2
y:	−1	1	3	35

Solution: Hint :

$$S(x) = y(x) = \begin{cases} S_1(x) = y_1(x) = -2x^3 - 6x^2 - 2x + 1, & x \in [-1,0] \\ S_2(x) = y_2(x) = 10x^3 - 6x^2 - 2x + 1, & x \in [0,1] \\ S_3(x) = y_3(x) = -8x^3 + 48x^2 - 56x + 19, & x \in [1,2] \end{cases}$$

6. Using cubic spline, compute $y(1.5)$ from the given data. (MJ13)

$$\begin{array}{cccc} x: & 1 & 2 & 3 \\ y: & -8 & -1 & 18 \end{array}$$

Solution: Hint :

$$S(x) = y(x) = 3x^3 - 9x^2 + 13x - 15$$

$$y(1.5) = 3(1.5)^3 - 9(1.5)^2 + 13(1.5) - 15 = -\frac{45}{8} = -5.625, \quad x \in [1, 2]$$

9. Obtain the cubic spline approximation for the function $y = f(x)$ from the following data, given that $y_0'' = y_3'' = 0$. (ND14)

$$\begin{array}{ccccc} x & -1 & 0 & 1 & 2 \\ y & -1 & 1 & 3 & 35 \end{array}$$

10. Find the cubic Spline interpolation. (AU N/D, 2007,AM2014)

x	1	2	3	4	5
$f(x)$	1	0	1	0	1

11. Given the following table, find $f(2.5)$ using cubic spline functions : (AU May/June 2007)

x	1	2	3	4
$f(x)$	0.5	0.3333	0.25	0.2

Solution : [Ans: $S_2(2.5) = 0.2829$]