CONSTRUCTION OF ANALYTIC FUNCTION

Method: [Milne – Thomson method]

(i) To find $f(z)$ when u is given

Let $f(z) = u + iv$

$f'(z) = u_x + iv_x$

$= u_x - iv_y$ [by C−R condition]

$\therefore f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

(ii) To find $f(z)$ when v is given

Let $f(z) = u + iv$

$f'(z) = u_x + iv_x$

$= v_y + iv_x$ [by C−R condition]

$\therefore f(z) = \int v_y(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

Example: Construct the analytic function $f(z)$ for which the real part is $e^x \cos y$.

Solution:

Given $u = e^x \cos y$

$\Rightarrow u_x = e^x \cos y$ [$\therefore \cos 0 = 1$]

$\Rightarrow u_x(z,0) = e^x$

$\Rightarrow u_y = e^x \cos y$ [$\therefore \sin 0 = 0$]

$\Rightarrow u_y(z,0) = 0$

$\therefore f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$\therefore f(z) = \int e^z\, dz - i\int 0\, dz + C$

$= e^z + C$

Example: Determine the analytic function $w = u + iv$ if $u = e^{2x}(x\cos 2y - y\sin 2y)$

Solution:

Given $u = e^{2x}(x\cos 2y - y\sin 2y)$

$u_x = e^{2x}[\cos 2y] + (x cos2y - y\sin 2y)[2\, e^{2x}]$

$u_x(z,0) = e^{2z}[1] + [z(1) - 0][2e^{2z}]$

$$= e^{2z} + 2ze^{2z}$$

$$= (1+2z)e^{2x}$$

$$u_y = e^{2x}[-2x\sin 2y - (y2cos2y\ + \sin 2y)]$$

$$u_y(z,0) = e^{2z}[-0-(0+0)] = 0$$

$\therefore f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$$f(z) = \int (1+2z)e^{2z}dz - i\int 0 + dz + C$$

$$= \int (1+2z)e^{2z}dz + C$$

$$= (1+2z)\frac{e^{2z}}{2} - 2\frac{e^{2z}}{4} + C \quad [\because \int uv\,dz = uv_1 - u'v_2 + u''v_3 - \cdots]$$

$$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + C$$

$$= ze^{2z} + C$$

Example: Determine the analytic function where real part is
$\boldsymbol{u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1}$.

Solution:

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$u_x = 3x^2 - 3y^2 + 6x$$

$$\Rightarrow u_x(z,0) = 3z^2 - 0 + 6z$$

$$u_y = 0 - 6xy + 0 - 6y$$

$$\Rightarrow u_y(z,0) = 0$$

$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$$f(z) = \int (3z^2 + 6z)dz - i\int 0 + dz + C$$

$$= 3\frac{z^2}{3} + 6\frac{z^2}{2} + C$$

$$= z^3 + 3z^2 + C$$

Example: Determine the analytic function whose real part in $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

Solution:

Given $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x = \frac{(\cosh 2y - \cos 2x)[2\cos 2x] - \sin 2x[2\sin 2x]}{[\cosh 2y - \cos 2x]^2}$$

$$u_x(z,0) = \frac{(1-\cos 2z)(2\cos 2z) - 2\sin^2 2z}{[\cosh 0 - \cos 2z]^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2[\cos^2 2z + \sin^2 2z]}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2}$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)}$$

$$= \frac{-2}{2sin^2 2}$$

$$= -cosec^2 z$$

$$u_y = \frac{(\cosh 2y - \cos 2x)\,(0) - \sin 2x[2\sin 2y]}{[\cosh 2y - \cos 2x]^2}$$

$$\Rightarrow\ u_y(z,0) = 0$$

$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule], where C is a complex constant.

$$f(z) = \int(-cosec^2 z)dz - i\int 0\,dz + C$$

$$= cot\ z + C$$

Example: Show that the function $u = \frac{1}{2}\log(x^2+y^2)$ is harmonic and determine its conjugate. Also find $f(z)$

Solution:

Given $u = \frac{1}{2}\log(x^2+y^2)$

$$u_x = \frac{1}{2}\frac{1}{(x^2+y^2)}(2x) = \frac{x}{x^2+y^2},$$

$$\Rightarrow u_x(z,0) = \frac{z}{z^2} = \frac{1}{z}$$

$$u_{xx} = \frac{(x^2+y^2)[1] - x[2x]}{[x^2+y^2]^2} = \frac{x^2+y^2-2x^2}{[x^2+y^2]^2} = \frac{y^2-x^2}{[x^2+y^2]^2} \quad \ldots(1)$$

$$u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$$

$$\Rightarrow u_y(z,0) = 0$$

$$u_{yy} = \frac{(x^2+y^2)[1] - y[2y]}{[x^2+y^2]^2} = \frac{x^2-y^2}{[x^2+y^2]^2} \quad \ldots(2)$$

To prove u is harmonic:

$$\therefore u_{xx} + u_{yy} = \frac{(y^2-x^2)+(x^2-y^2)}{[x^2+y^2]^2} = 0 \qquad by\ (1)\&(2)$$

$\Rightarrow u$ is harmonic.

To find $f(z)$:

$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$f(z) = \int \frac{1}{z} dz - i\int 0\, dz + C$

$= log\, z + C$

To find v :

$f(z) = log\,(re^{i\theta})$ $[\because z = re^{i\theta}]$

$u + iv = \log r + \log\, e^{i\theta} = \log r + i\theta$

$\Rightarrow u = logr, v = \theta$

Note: $z = x + iy$

$r = |z| = \sqrt{x^2 + y^2}$

$\log r = \frac{1}{2}\log(\,x^2 + y^2)$

$\tan\theta = \frac{y}{x}$

$\theta = tan^{-1}\left(\frac{y}{x}\right)$ $i.e., v = tan^{-1}\left(\frac{y}{x}\right)$

Example: Construct an analytic function $f(z) = u + iv$, given that $u = e^{x^2-y^2}\cos 2xy$. Hence find v.

Solution:

Given $u = e^{x^2-y^2}\cos 2xy = e^{x^2}e^{-y^2}\cos 2xy$

$u_x = e^{-y^2}\left[e^{x^2}(-2y\sin 2xy) + \cos 2xy\;\; e^{x^2}2x\right]$

$u_x(z,0) = 1\left[e^{z^2}(0) + 2ze^{z^2}\right] = 2ze^{z^2}$

$u_y = e^{x^2}\left[e^{-y^2}(-2x\sin 2xy) + \cos 2xy e^{-y^2}(-2y)\right]$

$u_y(z,0) = e^{z^2}[0+0] = 0$

$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule]

$= \int 2z\, e^{z^2} dz + C$

$= 2\int z\, e^{z^2} dz + C$

put $t = z^2$, dt= $2z\, dz$

$= \int e^t\, dt + C$

$= e^t + C$

$f(z) = e^{z^2} + C$

To find v :

$u + iv = e^{(x+iy)^2} = e^{x^2-y^2+i\,2xy} = e^{x^2-y^2}e^{i2\,xy}$

$$= e^{x^2-y^2}\ [\cos(2xy) + i\sin(2xy)]$$

$$v = e^{x^2-y^2} \sin 2xy \quad [\because \text{equating the imaginary parts}]$$

Example: Find the regular function whose imaginary part is $e^{-x}(x\cos y + y\sin y)$.

Solution:

Given $v = e^{-x}(x\cos y + y\sin y)$

$$v_x = e^{-x}[\cos y] + (x\cos y + y\sin y)[-e^{-x}]$$

$$v_x(z,0) = e^{-z} + (z)(-e^{-z}) = (1-z)e^{-z}$$

$$v_y = e^{-x}[-x\sin y + (y\cos y + \sin y\,(1))]$$

$$v_x(z,0) = e^{-z}[0+0+0] = 0$$

$$\therefore f(z) = \int v_y(z,0)dz + i\int v_x(z,0)dz + C \quad \text{[by Milne−Thomson rule]}$$

Where, C is a complex constant.

$$f(z) = \int 0dz + i\int(1-z)e^{-z}dz + C$$

$$= i\int(1-z)e^{-z}dz + C$$

$$= i\left[(1-z)\left[\frac{e^{-z}}{-1}\right] - (-1)\left[\frac{e^{-z}}{(-1)^2}\right]\right] + C$$

$$= i[-(1-z)e^{-z} + e^{-z}] + C$$

$$= ize^{-z} + C$$

Example: In a two dimensional flow, the stream function is $\psi = tan^{-1}\left(\frac{y}{x}\right)$. Find the velocity potential φ.

Solution:

Given $\psi = tan^{-1}(y/x)$

We should denote, ϕ by u and ψ by v

$$\therefore v = tan^{-1}(y/x)$$

$$v_x = \frac{1}{1+(y/x)^2}\left[\frac{-y}{x^2}\right] = \frac{-y}{x^2+y^2}, \qquad v_x(z,0) = 0$$

$$v_y = \frac{1}{1+(y/x)^2}\left[\frac{1}{x}\right] = \frac{x}{x^2+y^2} \qquad v_x(z,0) = \frac{z}{z^2} = \frac{1}{z}$$

$$\therefore f(z) = \int v_y(z,0)dz + i\int v_x(z,0)dz + C$$

$$f(z) = \int \frac{1}{z}dz + i\int 0\,dz + C = \log z + C$$

To find φ:

$$f(z) = log\,(re^{i\theta}) \qquad [\because z = re^{i\theta}]$$

$$u + iv = \log r + \log e^{i\theta}$$

$u + iv = \log r + i\theta$

$\Rightarrow u = \log r$

$z = x + iy, \lvert z \rvert = \sqrt{x^2 + y^2}$

$\Rightarrow u = log\sqrt{x^2 + y^2}$

$= \frac{1}{2}\log(x^2 + y^2)$

So, the velocity potential φ is

$$\varphi = \frac{1}{2}\log(x^2 + y^2)$$

Example: If $f(z) = u + iv$ is an analytic function and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ interms of z.

Solution:

Given $u - v = e^x(\cos y - \sin y)$, $\quad \ldots (A)$

Differentiate (A) p.w.r. to x, we get

$$u_x - v_x = e^x(\cos y - \sin y),$$

$$u_x(z,0) - v_x(z,0) = e^z \quad \ldots (1)$$

Differentiate (A) p.w.r. to y, we get

$$u_y - v_y = e^x(-\sin y - \cos y)$$

$$u_y(z,0) - v_y(z,0) = e^z[-1]$$

i.e., $u_y(z,0) - v_y(z,0) = -e^z$

$-v_x(z,0) - u_x(z,0) = -e^z \quad \ldots (2)$ [by C−R conditions]

$(1) + (2) \Rightarrow -2v_x(z,0) = 0$

$\Rightarrow v_x(z,0) = 0$

$(1) \Rightarrow u_x(z,0) = e^z$

$f(z) = \int u_x(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne−Thomson rule]

$f(z) = \int e^z dz + i0 + C$

$= e^z + C$

Example: Find the analytic functions $f(z) = u + iv$ given that

(i) $2u + v = e^x(\cos y - \sin y)$

(ii) $u - 2v = e^x(\cos y - \sin y)$

Solution:

Given (i) $2u + v = e^x(\cos y - \sin y) \quad \ldots (A)$

Differentiate (A) p.w.r. to x, we get

$$2u_x + v_x = e^x(\cos y - \sin y)$$

$2u_x - u_y = e^x(\cos y - \sin y)$ [by C−R condition]

$$2u_x(z,0) - u_y(z,0) = e^z \quad \ldots(1)$$

Differentiate (A) p.w.r. to y, we get

$$2u_y + v_y = e^x[-\sin y - \cos y]$$

$$2u_y + u_x = e^x[-\sin y - \cos y] \quad \text{[by C−R condition]}$$

$$2u_y(z,0) + u_x(z,0) = e^z(-1) = -e^z \quad \ldots(2)$$

$$(1)\times(2) \quad \Rightarrow 4u_x(z,0) - 2u_y(z,0) = 2e^z \quad \ldots(3)$$

$$(2)+(3) \quad \Rightarrow 5u_x(z,0) = e^z$$

$$\Rightarrow u_x(z,0) = \tfrac{1}{5}e^z$$

$$(1) \quad \Rightarrow u_y(z,0) = \tfrac{2}{5}e^z - e^z = -\tfrac{3}{5}e^z$$

$$\Rightarrow u_y(z,0) = -\tfrac{3}{5}e^z$$

$$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C \quad \text{[by Milne−Thomson rule]}$$

Where, C is a complex constant.

$$f(z) = \int \tfrac{1}{5}e^z dz - i\int -\tfrac{3}{5}e^z dz + C$$

$$= \tfrac{2}{5}e^z + \tfrac{3}{5}ie^z + C$$

$$= \tfrac{1+3i}{5}e^z + C$$

(ii) $u - 2v = e^x(\cos y - \sin y) \quad \ldots(B)$

Differrentiate (B) p.w.r. to x, we get

$$u_x - 2v_x = e^x(\cos y - \sin y)$$

$$u_x + 2u_y = e^x(\cos y - \sin y) \quad \text{[by C−R condition]}$$

$$u_x(z,0) + 2u_y(z,0) = e^z \quad \ldots(1)$$

Differentiate (B) p.w.r. to y, we get

$$u_y - 2v_y = e^x[-\sin y - \cos y]$$

$$u_y - 2u_x = e^x[-\sin y - \cos y] \quad \text{[by C−R condition]}$$

$$u_y(z,0) - 2u_x(z,0) = -e^z \quad \ldots(2)$$

$$(1)\times(2) \quad \Rightarrow 2u_x(z,0) + 4u_y(z,0) = 2e^z \quad \ldots(3)$$

$$(2)+(3) \quad \Rightarrow 5u_y(z,0) = e^z$$

$$\Rightarrow u_y(z,0) = \tfrac{1}{5}e^z$$

$$(1) \quad \Rightarrow u_x(z,0) = -\frac{2}{5}e^z + e^z$$

$$= \tfrac{3}{5}e^z$$

$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule]

Where, C is a complex constant.

$$f(z) = \int \frac{3}{5} e^z dz - i\int \frac{1}{5} e^z dz + C$$

$$= \frac{3}{5} e^z - i\frac{1}{5} e^z + C = \frac{3-i}{5} e^z + C$$