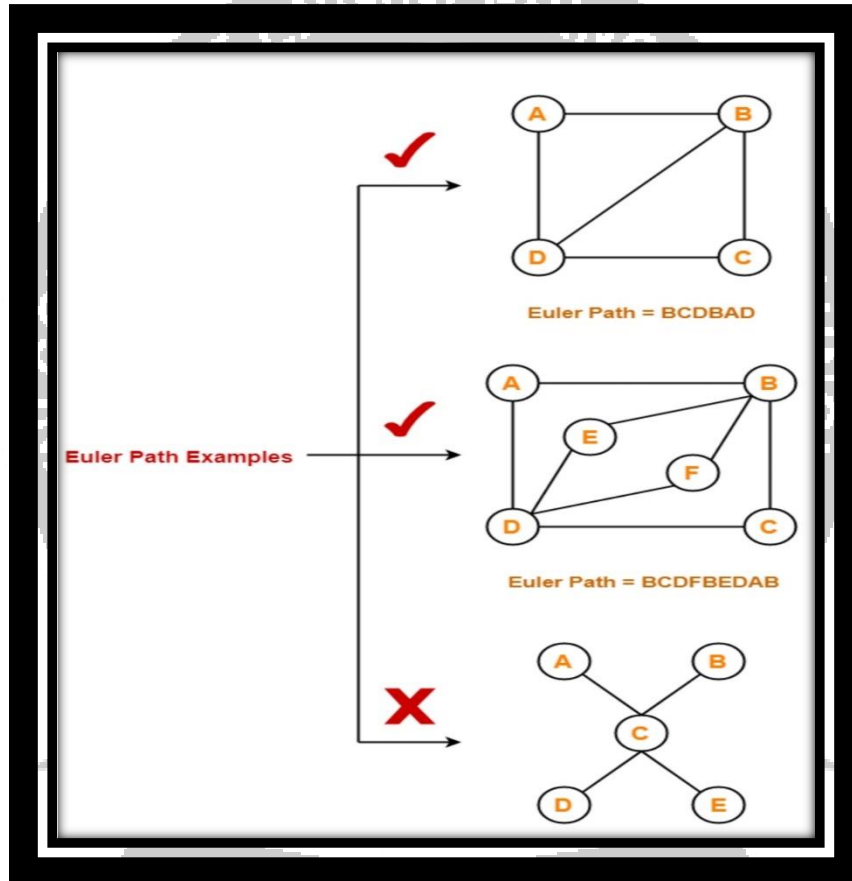


Euler graph and Hamilton graph:

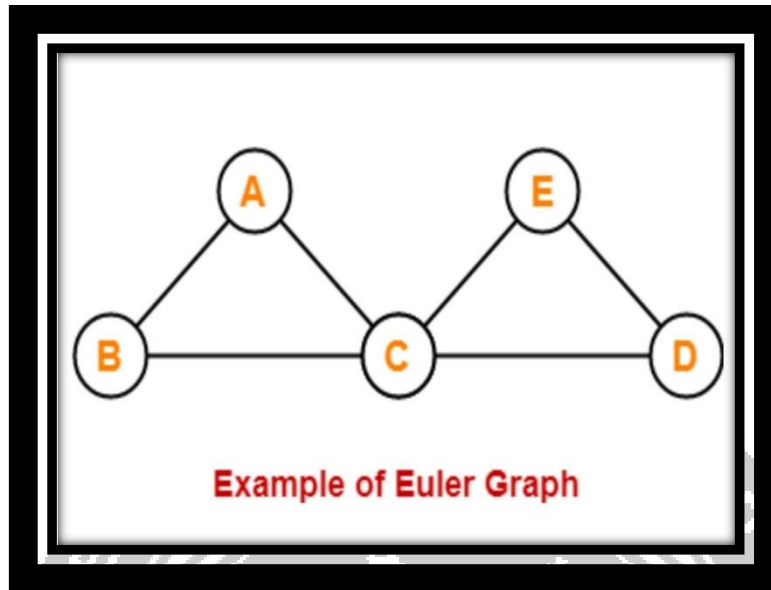
Euler path:

A path of a graph G is called an Eulerian path, if it contains each edge of the graph exactly once.



Euler graph:

A path of a graph G is called an Eulerian path, if it contains each edge of the graph exactly once.

**Eulerian Circuit or Eulerian Cycle:**

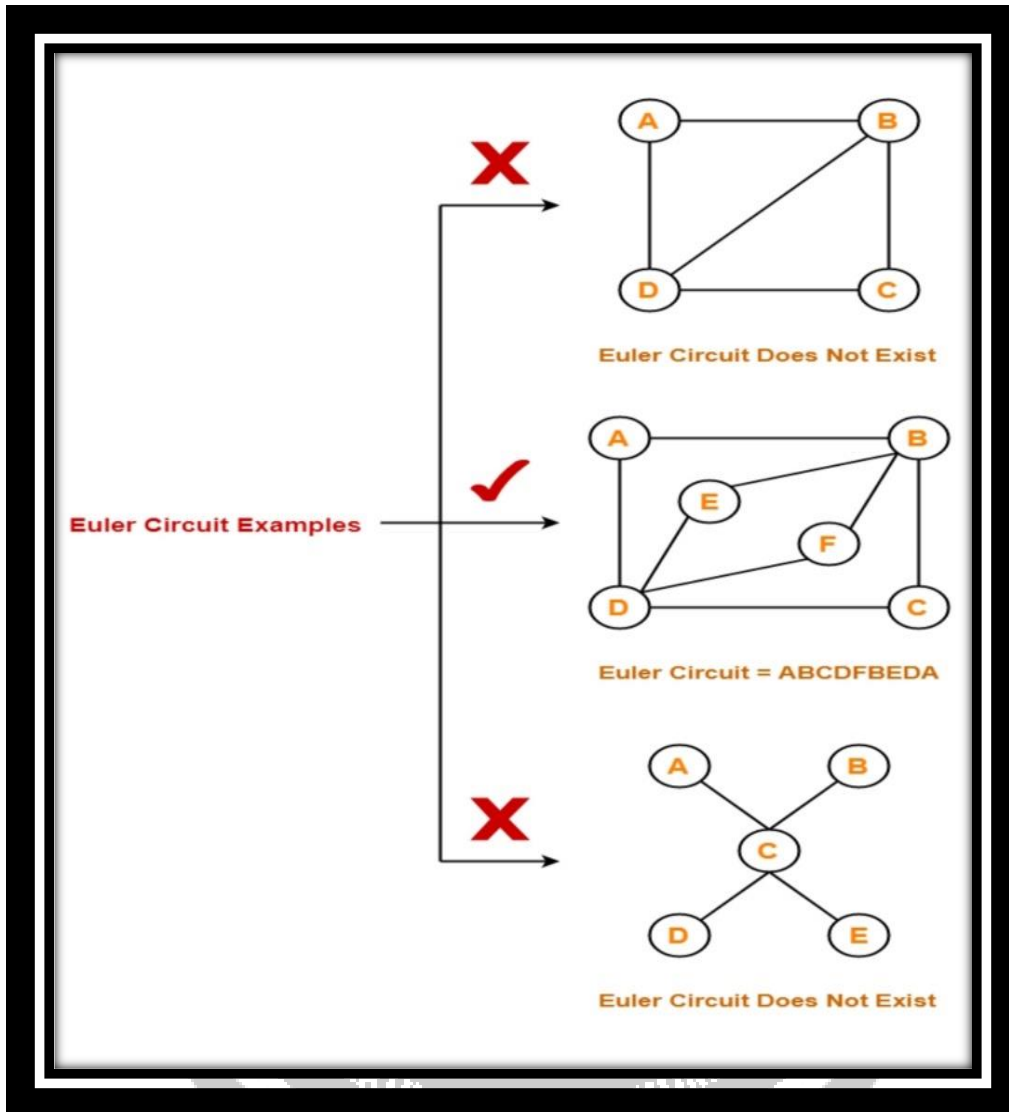
A circuit or cycle of a graph G is called Eulerian circuit or cycle, if it includes each edge of G exactly once.

Here starting and ending vertex are same.

An Eulerian circuit or cycle should satisfies the following conditions:

Starting and ending points (vertices) are same.

Cycle should contain all the edges of graph but exactly once.



Eulerian Graph or Euler graph:

Any graph containing an Eulerian circuit or cycle is called an Eulerian graph.

Theorem:1

A connected graph is Euler graph (contains Eulerian circuit) if and only if each of its vertices is of even degree.

Proof:

Let G be any graph having an Eulerian circuit and let " C " be an Eulerian circuit of G with origin vertex as u . Each time a vertex occurs as an internal vertex of C , then two of the edges incident with v are accounted for degree.

We, get, for internal vertex $v \in v(G)$

$$d(v) = 2 \times \text{number of times } v \text{ occur inside the Euler circuit } C$$

= even degree

And, since an Euler circuit C contains every edge of G and C starts and ends at u .

$$d(u) = 2 + 2 \times \text{number of times } u \text{ occur inside } C.$$

= even degree Hence G has all the vertices of even degree.

Conversely, assume each of its vertices has an even degree.

Claim:

G has an Eulerian circuit.

Assume G be a connected graph which is not having an Euler circuit, with all vertices of even degree and less number of edges. That is, any graph having less number of edges than G , then it has an Eulerian circuit. Since each vertex of G has atleast two, therefore G contains closed path. Let C be a closed path of maximum

possible length in G . If C itself has all the edges of G , then G itself an Euler circuit in G .

By assumption, C is not an Euler circuit of G and $G - E(C)$ has some component G' with $|E(G')| > 0$. C has less number of edges than G , therefore C itself is an Eulerian, and C has all the vertices of even degree.

Since $|E(G')| < |E(G)|$, therefore G' has an Euler circuit C' . Because G is connected, there is a vertex v in both C and C' . Now join C and C' and traverse all the edges of C and C' with common vertex v , we get CC' is a closed path in G and $E(CC') > E(C)$ which is not possible choices of C .

Hence G has an Eulerian circuit.

Hence G is a Euler graph.

Hence the proof.

Theorem:2

Prove that if a graph G has not more than two vertices of odd degree, then there can be Euler path in G .

Proof:

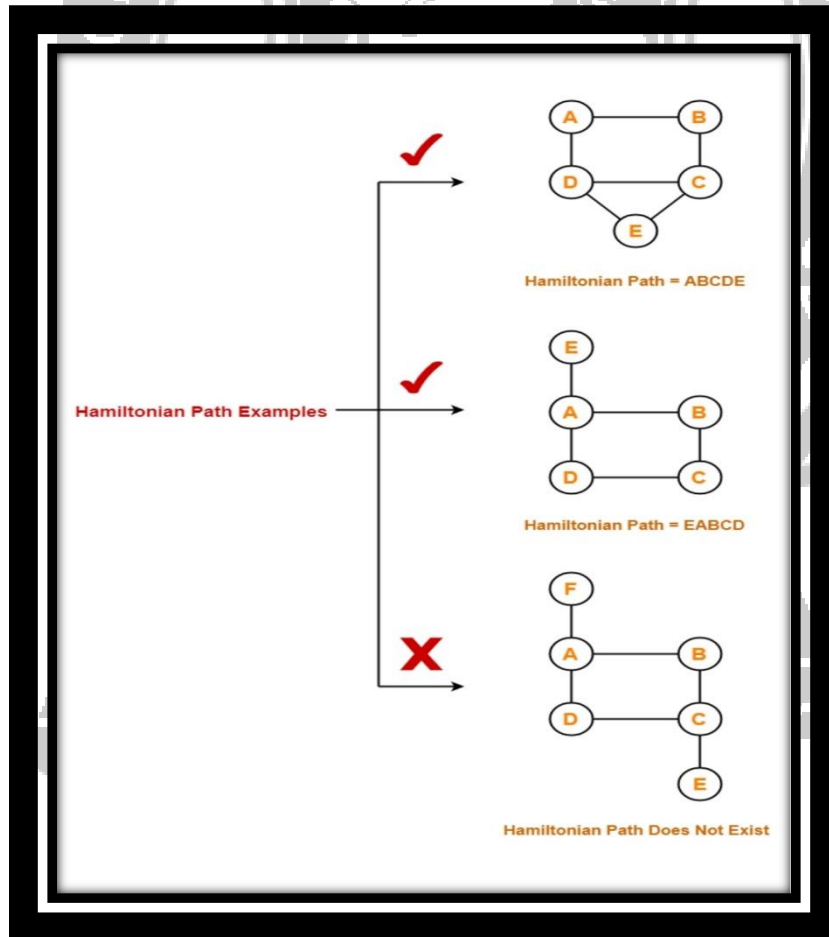
Let the odd degree vertices be labelled as V and W in any arbitrary order. Add an edge of G between the vertex pair (V, W) to form a new graph G' .

Now every vertex of G' is of even degree and hence G' has an Eulerian trail T . If the edge that we added to G is now removed from T , it will split into an open trail containing all edges of G which is nothing but an Euler path in G .

Hamiltonian Graph:

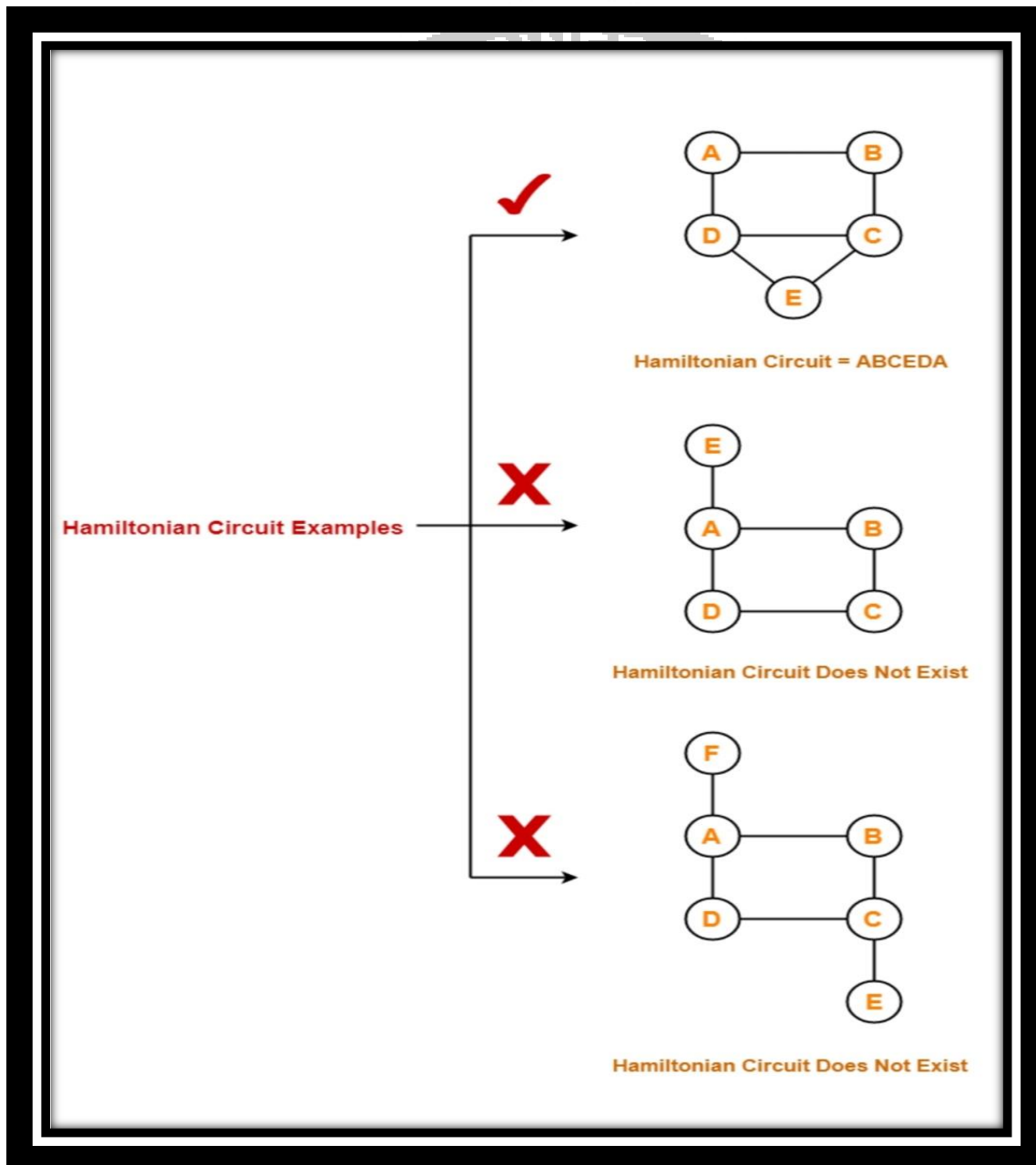
Hamiltonian Path:

A path of a graph G is called a Hamiltonian path, if it includes each vertex of G exactly once.



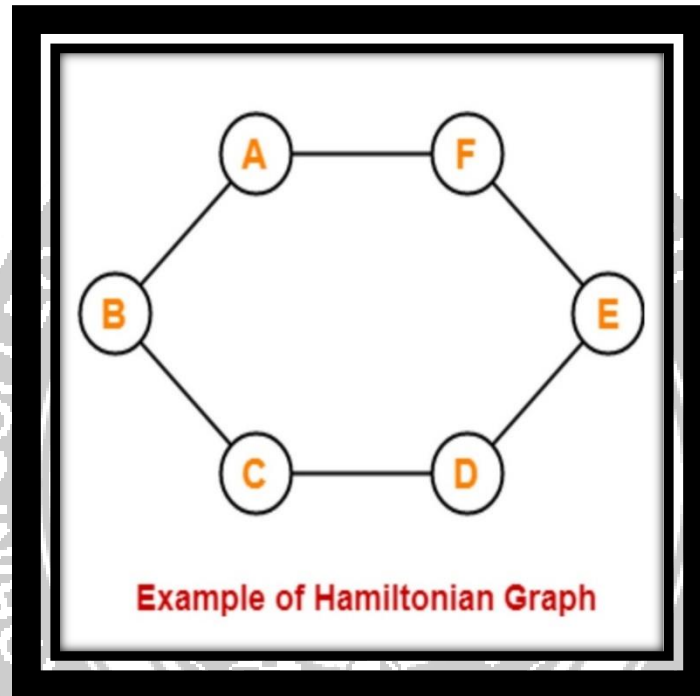
Hamiltonian Circuit or Cycle:

A circuit of a graph G is called a Hamiltonian circuit, if it includes each vertex of G exactly once, except the starting and ending vertices.



Hamiltonian graph:

Any graph containing a Hamiltonian circuit or cycle is called a Hamiltonian graph.

**Properties:**

- (i) A Hamiltonian circuit contains a Hamiltonian path, but a graph containing a Hamiltonian path need not have a Hamiltonian cycle.
- (ii) By deleting any one edge from Hamiltonian cycle, we can get Hamiltonian path.
- (iii) A graph may contain more than one Hamiltonian cycle.
- (iv) A complete graph k_n , will always have a Hamiltonian cycle, when $n \geq 3$.

(v) A graph with a vertex of degree one cannot have a Hamiltonian cycle.

Theorem: 1

Let G be a simple undirected graph with n vertices. Let u and v be two nonadjacent vertices in G such that $\deg(u) + \deg(v) \geq n$ in G . Show that G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Proof:

If G is Hamiltonian, then obviously $G + uv$ is Hamiltonian.

Conversely, suppose that $G + uv$ is Hamiltonian, but G is not.

Then by Dirac theorem, we have $\deg(u) + \deg(v) < n$

Which is a contradiction to our assumption.

Thus $G + uv$ is Hamiltonian implies G is Hamiltonian.

Hence the proof.

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