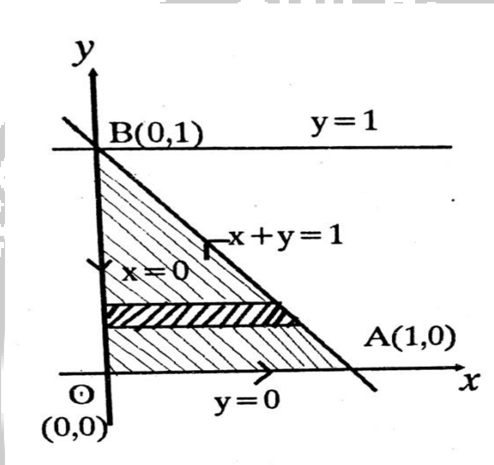
Verification and Application in evaluating line,surface and volume integrals

Example: Verify Green's theorem in the plane for $\int_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$

where C is the boundary of the region defined by $x = 0, y = 0, x + y = 1$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = 3x^2 - 8y^2$ and $N = 4y - 6xy$

$$\Rightarrow \frac{\partial M}{\partial y} = -16y \qquad \Rightarrow \frac{\partial N}{\partial x} = -6y$$

$$\therefore \int_c (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_c M\,dx + N\,dy$$

By Green's theorem in the plane,

$$\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_0^{1-x} (10y)\,dy\,dx$$

$$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= 5 \int_0^1 (1-x)^2\,dx$$

$$= 5 \left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{5}{3} \dots (1)$$

Consider $\int M\,dx + N\,dy = \int_{OA} + \int_{AB} + \int_{BO}$

Along $OA, y = 0 \Rightarrow dy = 0, x$ varies from 0 to 1

$$\therefore \int_{OA} M\,dx + N\,dy = \int_0^1 3x^2 dx = [x^3]_0^1 = 1$$

Along $AB, y = 1 - x \Rightarrow dy = -dx \; and \; x$ varies from 1 to 0

$$\therefore \int_{AB} M\,dx + N\,dy = \int_1^0 [3x^2 - 8(1-x)^2 - 4(1-x) + 6x(1-x)]dx$$

$$= \left[\frac{3x^3}{3} - \frac{8(1-x)^3}{-3} - \frac{4(1-x)^2}{-2} + 3x^2 - 2x^3\right]_1^0$$

$$= \frac{8}{3} + 2 - 1 - 3 + 2 = \frac{8}{3}$$

Along $BO, x = 0 \Rightarrow dx = 0 \; and \; y$ varies from 1 to 0

$$\therefore \int_{BO} M\,dx + N\,dy = \int_1^0 4y\,dy = [2y^2]_1^0 = -2$$

$$\therefore \int_c M\,dx + N\,dy = 1 + \frac{8}{3} - 2 = \frac{5}{3} \quad \ldots(2)$$

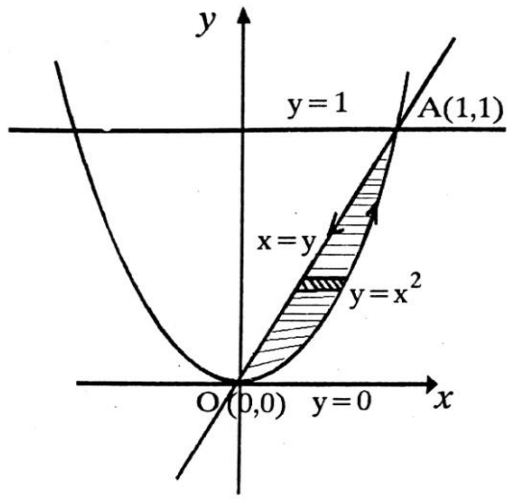
$\therefore$ From (1) and (2)

$$\therefore \int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence Green's theorem is verified.

Example: Verify Green's theorem in the XY −plane for $\int_c (xy + y^2)dx + x^2 dy$ where C is the closed curve of the region bounded by $y = x, y = x^2$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = xy + y^2$ and $N = x^2$

$\Rightarrow \frac{\partial M}{\partial y} = x + 2y \qquad \Rightarrow \frac{\partial N}{\partial x} = 2x$

R.H.S = $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Limits:

x varies from y to $\sqrt{y}$

y varies from 0 to 1

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_y^{\sqrt{y}} 2x - (x + 2y)\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy\right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y\sqrt{y}\right) - \left(\frac{y^2}{2} - 2y^2\right) dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y^{\frac{3}{2}} + 3\frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^2}{2} - \frac{4y^{\frac{5}{2}}}{5} + \frac{y^3}{2}\right]_0^1$$

$$= \frac{1}{4} - \frac{4}{5} + \frac{1}{2} = -\frac{1}{20}$$

L.H.S = $\int_c M\,dx + N\,dy$

Consider $\int M\,dx + N\,dy = \int_{OA} + \int_{AO}$

Along $OA, y = x^2 \;\Rightarrow dy = 2x\,dx, x$ varies from 0 to 1

$$\therefore \int_{OA} M\,dx + N\,dy = \int_0^1 [(x(x^2) + (x^2)^2)dx + x^2 \cdot 2x\,dx]$$

$$= \int_0^1 (3x^3 + x^4)\,dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Along $AO, y = x \;\Rightarrow dy = dx \; and \; x$ varies from 1 to 0

$$\therefore \int_{A0} M\,dx + N\,dy = \int_1^0 (x^2 + x^2)dx + x^2\,dx$$

$$= \int_1^0 3x^2 dx = [x^3]_1^0 = -1$$

L.H.S = $\int_c M\,dx + N\,dy = \frac{19}{20} - 1 = -\frac{1}{20}$

$\therefore$ L.H.S = R.H.S

Hence Green's theorem is verified.

Example: Verify Green's theorem in the plane for the integral $\int_c (x - 2y)dx + xdy$ taken around the circle $x^2 + y^2 = 1$.

Solution:

We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = x - 2y \; and \; N = x$

$\Rightarrow \frac{\partial M}{\partial y} = -2 \qquad \Rightarrow \frac{\partial N}{\partial x} = 1$

R.H.S = $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1+2)dxdy$$

$$= 3 \iint_R dxdy$$

$= 3$ (Area of the circle)

$= 3\pi r^2$

$= 3\pi \quad (\because radius = 1)$

L.H.S = $\int_c M\,dx + N\,dy$

Given C is $x^2 + y^2 = 1$

The parametric equation of circle is

$x = \cos\theta,\ y = \sin\theta$

$dx = -\sin\theta d\theta,\ dy = \cos\theta\, d\theta$

Where θ varies from 0 to 2π

$$\therefore \int_c M\,dx + N\,dy = \int_0^{2\pi}(\cos\theta - 2\sin\theta)\,(-\sin\theta\, d\theta) + \cos\theta\,(\cos\theta\, d\theta)$$

$$= \int_0^{2\pi}(-\sin\theta\cos\theta + 2\sin^2\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi}(-\sin\theta\cos\theta + \sin^2\theta + 1)\, d\theta \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= \int_0^{2\pi}\left(-\frac{\sin 2\theta}{2} + \frac{1-\cos 2\theta}{2} + 1\right) d\theta$$

$$= \left[-\frac{1}{2}\left(-\frac{\cos 2\theta}{2}\right) + \frac{\theta}{2} - \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right) + \theta\right]_0^{2\pi}$$

$$= \left[\frac{\cos(4\pi)}{4} + \frac{2\pi}{2} - \frac{\sin 4\pi}{4} + 2\pi\right] - \left[\frac{\cos 0}{4} + \frac{0}{2} - \frac{\sin 0}{4} + 0\right]$$

$$= \frac{1}{4} + \pi + 2\pi - \frac{1}{4} = 3\pi \quad [\because \sin n\pi = 0, \sin 0 = 0, \cos 0 = 1],$$

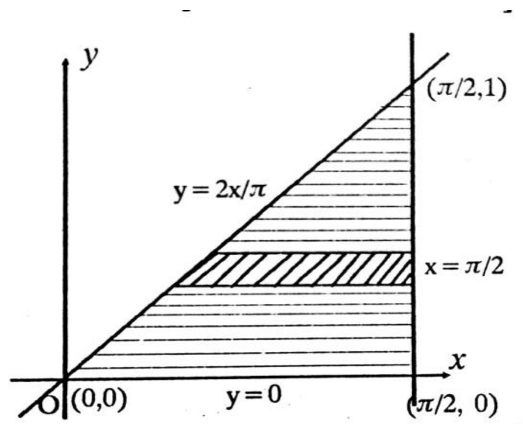
$[\cos n\pi = (-1)^n]$

$\therefore$ L.H.S = R.H.S

Hence Green's theorem is verified.

Example: Using Green's theorem evaluate $\int_c (y - \sin x)dx + \cos x\, dy$ where C is the triangle bounded by $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = y - \sin x$ and $N = \cos x$

$\Rightarrow \frac{\partial M}{\partial y} = 1 - 0$ $\Rightarrow \frac{\partial N}{\partial x} = -\sin x$

Limits:

x varies from $\frac{y\pi}{2}$ to $\frac{\pi}{2}$

y varies from 0 to 1

Hence $\int_c (y - sin\,x)dx + cos\,x\,dy = \int_0^1 \int_{\frac{y\pi}{2}}^{\frac{\pi}{2}} (-\sin x - 1)\,dx\,dy$

$$= \int_0^1 (\cos x - x)_{\frac{y\pi}{2}}^{\frac{\pi}{2}}\,dy$$

$$= \int_0^1 \left[\left(\cos\frac{\pi}{2} - \frac{\pi}{2}\right) - \left(\cos\left(\frac{y\pi}{2}\right) - \frac{y\pi}{2}\right)\right] dy$$

$$= \int_0^1 \left[0 - \frac{\pi}{2} - \cos\frac{y\pi}{2} + \frac{y\pi}{2}\right] dy$$

$$= \left[-\frac{\pi}{2}y - \frac{\sin\frac{y\pi}{2}}{\frac{\pi}{2}} + \frac{\pi}{2}\frac{y^2}{2}\right]_0^1$$

$$= -\frac{\pi}{2} - \frac{2}{\pi}\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{4}$$

$$= -\frac{\pi}{2} - \frac{2}{\pi} + \frac{\pi}{4}$$

$$= -\frac{\pi}{4} - \frac{2}{\pi} = -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

Example: Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\int_c (xdy - ydx)$. Hence find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by using Green's theorem.

Solution:

By Green theorem, $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Let $M = -y$ and $N = x$

$\Rightarrow \frac{\partial M}{\partial y} = -1 \qquad \Rightarrow \frac{\partial N}{\partial x} = 1$

$$\therefore \int_c (xdy - ydx) = \iint_R (1+1)\,dx\,dy$$

$$= 2\iint_R dx\,dy = 2 \quad \text{(Area enclosed by C)}$$

$$\therefore \text{Area enclosed by } C = \frac{1}{2}\int_c (xdy - ydx)$$

Equation of ellipse in parametric form is $x = a\cos\theta$ and $y = b\sin\theta$ where $0 \le \theta \le 2\pi$.

$$\therefore \text{Area of the ellipse} = \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\,d\theta$$

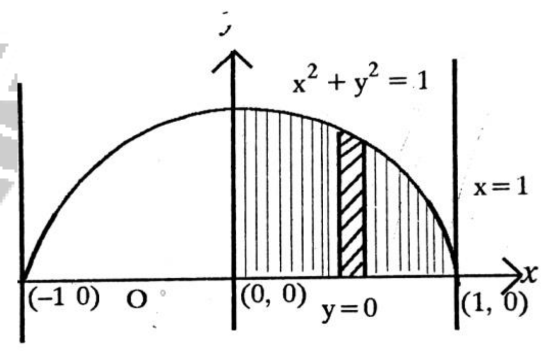
$$= \frac{1}{2}\,ab\int_0^{2\pi} (\cos^2\theta + \sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\,ab\int_0^{2\pi} d\theta = \frac{1}{2}\,ab\,[\theta]_0^{2\pi} = \pi ab$$

Example: Evaluate the integral using Green's theorem

$\int_c (2x^2 - y^2)dx + (x^2 + y^2)dy$ **where C is the boundary in the** xy **– plane of the area enclosed by the** x **– axis and the semicircle** $x^2 + y^2 = a^2$ **in the upper half** xy **– plane.**

Solution:



In this figure $'a'$ is represented as 1

By Green theorem, $\int_c \; M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Let $M = 2x^2 - y^2$ and $N = x^2 + y^2$

$$\Rightarrow \frac{\partial M}{\partial y} = -2y \qquad \Rightarrow \frac{\partial N}{\partial x} = 2x$$

Limits:

y varies from 0 to $\sqrt{a^2 - x^2}$

x varies from $-a$ to a

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (2x + 2y)\,dy\,dx$$

$$= 2\int_{-a}^{a}\left[xy + \frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a}\left[x\sqrt{a^2 - x^2} + \frac{a^2 - x^2}{2}\right] dx$$

In the first integral, the function is odd function.

∴ The value is zero.

∴ we get $2\int_{-a}^{a}\frac{a^2 - x^2}{2}\,dx$

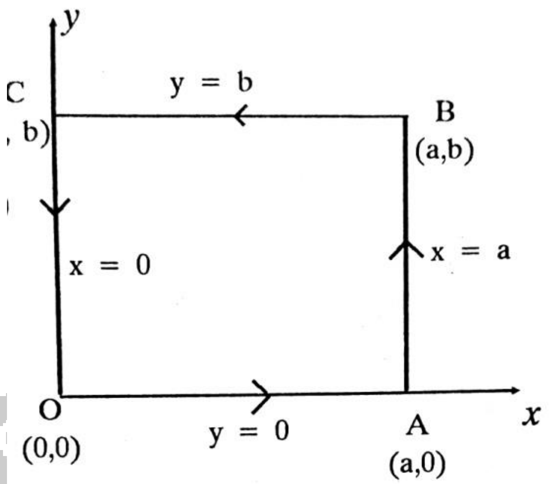
$$= \left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$= \left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)$$

$$= \frac{4a^3}{3}$$

Example: Verify stokes theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in a rectangular region in the xoy plane bounded by the lines $x = 0, x = a, y = 0, y = b$.

Solution:



By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

To evaluate: $\iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

Given $\vec{F} = (x^2 - y^2)\vec{i} + 2\,x\,y\vec{j}$

Curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$= \vec{i}\,(0) - \vec{j}\,(0 - 0) + \vec{k}[\,2y - (0 - 2y)]$

$= 4y\,\vec{k}$

Since the surface is a rectangle in the xy plane, $\hat{n} = \vec{k}$, $dS = dxdy$

Curl $\vec{F} \cdot \hat{n} = 4y\ \vec{k} \cdot \vec{k} = 4y$

Order of integration is $dxdy$

x varies from $x = 0$ to $x = a$

y varies from $y = 0$ to $y = b$

$$\Rightarrow \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS = \int_0^b \int_0^a 4y\, dx\, dy$$

$$= \int_0^b 4y\,[x]_0^a dy$$

$$= \int_0^b 4ay dy$$

$$= \left[\frac{4ay^2}{2}\right]_0^b$$

$$= 2ab^2$$

$$\Rightarrow \iint_S Curl\,\vec{F} \cdot \hat{n}\,dS = 2ab^2 \qquad \dots (1)$$

Here the line integral over the simple closed curve C bounding the surface $OABCO$ consisting of the edges OA, AB ,BC and CO.

Curve	Equation	Limit
OA	$y = 0$	$x = 0$ to $x = a$
AB	$x = a$	$y = 0$ to $y = b$
BC	$y = b$	$x = a$ to $x = 0$
CO	$x = 0$	$y = b$ to $y = 0$

Therefore, $\int_c \vec{F} \cdot d\vec{r} = \int_{OABCO} \vec{F} \cdot d\vec{r}$

$$\int_c \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$\vec{F} \cdot d\vec{r} = (x^2 - y^2) + 2xy\text{dy} \qquad \dots (2)$$

On OA: $y = 0, dy = 0, x$ varies from 0 to a

(2) $\Rightarrow \vec{F}\,.\,\text{d}\vec{r} = x^2dx$

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^a x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

On AB: $x = a, dx = 0, y$ varies from 0 to b

(2) $\Rightarrow \vec{F}\,.\,\text{d}\vec{r} = 2ay\,dy$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^b 2ay\,dy$$

$$= \left[\frac{2ay^2}{2}\right]_0^b = ab^2$$

On BC: $y = b, dy = 0, x$ varies from a to 0

(2) $\Rightarrow \vec{F}\,.\,\text{d}\vec{r} = (x^2 - b^2)dx$

$\int_{BC} \vec{F} \cdot d\vec{r} = \int_a^0 x^2 - b^2 \, dx$

$= \left[\frac{x^3}{3} - b^2 x\right]_a^0$

$= -\frac{a^3}{3} + a\, b^2$

On CO: $x = 0, dx = 0, y$ varies from $b \, to \, 0$

(2) $\Rightarrow \vec{F} \,.\, \mathrm{d}\vec{r} = 0$

$\int_{CO} \vec{F} \cdot d\vec{r} = 0$

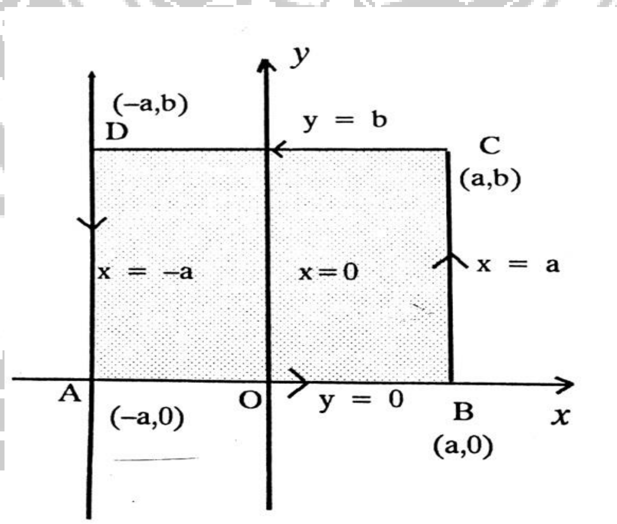
(2) $\Rightarrow \vec{F} \,.\, \mathrm{d}\vec{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 = 2ab^2$ ... (3)

From (3) and (1) $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\, \vec{F} \cdot \hat{n}\, dS$

Hence Stokes theorem is verified.

Example: Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\vec{\imath} - 2xy\,\vec{\jmath}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0, y = b$.

Solution:



By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\, \vec{F} \cdot \hat{n}\, dS$

Given $\vec{F} = (x^2 + y^2)\vec{\imath} - 2xy\,\vec{\jmath}$

$$Curl\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}\,[0-0] + \vec{k}\,[-2y-2y]$$

$$= -4y\,\vec{k}$$

Since the region is in xoy plane we can take $\hat{n} = \vec{k}\ and\ dS = dx\ dy$

Limits:

x varies from – a to a.

y varies from 0 to b.

$$\therefore \iint_S Curl\,\vec{F}\cdot\hat{n}\,dS = -4\int_0^b\int_{-a}^a y\,dx\,dy$$

$$= -4\int_0^b [xy]_{-a}^a\,dy$$

$$= -8a\left[\frac{y^2}{2}\right]_0^b = -4ab^2 \quad \dots(1)$$

$$\int_c \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB: $y=0, dy=0, x$ varies from – $a\ to\ a$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$$\int_{AB}\vec{F}\cdot d\vec{r} = \int_{-a}^a x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{-a}^a = \frac{2a^3}{3}$$

Along $BC, x=a, dx=0,\ y$ varies from $0\ to\ b$

$$\int_{BC}\vec{F}\cdot d\vec{r} = \int_0^b(-2ay)\,dy$$

$$= -a[y^2]_0^b = -ab^2$$

Along CD: $y=b, dy=0, x$ varies from $a\ to -a$

$$\int_{CD}\vec{F}\cdot d\vec{r} = \int_a^{-a}(x^2+b^2)\,dx = \left[\frac{x^3}{3}+b^2x\right]_a^{-a}$$

$$= -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -\frac{2a^3}{3} - 2ab^2$$

Along DC: $x=-a, dx=0, y$ varies from $b\ to\ 0$

$$\int_{DC} \vec{F} \cdot d\vec{r} = \int_b^0 2ay\, dy$$

$$= a[y^2]_b^0 = -b^2 a$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - b^2 a$$

$$= -4ab^2 \quad \ldots.(2)$$

From (1) and (2) $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\, \vec{F} \cdot \vec{n}\, dS$

Hence Stoke's theorem is verified.

Example: Verify Stoke's theorem for $\vec{F} = y^2 z\vec{i} + z^2 x\vec{j} + x^2 y\,\vec{k}$, where S is the open surface of the cube formed by the planes $x = \pm a$, $y = \pm a$, and $z = \pm a$ in which the plane $z = -a$ is a cut.

Solution:

Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$

Given $\vec{F} = y^2 z\vec{i} + z^2 x\vec{j} + x^2 y\,\vec{k}$

$\vec{F} \cdot d\vec{r} = y^2 z dx + z^2 x dy + x^2 y\, dz$

This square ABCD lies in the plane $z = -a \Rightarrow dz = 0$

$\therefore \vec{F} \cdot d\vec{r} = -ay^2 dx + a^2 x\, dy$

L.H.S $= \int_c \vec{F} \cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

On AB: $y = -a \Rightarrow dy = 0$, x varies from $-a$ to a.

$$\Rightarrow \int_{AB} \vec{F} \cdot d\vec{r} = \int_{-a}^{a} -a^3\, dx$$

$$= -a^3\, [x]_{-a}^{a}$$

$$= -a^3(2a) = -2a^4$$

On BC: $x = a \Rightarrow dx = 0$, y varies from $-a$ to a.

$$\Rightarrow \int_{BC} \vec{F} \cdot d\vec{r} = \int_{-a}^{a} a^3\, dy$$

$$= a^3\, [y]_{-a}^{a}$$

$$= a^3(2a) = 2a^4$$

On CD: $y = a \Rightarrow dy = 0$, x varies from a to $-a$.

$$\Rightarrow \int_{CD} \vec{F} \cdot d\vec{r} = \int_a^{-a} -a^3\, dx$$

$$= -a^3\, [x]_a^{-a}$$

$$= -a^3(-2a) = 2a^4$$

On DA: $x = -a \Rightarrow dx = 0$, y varies from a to $-a$.

$$\Rightarrow \int_{DA} \vec{F} \cdot d\vec{r} = \int_a^{-a} -a^3\, dy$$

$$= -a^3\, [y]_a^{-a}$$

$$= -a^3(-2a) = 2a^4$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = -2a^4 + 2a^4 + 2a^4 + 2a^4 = 4a^4 \quad \ldots (1)$$

R.H.S $= \iint_S curl\, \vec{F} \cdot \hat{n}\; ds$

$$curl\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$$

$$= \vec{i}\,(x^2 - 2xz) - \vec{j}\,(y^2 - 2xy) + \vec{k}\,(z^2 - 2yz)$$

Given S is an open surface consisting of the 5 faces of the cube except, $z = -a$.

$$\int\int curl\, \vec{F} \cdot \hat{n}\; ds = \iint_{S1} + \iint_{S2} + \cdots + \iint_{S5}$$

$$curl\, \vec{F} = 2y\vec{i} + z\,\vec{j} - x\,\vec{k}$$

Faces	Plane	ds	$\hat{n}$	Eqn	$curl\, \vec{F} \cdot \hat{n}$	$\nabla \times \vec{F} \cdot \hat{n}$
Top (S_1)	xy	$dxdy$	$\vec{k}$	$z = a$	$z^2 - 2yz$	$a^2 - 2ay$
Left (S_2)	xz	$dxdz$	$-\vec{j}$	$y = -a$	$y^2 - 2xy$	$a^2 + 2ax$
Right (S_3)	xz	$dxdz$	$\vec{j}$	$y = a$	$-(y^2 - 2xy)$	$-(a^2 - 2ax)$
Back (S_4)	yz	$dydz$	$-\vec{i}$	$x = -a$	$-(x^2 - 2xz)$	$-(a^2 + 2az)$
Front (S_5)	yz	$dydz$	$\vec{i}$	$x = a$	$x^2 - 2xz$	$a^2 - 2az$

On S_1: $\int_{-a}^{a}\int_{-a}^{a}(a^2 - 2ay)\, dxdy$

$= \int_{-a}^{a}[(a^2x - 2ayx)]_{-a}^{a}\, dy$

$= \int_{-a}^{a}(a^3 - 2a^2y) - (-a^3 + 2a^2y)\, dy$

$= \int_{-a}^{a} 2a^3 - 4a^2y\, dy$

$= \left[2a^3y - 4a^2\frac{y^2}{2}\right]_{-a}^{a}$

$= (2a^4 - 2a^4) - (-2a^4 - 2a^4)$

$= 2a^4 - 2a^4 + 2a^4 + 2a^4$

$= 4a^4$

On $S_2 + S_3$: $\int_{-a}^{a}\int_{-a}^{a}(a^2 + 2ax)\, dxdz + \int_{-a}^{a}\int_{-a}^{a} -(a^2 - 2ax)\, dxdz$

$= \int_{-a}^{a}\int_{-a}^{a}(a^2 + 2ax - a^2 + 2ax)\, dx\, dz$

$= \int_{-a}^{a}\int_{-a}^{a} 4ax\, dx\, dz$

$= 4a\int_{-a}^{a}\left[\frac{x^2}{2}\right]_{-a}^{a} dz$

$= 2a^3\int_{-a}^{a} dz$

$= 2a^3\,[z]_{-a}^{a}$

$= 2a^3(0) = 0$

On $S_4 + S_5$: $\int_{-a}^{a}\int_{-a}^{a} -(a^2 + 2az)\, dydz + \int_{-a}^{a}\int_{-a}^{a}(a^2 - 2az)\, dydz$

$= \int_{-a}^{a}\int_{-a}^{a}(-a^2 - 2az + a^2 - 2az)\, dy\, dz$

$= \int_{-a}^{a}\int_{-a}^{a} -4az\, dy\, dz$

$= -4a\int_{-a}^{a}[zy]_{-a}^{a} dz$

$= -4a\int_{-a}^{a} z(2a) dz$

$= -6a^2\left[\frac{z^2}{2}\right]_{-a}^{a}$

$= -3a^2(a^2 - a^2) = 0$

$\therefore \iint_S curl\,\vec{F}\cdot\hat{n}\, ds = 4a^4 + 0 + 0 = 4a^4 \quad \ldots (2)$

From (1) and (2) $\int_c \vec{F}\cdot d\vec{r} = \iint_S curl\,\vec{F}\cdot\hat{n}\, ds$

Hence Stoke's theorem is verified.

Example: Evaluate $\int_c \vec{F} \cdot d\vec{r}$ by stoke's theorem, where $\vec{F} = y^2\vec{i} + x^2\vec{j} + (x+z)\vec{k}$, and C is the boundary of the triangle with vertices at $(0,0,0), (1,0,0)$ *and* $(1,1,0)$.

Solution:

Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n}\, ds \quad \ldots (1)$

Given $\vec{F} = y^2\vec{i} + x^2\vec{j} + (x+z)\vec{k}$

And C is triangle $(0,0,0), (1,0,0)$ *and* $(1,1,0)$.

Since z −coordinate of each vertex is zero the triangle lies in xy − plane with corners $(0,0), (1,0)$ *and* $(1,1)$.

To evaluate : $\iint_S curl\,\vec{F} \cdot \hat{n}\, ds$

$In\ xy - plane\ \hat{n} = \vec{k}, ds = dxdy$

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(-1) + \vec{k}(2x - 2y)$$

$$= \vec{j} + 2(x-y)\vec{k}$$

$$curl\,\vec{F} \cdot \hat{n} = \left(\vec{j} + 2(x-y)\vec{k}\right) \cdot \vec{k}$$

$$= 2(x-y)$$

Limits:

x varies from y to 1.

y varies from 0 to 1.

$$\therefore \iint_S curl\,\vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_y^1 2(x-y)dxdy$$

$$= 2\int_0^1 \left[\frac{x^2}{2} - xy\right]_y^1 dy$$

$$= 2\int_0^1 \left(\frac{1}{2} - y - \frac{y^2}{2} + y^2\right) dy$$

$$= 2\left[\frac{y}{2} - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^3}{3}\right]_0^1$$

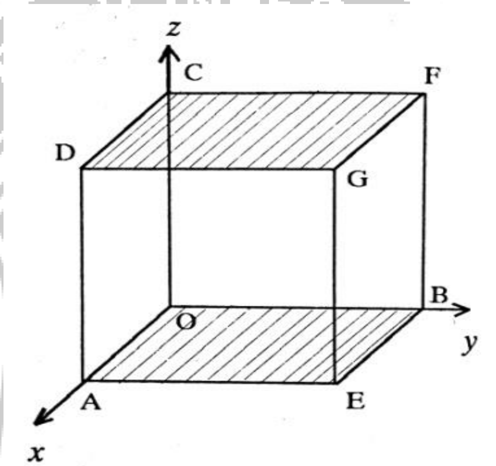
$$= 2\left[\frac{1}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{3}\right]$$

$$= 2\left[\frac{1}{6}\right] = \frac{1}{3}$$

From (1), $\int_c \vec{F} \cdot d\vec{r} = \frac{1}{3}$

Example: Verify the G.D.T for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

Solution:



Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$\nabla \cdot \vec{F} = 4z - 2y + y$

$= 4z - y$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$= \int_0^1 \int_0^1 \int_0^1 (4z - y)\, dxdydz$

$= \int_0^1 \int_0^1 [(4xz - yz)]_0^1\, dydz$

$= \int_0^1 \int_0^1 (4z - y)\, dydz$

$= \int_0^1 \left(4zy - \frac{y^2}{2}\right)_0^1 dz$

$= \int_0^1 \left(4z - \frac{1}{2}\right) dz$

$= \left[4\frac{z^2}{2} - \frac{1}{2}z\right]_0^1 = \left(2 - \frac{1}{2}\right) - 0 = \frac{3}{2}$

Now, L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	**Plane**	dS	$\hat{n}$	$\vec{F}\cdot\hat{n}$	**Equation**	$\vec{F}\cdot\hat{n}$ **on S**	$= \iint_S \vec{F} \cdot \hat{n}\, ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-yz$	$z = 0$	0	$\int_0^1 \int_0^1 0\, dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	yz	$z = 1$	y	$\int_0^1 \int_0^1 y\, dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	y^2	$y = 0$	0	$\int_0^1 \int_0^1 0\, dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	$-y^2$	$y = 1$	-1	$\int_0^1 \int_0^1 -1 dxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-4xz$	$x = 0$	0	$\int_0^1 \int_0^1 0\, dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	$4xz$	$x = 1$	$4z$	$\int_0^1 \int_0^1 4z\, dydz$

$(i) \iint_{S1} \vec{F} \cdot \hat{n}\, ds + \iint_{S2} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dxdy + \int_0^1 \int_0^1 y\, dxdy$

$$= 0 + \int_0^1 \int_0^1 y\, dxdy$$

$$= \int_0^1 [yx]_0^1\, dy$$

$$= \int_0^1 y\, dy$$

$$= \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2} - 0 = \frac{1}{2}$$

$(ii) \iint_{S3} \vec{F} \cdot \hat{n}\, ds + \iint_{S4} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dxdz + \int_0^1 \int_0^1 -1\, dxdz$

$$= 0 + \int_0^1 \int_0^1 -1\, dxdz$$

$$= -\int_0^1 [x]_0^1\, dz$$

$$= -\int_0^1 dz$$

$$= -[z]_0^1 = -[1]$$

$$(iii) \iint_{S5} \vec{F} \cdot \hat{n}\, ds + \iint_{S6} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dydz + \int_0^1 \int_0^1 4z dydz$$

$$= 0 + \int_0^1 \int_0^1 4z\, dydz$$

$$= \int_0^1 [4zy]_0^1\, dz$$

$$= \int_0^1 4z\, dz$$

$$= 4\left[\frac{z^2}{2}\right]_0^1 = 4\left(\frac{1}{2} - 0\right) = 2$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

$$= \frac{1}{2} - 1 + 2 = \frac{3}{2}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Hence Gauss divergence theorem is verified.

Example: Verify the G.D.T for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. (OR)

Verify the G.D.T for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ over the rectangular parallelopiped bounded by $x = 0, x = a, y = 0, y = b, z = 0, z = c$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$

$$\nabla \cdot \vec{F} = 2x + 2y + 2z = 2(x + y + z)$$

Now, R.H.S = $\iiint_V \nabla \cdot \vec{F}\, dv$

$$= 2\int_0^c \int_0^b \int_0^a (x + y + z) dxdydz$$

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$$= 2\int_0^c \int_0^b \left[\left(\frac{x^2}{2} + xy + xz\right)\right]_0^a dydz$$

$$= 2\int_0^c \int_0^b \left(\frac{a^2}{2} + ay + az\right) dydz$$

$$= 2\int_0^c \left(\frac{a^2y}{2} + \frac{ay^2}{2} + azy\right)_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + azb\right) dz$$

$$= 2\left[\frac{a^2bz}{2} + \frac{ab^2z}{2} + \frac{abz^2}{2}\right]_0^c$$

$$= 2\left(\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right)$$

$$= abc(a + b + c)$$

Now, L.H.S $= \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	**Plane**	dS	$\hat{n}$	$\vec{F}\cdot\hat{n}$	**Eqn**	$\vec{F}\cdot\hat{n}$ **on S**	$= \iint_S \vec{F}\cdot\hat{n}\,ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-(z^2 - xy)$	$z = 0$	xy	$\int_0^b\int_0^a xy\,dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	$(z^2 - xy)$	$z = c$	$c^2 - xy$	$\int_0^b\int_0^a c^2 - xy\,dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	$-(y^2 - xz)$	$y = 0$	xz	$\int_0^c\int_0^a xz\,dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	$(y^2 - xz)$	$y = b$	$b^2 - xz$	$\int_0^c\int_0^a b^2 - xz\,dxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-(x^2 - yz)$	$x = 0$	yz	$\int_0^c\int_0^b yz\,dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	$(x^2 - yz)$	$x = a$	$a^2 - yz$	$\int_0^c\int_0^b a^2 - yz\,dydz$

$(i) \iint_{S1} \vec{F}\cdot\hat{n}\,ds + \iint_{S2} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a xy\,dxdy + \int_0^b\int_0^a c^2 - xy\,dxdy$

$$= \int_0^b\int_0^a c^2\,dxdy$$

$$= c^2\int_0^a dx\int_0^b dy$$

$$= c^2[x]_0^a[y]_0^b \quad = c^2ab$$

$(ii) \iint_{S3} \vec{F}\cdot\hat{n}\,ds + \iint_{S4} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a xz\,dxdz + \int_0^c\int_0^a b^2 - xz\,dxdz$

$$= \int_0^c \int_0^a b^2 \, dxdz$$

$$= b^2 \int_0^a dx \int_0^c dz$$

$$= b^2 [x]_0^a [z]_0^c \quad = b^2 ac$$

$$(iii) \iint_{S5} \vec{F} \cdot \hat{n} \, ds + \iint_{S6} \vec{F} \cdot \hat{n} \, ds = \int_0^c \int_0^b yz \, dydz + \int_0^c \int_0^b a^2 - yz \, dydz$$

$$= \int_0^c \int_0^b a^2 \, dydz$$

$$= a^2 \int_0^b dy \int_0^c dz$$

$$= a^2 [y]_0^b [z]_0^c \quad = a^2 bc$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

$$= abc(a + b + c)$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$$

Hence Gauss divergence theorem is verified.

Example: Verify divergence theorem for $\vec{F} = (2x - z)\vec{i} + x^2 y\vec{j} - xz^2\vec{k}$ over the cube bounded by $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$

Given $\vec{F} = (2x - z)\vec{i} + x^2 y\vec{j} - xz^2\vec{k}$

$$\nabla \cdot \vec{F} = 2 + x^2 - 2xz$$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F} \, dv$

$$= \int_0^1 \int_0^1 \int_0^1 (2 + x^2 - 2xz) dxdydz$$

$$= \int_0^1 \int_0^1 \left[\left(2x + \frac{x^3}{3} - \frac{2zx^2}{2}\right)\right]_0^1 dydz$$

$= \int_0^1 \int_0^1 \left(2 + \frac{1}{3} - z\right) dydz$

$= \int_0^1 \left(2y + \frac{1}{3}y - zy\right)_0^1 dz$

$= \int_0^1 \left(2 + \frac{1}{3} - z\right) dz$

$= \left[2z + \frac{1}{3}z - \frac{z^2}{2}\right]_0^1$

$= \left(2 + \frac{1}{3} - \frac{1}{2}\right) - 0 = \frac{11}{6}$

Now, L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	Plane	dS	$\hat{n}$	$\vec{F}\cdot\hat{n}$	Equation	$\vec{F}\cdot\hat{n}$ on S	$= \iint_S \vec{F}\cdot\hat{n}\, ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	xz^2	$z = 0$	0	$\int_0^1 \int_0^1 0\, dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	$-xz^2$	$z = 1$	$-x$	$\int_0^1 \int_0^1 (-x)\, dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	$-x^2y$	$y = 0$	0	$\int_0^1 \int_0^1 0\, dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	x^2y	$y = 1$	x^2	$\int_0^1 \int_0^1 x^2 dxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-(2x - z)$	$x = 0$	z	$\int_0^1 \int_0^1 z\, dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	$(2x - z)$	$x = 1$	$2 - z$	$\int_0^1 \int_0^1 2 - z\, dydz$

$(i) \iint_{S1} \vec{F} \cdot \hat{n}\, ds + \iint_{S2} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dxdy + \int_0^1 \int_0^1 (-x)\, dxdy$

$= \int_0^1 \int_0^1 (-x)\, dxdy$

$= -\int_0^1 \left[\frac{x^2}{2}\right]_0^1 dy$

$= -\int_0^1 \frac{1}{2}\, dy$

$= -\left[\frac{1}{2}y\right]_0^1 = -\left(\frac{1}{2} - 0\right) = \frac{-1}{2}$

$$(ii) \iint_{S3} \vec{F} \cdot \hat{n}\, ds + \iint_{S4} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dxdz + \int_0^1 \int_0^1 x^2\, dxdz$$

$$= \int_0^1 \int_0^1 x^2\, dxdz$$

$$= \int_0^1 \left[\frac{x^3}{3}\right]_0^1 dz$$

$$= \int_0^1 \frac{1}{3}\, dz$$

$$= \left[\frac{1}{3}z\right]_0^1 = \left(\frac{1}{3} - 0\right) = \frac{1}{3}$$

$$(iii) \iint_{S5} \vec{F} \cdot \hat{n}\, ds + \iint_{S6} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 z\, dydz + \int_0^1 \int_0^1 (2 - z) dydz$$

$$= \int_0^1 \int_0^1 2\, dydz$$

$$= 2 \int_0^1 [y]_0^1\, dz$$

$$= 2 \int_0^1 dz$$

$$= 2\, [z]_0^1 = 2$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

$$= -\frac{1}{2} + \frac{1}{3} + 2 = \frac{11}{6}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Hence Gauss divergence theorem is verified.

Example: Verify divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube bounded by $x = \pm 1,\ y = \pm 1,\ z = \pm 1$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$

$$\nabla \cdot \vec{F} = 2x + y$$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(2x+y)dxdydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left[\left(2\frac{x^2}{2}+yx\right)\right]_{-1}^{1} dydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}[(1+y)-(1-y)]dydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}[2y]dydz$$

$$= \int_{-1}^{1}\left(2\frac{y^2}{2}\right)_{-1}^{1} dz$$

$$= \int_{-1}^{1}[(1)-((-1)^2)]\, dz$$

$$= \int_{-1}^{1}[0]\, dz$$

$$= 0$$

Now, L.H.S $= \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	**Plane**	dS	$\hat{n}$	$\vec{F}\cdot\hat{n}$	**Equation**	$\vec{F}\cdot\hat{n}$ **on S**	$= \iint_S \vec{F}\cdot\hat{n}\,ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-yz$	$z=-1$	y	$\int_{-1}^{1}\int_{-1}^{1} y\,dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	yz	$z=1$	y	$\int_{-1}^{1}\int_{-1}^{1} y\,dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	$-z$	$y=-1$	$-z$	$\int_{-1}^{1}\int_{-1}^{1} -z\,dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	z	$y=1$	z	$\int_{-1}^{1}\int_{-1}^{1} zdxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-x^2$	$x=-1$	-1	$\int_{-1}^{1}\int_{-1}^{1} -1\,dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	x^2	$x=1$	1	$\int_{-1}^{1}\int_{-1}^{1} dydz$

$(i) \iint_{S1} \vec{F}\cdot\hat{n}\,ds + \iint_{S2} \vec{F}\cdot\hat{n}\,ds = \int_{-1}^{1}\int_{-1}^{1} y\,dxdy + \int_{-1}^{1}\int_{-1}^{1} y\,dxdy$

$$= \int_{-1}^{1}\int_{-1}^{1} 2y\,dxdy$$

$$= 2\int_{-1}^{1}[xy]_{-1}^{1}\, dy$$

$$= 2\int_{-1}^{1}[(y)-(-y)]\, dy$$

$$= 2\int_{-1}^{1} 2y dy$$

$$= 4\left[\frac{y^2}{2}\right]_{-1}^{1} = 4\left[\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\right] \quad = 0$$

$$(ii) \iint_{S3} \vec{F} \cdot \hat{n}\, ds + \iint_{S4} \vec{F} \cdot \hat{n}\, ds = \int_{-1}^{1}\int_{-1}^{1} -z\, dxdz + \int_{-1}^{1}\int_{-1}^{1} z\, dxdz$$

$$= \int_{-1}^{1}\int_{-1}^{1} 0\, dxdz$$

$$= 0$$

$$(iii) \iint_{S5} \vec{F} \cdot \hat{n}\, ds + \iint_{S6} \vec{F} \cdot \hat{n}\, ds = -\int_{-1}^{1}\int_{-1}^{1} dxdz + \int_{-1}^{1}\int_{-1}^{1} dxdz$$

$$= 0$$

$$\therefore \iint_{S} \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

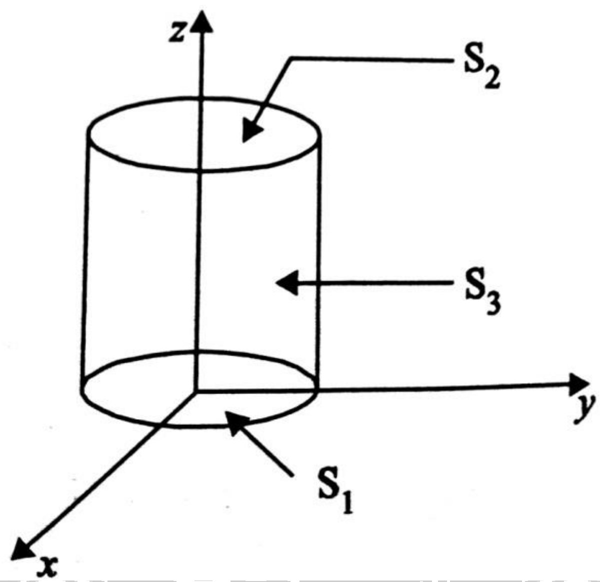
$$= 0$$

$$\therefore \iint_{S} \vec{F} \cdot \hat{n}\, ds = \iiint_{V} \nabla \cdot \vec{F}\, dv$$

Hence , Gauss divergence theorem is verified.

Example: Verify divergence theorem for the function $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the surface bounded by the cylinder $x^2 + y^2 = 4$ and $z = 0, z = 3$.

Solution:



Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$

$\nabla \cdot \vec{F} = 4 - 4y + 2z$

Limits:

$z = 0 \text{ to } 3$

$x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2$

$\Rightarrow y = \pm\sqrt{4 - x^2}$

$\therefore y = -\sqrt{4 - x^2} \text{ to } \sqrt{4 - x^2}$

Put $y = 0 \Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

$\therefore y = -2 \text{ to } 2$

$\therefore$ R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4 - 4y + 2z)\, dzdydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z - 4yz + 2\frac{z^2}{2}\right]_0^3 dydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)dydx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)dydx$$

$$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} 21\, dydx \qquad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x) \text{ if } f(x) \text{ is even} \atop = 0 \text{ if } f(x) \text{ is odd}\right]$$

$$= 42\int_{-2}^{2}[y]_0^{\sqrt{4-x^2}}\, dx$$

$$= 42\int_{-2}^{2}\sqrt{4-x^2}\, dx$$

$$= 42 \times 2\int_{0}^{2}\sqrt{4-x^2}\, dx \qquad [\because \text{even function}]$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\,[0 + 2\,sin^{-1}(1)]$$

$$= 84\left[2 \times \frac{\pi}{2}\right]$$

$$= 84\,\pi$$

L.H.S $= \iint_S \vec{F}\cdot\hat{n}\,ds$

$$= \iint_{S1} + \iint_{S2} + \iint_{S3}$$

Along S_1 (bottom):

xy −plane $\Rightarrow z = 0, dz = 0$

And $ds = dxdy,\ \hat{n} = -\vec{k}$

$\therefore \vec{F}\cdot\hat{n} = (4x\,\vec{i} - 2y^2\,\vec{j} + z^2\,\vec{k})\cdot(-\vec{k})$

$$= -z^2 = 0$$

$\therefore \iint_{S1} \vec{F}\cdot\hat{n}\,ds = \iint_{S1} 0 = 0$

Along S_2 (top):

xy −plane $\Rightarrow z = 3, dz = 0$

And $ds = dxdy,\ \hat{n} = \vec{k}$

$\therefore \vec{F}\cdot\hat{n} = (4x\,\vec{i} - 2y^2\,\vec{j} + z^2\,\vec{k})\cdot(-\vec{k})$

$$= z^2 = 9$$

$\therefore \iint_{S2} \vec{F}\cdot\hat{n}\,ds = \iint_{S2} 9\,dxdy$

$$= \iint_R 9\,dxdy$$

$$= 9\ (Area\ of\ the\ circle)$$

$$= 9\ (\pi r^2) \qquad [\because r = 2]$$

$$= 36\,\pi$$

Along S_3 (curved surface):

Given $x^2 + y^2 = 4$

Let $\varphi = x^2 + y^2 - 4$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{4} = 4$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2(x\vec{i}+y\vec{j})}{4}$$

$$= \frac{x\vec{i}+y\vec{j}}{2}$$

The cylindrical coordinates are

$x = 2\cos\theta,\ y = 2\sin\theta\ ds = 2dzd\theta$

Where z varies from 0 to 3

θ varies from 0 to 2π

Now $\vec{F}\cdot\hat{n} = \left(4x\,\vec{i} - 2y^2\,\vec{j} + z^2\,\vec{k}\right)\cdot\left(\frac{x\vec{i}+y\vec{j}}{2}\right)$

$$= 2x^2 - y^3$$

$$= 2(2\cos\theta)^2 - (2\sin\theta)^3$$

$$= 8\cos^2\theta - 8\sin^3\theta$$

$$= 8\left[\frac{1+\cos 2\theta}{2} - \left(\frac{3\sin\theta - \sin 3\theta}{4}\right)\right]$$

$$\therefore \iint_{S3} \vec{F}\cdot\hat{n}\,ds = 8\int_0^{2\pi}\int_0^3 \left(\frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4}\right) 2dzd\theta$$

$$= 16\int_0^{2\pi}\left(\frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4}\right)[z]_0^3\,d\theta$$

$$= 48\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} - \frac{3\cos\theta}{4} - \frac{\cos 3\theta}{12}\right]_0^{2\pi}$$

$$= 48\left[\left(\frac{2\pi}{2} + \frac{3}{4} - \frac{1}{12}\right) - \left(\frac{3}{4} - \frac{1}{12}\right)\right]$$

$$= 48\,\pi$$

$$\text{L.H.S} = \iint_S \vec{F} \cdot \hat{n}\, ds = 0 + 36\,\pi + 48\pi$$

$$= 84\,\pi$$

∴ L.H.S = R.H.S

$$(i.e) \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Hence Gauss divergence theorem is verified.