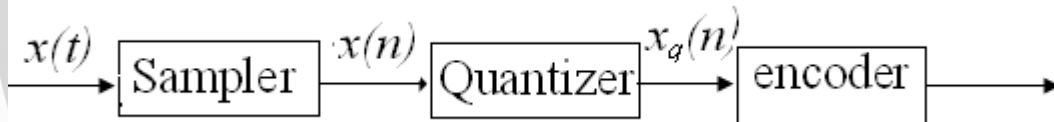


QUANTIZATION NOISE, COEFFICIENT QUANTIZATION ERROR

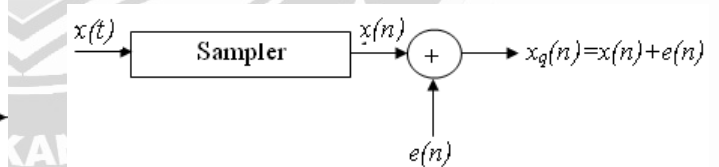
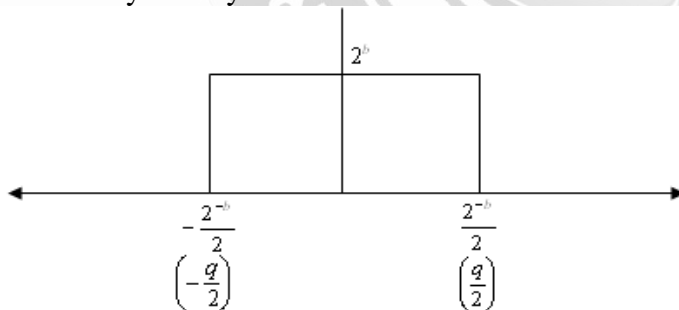
Quantization Noise power:

Input Quantization error:

***Derive the equation for quantization noise power (or) Steady state Input Noise Power.**



Probability density function for round off error in A/D conversion is



If rounding is used for quantization, which is bounded by $-\frac{q}{2} \leq e(n) \leq \frac{q}{2}$, then the error lies between $-\frac{q}{2}$ to $\frac{q}{2}$ with equal probability, where $q \rightarrow$ quantization step size.

Properties of analog to digital conversion error, e(n):

1. The error sequence $e(n)$ is a sample sequence of a stationary random process.
2. The error sequence is uncorrelated with $x(n)$ and other signals in the system.
3. The error is a white noise process with uniform amplitude probability distribution over the range of quantization error.

The variance of $e(n)$ is given by

$$\sigma_e^2 = E[e^2(n)] - E^2[e(n)] \text{ -----} >(1)$$

Where $E[e^2(n)] \rightarrow$ Average of $e^2(n)$

$E[e(n)] \rightarrow$ Mean value of $e(n)$.

For rounding, $e(n)$ lies between $-\frac{q}{2}$ and $\frac{q}{2}$ with equal probability

$$E[e^2(n)] = \int_{-\infty}^{\infty} e^2(n)p(e)de \text{-----}>(2)$$

$$p(e) = \frac{1}{q}, -\frac{q}{2} \leq e(n) \leq \frac{q}{2} \text{-----}>(3)$$

Substituting (3) in (2)

$$E[e^2(n)] = \int_{-\frac{q}{2}}^{\frac{q}{2}} e^2(n) \frac{1}{q} de$$

$$E[e^2(n)] = \frac{1}{q} \int_{-\frac{q}{2}}^{\frac{q}{2}} e^2(n) de \text{-----}>(4)$$

$$E[e(n)] = 0$$

$$E^2[e(n)] = 0 \text{-----}>(5)$$

Substituting (4) and (5) in (1)

$$\sigma_e^2 = \frac{1}{q} \int_{-\frac{q}{2}}^{\frac{q}{2}} e^2(n) de - 0$$

$$= \frac{1}{q} \left[\frac{e^3}{3} \right]_{-\frac{q}{2}}^{\frac{q}{2}}$$

$$= \frac{1}{3q} \left[\left(\frac{q}{2} \right)^3 - \left(-\frac{q}{2} \right)^3 \right]$$

$$= \frac{1}{3q} \left[\left(\frac{q^3}{8} \right) - \left(-\frac{q^3}{8} \right) \right]$$

$$= \frac{1}{3q} \left[\left(\frac{q^3}{8} \right) + \left(\frac{q^3}{8} \right) \right]$$

$$= \frac{1}{3q} \left[\frac{2q^3}{8} \right]$$

$$\sigma_e^2 = \frac{q^2}{12} \text{-----}>(6)$$

In general,

$$\frac{1}{2^b} = 2^{-b} = q \text{-----}>(7)$$

$$\sigma_e^2 = \frac{(2^{-b})^2}{12}$$

$$\sigma_e^2 = \frac{2^{-2b}}{12} \text{-----}>(8)$$

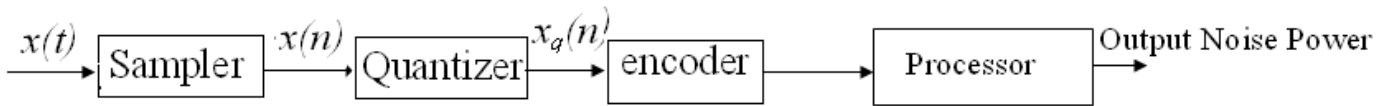
Equation (8) is known as the steady state noise power due to input quantization.

$$q = \frac{R}{2^b} \quad \rightarrow \text{in two's complement representation.}$$

$$q = \frac{R}{2^b - 1} \quad \rightarrow \text{in sign magnitude (or) one's complement representation.}$$

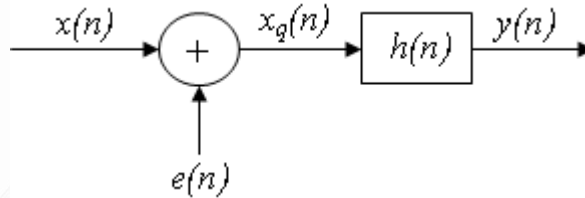
$$R \quad \rightarrow \text{Range of analog signal to be quantized.}$$

Steady state Output Noise power:



After quantization, we have noise power σ_e^2 as input noise power. Therefore, Output noise power of system is given by

$$\sigma_{eo}^2 = \sigma_e^2 \left[\sum_{n=0}^{\infty} h^2(n) \right] \text{----->(9)}$$



where $h(n) \rightarrow$ impulse response of the system.

Let error E(n) be output noise power due to quantization

Error

$$E(n) = e(n) * h(n) \\ = \sum_{k=0}^{\infty} h(n)e(n-k)$$

The variance of error E(n) is called output noise power, σ_e^2 .

By using Parseval's theorem,

$$\sigma_{eo}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n) \\ = \sigma_e^2 \frac{1}{2\pi j} \oint H(Z)H(Z^{-1}) \frac{dZ}{Z}$$

Where the closed contour integration is evaluated using the method of residue by taking only the poles that lie inside the unit circle.

Z transform of h(n), $H(Z) = \sum_{n=0}^{\infty} h(n)z^{-n}$

Z transform of $h^2(n) = Z[h^2(n)] = \sum_{n=0}^{\infty} h^2(n)z^{-n} = \sum_{n=0}^{\infty} h(n)h(n)z^{-n}$ ----->(10)

By Inverse Z transform, $h(n) = \frac{1}{2\pi j} \oint H(Z)Z^{n-1} dZ$ ----->(11)

Substituting (11) in (10)

$$\sum_{n=0}^{\infty} h^2(n)z^{-n} = \sum_{n=0}^{\infty} \frac{1}{2\pi j} \oint H(Z)Z^{n-1} dZ h(n)z^{-n} \\ = \frac{1}{2\pi j} \oint H(Z) \left[\sum_{n=0}^{\infty} h(n)Z^{-1} \right] dZ \\ \sum_{n=0}^{\infty} h^2(n) = \frac{1}{2\pi j} \oint H(Z) \left[\sum_{n=0}^{\infty} h(n)Z^{-1} \right] \frac{dZ}{Z^{-n}} \\ = \frac{1}{2\pi j} \oint H(Z) \left[\sum_{n=0}^{\infty} h(n)(Z^{-n})^{-1} Z^{-1} dZ \right] \\ \sum_{n=0}^{\infty} h^2(n) = \frac{1}{2\pi j} \oint H(Z)H(Z^{-1}) \frac{dZ}{Z} \text{----->(12)}$$

Substituting (12) in (9)

$$\sigma_{eo}^2 = \sigma_e^2 \left[\frac{1}{2\pi j} \oint H(Z)H(Z^{-1}) \frac{dZ}{Z} \right]$$

Problem:

The output signal of an A/D converter is passed through a first order low pass filter, with transfer function given by

$$H(z) = \frac{(1-a)z}{z-a} \text{ for } 0 < a < 1. \text{ Find the steady state output noise power due to quantization at the}$$

output of the digital filter. [Nov/Dec-2015]

Solution:

$$\sigma_e^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z)H(z^{-1})z^{-1}dz$$

$$\text{Given } H(z) = \frac{(1-a)z}{(z-a)} \quad H(z^{-1}) = \frac{(1-a)z^{-1}}{(z^{-1}-a)}$$

Substituting $H(z)$ and $H(z^{-1})$ in equation (1), we have

$$\sigma_e^2 = \frac{\sigma_e^2}{2\pi j} \oint_c \frac{(1-a)z}{(z-a)} \frac{(1-a)z^{-1}}{(z^{-1}-a)} z^{-1} dz = \frac{\sigma_e^2}{2\pi j} \oint_c \frac{(1-a)^2}{(z-a)(z^{-1}-a)} dz$$

$$= \sigma_e^2 \left[\text{residue of } H(z)H(z^{-1})z^{-1} \text{ at } z=a + \text{residue of } H(z)H(z^{-1})z^{-1} \text{ at } z=\frac{1}{a} \right]$$

$$= \sigma_e^2 \left[(z-a) \frac{(1-a)^2 z^{-1}}{(z-a)(z^{-1}-a)} + 0 \right]$$

$$= \sigma_e^2 \left[\frac{(1-a)^2}{(z^{-1}-a)} \right] = \sigma_e^2 \left[\frac{(1-a)}{(1+a)} \right]$$

$$\text{Where, } \sigma_e^2 = \frac{2^{-2b}}{12}$$

Find the steady state variance of the noise in the output due to quantization of input for the first order filter. [Apr/May'11] [Nov/Dec-2016]

$$y(n) = ay(n-1) + x(n)$$

Solution:

The impulse response for the above filter is given by $h(n) = a^n u(n)$

$$\begin{aligned} \sigma_e^2 &= \sigma_e^2 \sum_{k=0}^{\infty} h^2(n) \\ &= \sigma_e^2 \sum_{k=0}^{\infty} a^{2n} \\ &= \sigma_e^2 [1 + a^2 + a^4 + \dots \infty] \\ &= \sigma_e^2 \frac{1}{1-a^2} \\ &= \frac{2^{-2b}}{12} \left[\frac{1}{1-a^2} \right] \end{aligned}$$

(or) OBSERVE OPTIMIZE OUTSPREAD

Taking Z-transform on both sides we have

$$Y(z) = az^{-1}Y(z) + X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$H(z^{-1}) = \frac{z^{-1}}{z^{-1}-a}$$

We know

$$\sigma_e^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z)H(z^{-1})z^{-1} dz$$

Substituting $H(z)$ and $H(z^{-1})$ values in the above equation we get

$$\sigma_e^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c \frac{z}{z-a} \frac{z^{-1}}{z^{-1}-a} z^{-1} dz$$

$$\sigma_e^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c \frac{z^{-1}}{(z-a)(z^{-1}-a)} dz$$

$$= \sigma_e^2 \left[\begin{array}{l} \text{residue of } \frac{z^{-1}}{(z-a)(z^{-1}-a)} \text{ at } z=a \\ + \text{residue of } \frac{z^{-1}}{(z-a)(z^{-1}-a)} \text{ at } z=1/a \end{array} \right]$$

$$= \sigma_e^2 \left[(z-a) \frac{z^{-1}}{(z-a)(z^{-1}-a)} \Big|_{z=a} \right]$$

$$= \sigma_e^2 \frac{a^{-1}}{a^{-1}-a} = \sigma_e^2 \frac{1}{1-a^2}$$

The output of the A/D converter is applied to a digital filter with the system function

$$H(Z) = \frac{0.45Z}{Z-0.72}$$

Find the output noise power of the digital filter, when the input signal is quantized to 7 bits.

Given:

$$H(Z) = \frac{0.45Z}{Z-0.72}$$

Solution:

$$\begin{aligned} H(Z)H(Z^{-1})Z^{-1} &= \frac{0.45Z}{Z-0.72} \times \frac{0.45Z^{-1}}{Z^{-1}-0.72} \times Z^{-1} \\ &= \frac{0.45^2 Z^{-1}}{(Z-0.72) \left(\frac{1}{Z} - 0.72 \right)} \\ &= \frac{0.2025 Z^{-1}}{(Z-0.72) \left(\frac{1-0.72Z}{Z} \right)} \\ &= \frac{0.2025 Z^{-1} Z}{(Z-0.72) \left(Z - \frac{1}{0.72} \right)} \\ &= \frac{-0.28125}{(Z-0.72)(Z-1.3889)} \end{aligned}$$

Now the poles of $H(Z)H(Z^{-1})Z^{-1}$ are $p_1=0.72$, $p_2=1.3889$

Output noise power due to input quantization

$$\begin{aligned} \sigma_{eo}^2 &= \sigma_e^2 \left[\frac{1}{2\pi j} \oint H(Z)H(Z^{-1})Z^{-1} dZ \right] \\ &= \sigma_e^2 \sum_{i=1}^N \operatorname{Re} s \left[H(Z)H(Z^{-1})Z^{-1} \right] \Big|_{z=p_i} \end{aligned}$$

$$= \sigma_e^2 \sum_{i=1}^N \operatorname{Re} s \left[H(Z)H(Z^{-1})Z^{-1} \right]_{z=p_i}$$

Where p_1, p_2, \dots, p_n are the poles of $H(Z)H(Z^{-1})Z^{-1}$ that lies inside the unit circle in z -plane.

$$\begin{aligned} \sigma_{eo}^2 &= \sigma_e^2 \times (Z - 0.72) \times \frac{-0.28125}{(Z - 0.72)(Z - 1.3889)} \Big|_{Z=0.72} \\ &= \sigma_e^2 \times \frac{-0.28125}{0.72 - 1.3889} \\ &= 0.4205 \sigma_e^2 \end{aligned}$$

Consider the transfer function $H(z) = H_1(z)H_2(z)$ where $H_1(z) = \frac{1}{1 - a_1 z^{-1}}$ and $H_2(z) = \frac{1}{1 - a_2 z^{-1}}$

Find the output round off noise power. Assume $\alpha_1 = 0.5$ and $\alpha_2 = 0.6$ and find output round off noise power.

Solution:

The round off noise model for $H(z) = H_1(z)H_2(z)$ is given by,

From the realization we can find that the noise transfer function seen by noise source $e_1(n)$ is $H(z)$, where,

$$H(z) = \frac{1}{(1 - a_1 z^{-1})(1 - a_2 z^{-1})} \text{----- (1)}$$

Whereas, the noise transfer function seen by $e_2(n)$ is,

$$H_2(z) = \frac{1}{(1 - a_2 z^{-1})} \text{----- (2)}$$

The total steady state noise variance can be obtained, we have

$$\sigma_0^2 = \sigma_{01}^2 + \sigma_{02}^2 \text{----- (3)}$$

$$\sigma_{01}^2 = \frac{1}{2\pi j} \oint_c H(z)H(z^{-1})z^{-1} dz$$

$$= \sigma_e^2 \frac{1}{2\pi j} \oint_c \frac{1}{1 - a_1 z^{-1}} \frac{1}{1 - a_2 z^{-1}} \frac{1}{1 - a_1 z} \frac{1}{1 - a_2 z} z^{-1} dz$$

$$= \sigma_e^2 \left[\sum \text{of residue of } H(z)H(z^{-1})z^{-1} \text{ at poles } z = a_1, z = a_2, z = \frac{1}{a_1} \text{ and } z = \frac{1}{a_2} \right]$$

If a_1 and a_2 are less than the poles $z=1/a_1$ and $z=1/a_2$ lies outside of the circle $|z|=1$. So, the residue of $H(z)H(z^{-1})z^{-1}$ at $z=1/a_1$ and $z=1/a_2$ are zero. Consequently we have,

$$\sigma_{01}^2 = \left[\sum \text{of residue of } H(z)H(z^{-1})z^{-1} \text{ at poles } z = a_1, z = a_2 \right]$$

$$= \left[(z - a_1) \frac{z^{-1}}{(1 - a_1 z^{-1})(1 - a_2 z^{-1})(1 - a_1 z)(1 - a_2 z)} \Big|_{z=a_1} + (z - a_2) \frac{z^{-1}}{(1 - a_1 z^{-1})(1 - a_2 z^{-1})(1 - a_1 z)(1 - a_2 z)} \Big|_{z=a_2} \right]$$

$$= \sigma_e^2 \left[\frac{1}{\left(1 - \frac{a_2}{a_1}\right)(1 - a_2^2)(1 - a_1 a_2)} + \frac{1}{\left(1 - \frac{a_2}{a_1}\right)(1 - a_1 a_2)(1 - a_2^2)} \right]$$

$$\sigma_{01}^2 = \sigma_e^2 \left[\frac{a_1}{a_1 - a_2} \cdot \frac{1}{1 - a_1^2} \cdot \frac{1}{1 - a_1 a_2} + \frac{a_2}{a_2 - a_1} \cdot \frac{1}{1 - a_2^2} \cdot \frac{1}{1 - a_1 a_2} \right] \text{----- (4)}$$

In the same way,

$$\begin{aligned} \sigma_{02}^2 &= \frac{\sigma_e^2}{2\pi j} \oint_c H_2(z)H_2(z^{-1})z^{-1}dz \\ &= \frac{\sigma_e^2}{2\pi j} \oint_c \frac{1}{1-a_2z^{-1}} \frac{1}{1-a_2z} z^{-1}dz \\ &= \sigma_e^2 \left[(z-a_2) \frac{z^{-1}}{(1-a_2z^{-1})(1-a_2z)} \Big|_{z=a_2} \right] \\ &= \sigma_e^2 \left[(z-a_2z^{-1}) \frac{z^{-1}}{(1-a_2z^{-1})(1-a_2z)} \Big|_{z=a_2} \right] \\ &= \sigma_e^2 \left[\frac{1}{1-a_2^2} \right] \text{-----(5)} \end{aligned}$$

$$\begin{aligned} \sigma_0^2 &= \sigma_e^2 \left[\frac{1}{1-a_2^2} + \frac{a_1}{a_1-a_2} \cdot \frac{1}{1-a_1^2} \cdot \frac{1}{1-a_1a_2} + \frac{a_2}{a_2-a_1} \cdot \frac{1}{1-a_2^2} \cdot \frac{1}{1-a_1a_2} \right] \\ &= \sigma_e^2 \left[\frac{1}{1-a_2^2} + \frac{a_1(1-a_2^2) - a_2^2(1-a_1^2)}{(1-a_1^2)(1-a_2^2)(1-a_1a_2)(a_1-a_2)} \right] \\ &= \sigma_e^2 \left[\frac{1}{1-a_2^2} + \frac{(a_1-a_2)(1+a_1a_2)}{(1-a_1^2)(1-a_2^2)(1-a_1a_2)(a_1-a_2)} \right] \\ &= \frac{2^{-2b}}{12} \left[\frac{1}{1-a_2^2} + \frac{(1+a_1a_2)}{(1-a_1^2)(1-a_2^2)(1-a_1a_2)} \right] \end{aligned}$$

The steady state noise power for $a_1 = 0.5, a_2 = 0.6$ is given by

$$\begin{aligned} &= \frac{2^{-2b}}{12} \left[\frac{1}{1-(0.6)^2} + \frac{1+(0.5)(0.6)}{(1-(0.5)^2)(1-(0.6)^2)(1-0.6(0.5))} \right] \\ &= \frac{2^{-2b}}{12} (5.4315) \end{aligned}$$

Draw the quantization noise model for a second order system $H(z) = \frac{1}{1-2r \cos \theta z^{-1} + r^2 z^{-2}}$ and find

the steady state output noise variance.

Solution:

Given:

$$H(z) = \frac{1}{1-2r \cos \theta z^{-1} + r^2 z^{-2}}$$

The quantization noise model is,

we know, $\sigma_0^2 = \sigma_{01}^2 + \sigma_{02}^2$

Both noise sources see the same transfer function

$$H(z) = \frac{1}{1-2r \cos \theta z^{-1} + r^2 z^{-2}}$$

The impulse response of the transfer function is given by

$$h(n) = r^n \frac{\sin(n+1)\theta}{\sin \theta} u(n)$$

Now the steady state output noise variance is,

$$\sigma_0^2 = \sigma_{01}^2 + \sigma_{02}^2$$

But $\sigma_{01}^2 = \sigma_{02}^2 = \sigma_e^2 \sum_{n=-\infty}^{\infty} h^2(n)$, which gives us

$$\begin{aligned}
 \sigma_0^2 &= 2 \cdot \frac{2^{-2b}}{12} \sum_{n=0}^{\infty} r^{2n} \frac{\sin^2(n+1)\theta}{\sin^2 \theta} \\
 &= 2 \cdot \frac{2^{-2b}}{12} \frac{1}{2 \sin^2 \theta} \sum_{n=0}^{\infty} r^{2n} [1 - \cos 2(n+1)\theta] \quad \therefore \cos 2\theta = 1 - 2\sin^2 \theta \\
 &= \frac{2^{-2b}}{6} \frac{1}{2 \sin^2 \theta} \left[\sum_{n=0}^{\infty} r^{2n} - \sum_{n=0}^{\infty} r^{2n} \cos 2(n+1)\theta \right] \\
 &= \frac{2^{-2b}}{6} \frac{1}{2 \sin^2 \theta} \left[\frac{1}{1-r^2} - \frac{1}{2} \left(\sum_{n=0}^{\infty} r^{2n} e^{j2(n+1)\theta} + \sum_{n=0}^{\infty} r^{2n} e^{-j2(n+1)\theta} \right) \right] \\
 &= \frac{2^{-2b}}{6} \frac{1}{2 \sin^2 \theta} \left[\frac{1}{1-r^2} - \frac{1}{2} \left(\frac{e^{j2\theta}}{1-r^2 e^{2j\theta}} + \frac{e^{-j2\theta}}{1-r^2 e^{-2j\theta}} \right) \right] \\
 &= \frac{2^{-2b}}{6} \frac{1}{2 \sin^2 \theta} \left[\frac{1}{1-r^2} - \frac{\cos 2\theta - r^2}{1-2r^2 \cos 2\theta + r^4} \right] \\
 &= \frac{2^{-2b}}{6} \frac{1}{2 \sin^2 \theta} \left[\frac{(1+r)^2 (1 - \cos 2\theta)}{(1-r^2)(1-2r^2 \cos 2\theta + r^4)} \right] \\
 &= \frac{2^{-2b}}{6} \frac{(1+r)^2}{(1-r^2)(1-2r^2 \cos 2\theta + r^4)}
 \end{aligned}$$

