Eigen values and Eigen vector

1. Let $T: R^2 \rightarrow R^2$ be a linear operator given given by $T(a, b) = (-2a + 3b, -10a + 9b)$. Let β be an ordered basis of R^2 with $A = [T]_B$. (i) Find the matrix A (ii)The eigen values and eigen vectors of T.

Solution

Given, $T(a, b) = (-2a + 3b, -10a + 9b)$.

Since β is the standard basis of R^2

$$A = [T]_B = \begin{bmatrix} -2 & 3 \\ -10 & 9 \end{bmatrix}$$

To find the Eigen values:

The characteristic equation is $|A - \lambda I| = 0$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

S_1 =Sum of the leading diagonal elements

$$= -2 + 9 = 7$$

$$S_2 = |A| = -18 + 30 = 12$$

$$\lambda^2 - 7\lambda + 12 = 0$$

$$\lambda = 3, \lambda = 4$$

$\lambda = 3,4$ are the Eigen values of A

To find Eigen vectors:

Solve the equation $(A - \lambda I)X = 0$ we get $\begin{pmatrix} -2-\lambda & 3 \\ -10 & 9-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \dots\dots (a)$

Case 1: When $\lambda = 3$, from (a) we get

$$\begin{pmatrix} -5 & 3 \\ -10 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-5x_1 + 3x_2 = 0$$

$$-10x_1 + 6x_2 = 0$$

Since the two equations are same, consider

$$-5x_1 + 3x_2 = 0$$

$$-5x_1 = -3x_2$$

$$\frac{x_1}{3} = \frac{x_2}{5}$$

$$x_1 = 3, x_2 = 5$$

Hence the Eigen vector corresponding to $\lambda = 3$ is $E_{\lambda_1} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

Case 2: When $\lambda = 4$, from (a) we get

$$\begin{pmatrix} -6 & 3 \\ -10 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-6x_1 + 3x_2 = 0$$

$$-10x_1 + 5x_2 = 0$$

Since the two equations are same, consider

$$-6x_1 + 3x_2 = 0$$

$$-6x_1 = -3x_2$$

$$\frac{x_1}{3} = \frac{x_2}{6}$$

$$\frac{x_1}{1} = \frac{x_2}{2}$$

$$x_1 = 1, x_2 = 2$$

Hence the Eigen vector corresponding to $\lambda = 4$ is $E_{\lambda_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

2. Let $T: P_2(R) \to P_2(R)$ be the linear operator defined by $T(f(x)) = f(x) + (x+1)f'(x)$. Let $\beta = \{1, x, x^2\}$ be an ordered basis of $P_2(R)$ with $A = [T]_\beta$. Find (i) The matrix A (ii)The eigen values and eigen vectors of T.

Solution

Given, $T: P_2(R) \to P_2(R)$ be the linear operator defined by $T(f(x)) = f(x) + (x+1)f'(x) \ldots. (1)$

Let $\beta = \{1, x, x^2\}$ be an ordered basis of $P_2(R)$

To find $A = [T]_\beta$

Let, $(f(x)) = 1$. Then $f'(x) = 0$

$$(1) \Rightarrow T(1) = 1 + (x+1).0 = 1 = 1.1 + 0.x + 0.x^2$$

The first column of $[T]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Let, $(f(x)) = x$. Then $f'(x) = 1$

$$(1) \Rightarrow T(x) = x + (x+1).1 = 1 + 2x = 1.1 + 2.x + 0.x^2$$

The second column of $[T]_\beta = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Let, $(f(x)) = x^2$. Then $f'(x) = 2x$

$$(1) \Rightarrow T(x^2) = x^2 + (x+1).2x = 2x + 3x^2$$

$$= 0.1 + 2.x + 3.x^2$$

The third column of $[T]_\beta = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

$$A = [T]_\beta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Since A is an upper triangular matrix, the eigen values are

$$\lambda = 1,2,3$$

To find Eigen vectors:

Solve the equation $(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \ldots\ldots (a)$$

Case 1: When $\lambda = 1$, from (a) we get

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$0x_1 + x_2 + 0x_3 = 0 \ldots. (1)$$

$$0x_1 + x_2 + 2x_3 = 0 \ldots (2)$$

$$0x_1 + 0x_2 + 2x_3 = 0 \ldots (3)$$

Solving the two distinct equations (1) and (2) by the rule of cross multiplication, we get

$$\Rightarrow \frac{x_1}{2-0} = \frac{x_2}{0-0} = \frac{x_3}{0-0}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$x_1 = 1, x_2 = 0, \ x_3 = 0$$

Hence the Eigen vector corresponding to $\lambda = 1$ is $E_{\lambda_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Case 2: When $\lambda = 2$, from (a) we get

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-x_1 + x_2 + 0x_3 = 0 \ldots . (4)$$

$$0x_1 + 0x_2 + 2x_3 = 0 \ldots (5)$$

$$0x_1 + 0x_2 + 1x_3 = 0 \ldots (6)$$

Solving the two distinct equations (4) and (5) by the rule of cross multiplication, we get

$$\Rightarrow \frac{x_1}{2-0} = \frac{x_2}{0+2} = \frac{x_3}{0-0}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$x_1 = 1, x_2 = 1, \ x_3 = 0$$

Hence the Eigen vector corresponding to $\lambda = 2$ is $E_{\lambda_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Case 3: When $\lambda = 3$, from (a) we get

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-2x_1 + x_2 + 0x_3 = 0 \;.... (7)$$

$$0x_1 - x_2 + 2x_3 = 0 \;... (8)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \;... (9)$$

Solving the two distinct equations (7) and (8) by the rule of cross multiplication, we get

$$\Rightarrow \frac{x_1}{2-0} = \frac{x_2}{0+4} = \frac{x_3}{2-0}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{4} = \frac{x_3}{2}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$x_1 = 1, x_2 = 2, x_3 = 1$

Hence the Eigen vector corresponding to $\lambda = 3$ is $E_{\lambda_3} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$