LINEAR SPAN

Definition:

Let V be a vector space over F and S be a non-empty subset of V. Then the set of all linear combination of the finite subset of S is called the linear span of set of and is denoted by $L(S)$.

$$\text{i.e., } L(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n / \alpha_i \in F, v_i \in S\}$$

Note:

- $L(S) \subseteq V$
- If $S = \emptyset$, then $L(S) = 0$.

Definition:

A subset S of a vector space V generates (or span) V, if $L(S) = V$

Theorem 1.13: Let S be a nonempty subset of a vector space $V(F)$.

i) $L(S)$ is a subspace of V and $S \subseteq L(S)$

ii) if W is a subspace of V such that $S \subseteq W$, then $L(S) \subseteq W$

Proof:

i) Let S be a nonempty subset of a vector space $V(F)$.

Let $u, v \in L(S)$ and $\alpha, \beta \in F$.

Then $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m$ and $v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$

where $\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_n \in F$ and

$u_1, u_2, \dots, u_m, v_1, v_2 \dots, v_n \in S$ and also m and n are finite.

$$\text{u} + \beta v = \alpha(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m) + \beta(\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n)$$

$$\alpha\alpha_1 u_1 + \alpha\alpha_2 u_2 + \cdots + \alpha\alpha_m u_m + \beta\beta_1 v_1 + \beta\beta_2 v_2 + \cdots + \beta\beta_n v_n \dots (1)$$

assume$\alpha\alpha_i = \gamma_i; \beta\beta_i = \gamma_{m+i}$ and $v_i = u_{m+i}$ in (1), we get

$u + \beta v$

$$\gamma_1 u_1 + \gamma_2 u_2 + \cdots + \gamma_m u_m + \gamma_{m+1} u_{m+1} + \gamma_{m+1} u_{m+1} + \cdots + \gamma_{m+n} u_{m+n} \in L(S)$$

$u+\beta v \in L(S)$

hence $L(S)$ is a subspace of V.

Let W be a subspace of V such that $S \subseteq W$

have to prove $L(S) \subseteq W$

v $\mathcal{E} L(S)$. Then $v=\alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_n v_n$ where the $\alpha_i \in F$ and $v_i \in S$

Since $S \subseteq W, v_1, v_2, \ldots, v_n \in W$

Since W is a subspace of V, m

$\alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_n v_n \in W$

$$\Rightarrow v \in W$$

$$v \in L(S) \Rightarrow v \in W$$

$\therefore L(S) \subseteq W$

Theorem 1.14: Let V be a vector space over a field $\boldsymbol{F}$.

Let $S, T \subseteq V$. Then

(a) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

(b) $L(S \cup T)=L(S)+L(T)$

(C) $L(S)=S$ if and only if S is a subspace of V.

Proof:

(a) Let $S \subseteq T$ and $v \in L(S)$,

Then $v=\alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_n v_n$ where $v_i \in S$ and $\alpha_i \in F$.

Now, since $S \subseteq T, v_1, v_2, \ldots, v_n \in T$

$\therefore \alpha_1 v_1+\alpha_2 v_2+\cdots+\alpha_n v_n \in L(T)$

$$\therefore v \in L(T)$$

$v \in L(S) \Rightarrow v \in L(T)$

$$\Rightarrow L(S) \subseteq L(T)$$

(ii) Let $v \in L(S \cup T)$

Then $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \ldots + \alpha_n v_n$ where $v_1, v_2, \ldots, v_n \in S \cup T$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$. Without loss of generality, we shall assume that $v_1, v_2, \ldots, v_m \in S$ and $v_{m+1}, v_{m+2}, \ldots v_n \in T$

Hence

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m \in L(S)$ and $\alpha_{m+1} v_{m+1} + \alpha_{m+2} v_{m+2} + \cdots + \alpha_n v_n \in L(T)$.

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \ldots + \alpha_n v_n$

$= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \alpha_{m+2} v_{m+2} + \cdots + \alpha_n v_n$

$$v \in L(S) + L(T)$$

$v \in L(S \cup T) \Rightarrow v \in L(S) + L(T)$

$\therefore L(S \cup T) \subseteq L(S) + L(T) \ldots (1)$

Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, we have $L(S) \subseteq L(S \cup T)$ and $L(T) \subseteq L(S \cup T)$.

$\therefore$ their linear sum $L(S) + L(T) \subseteq L(S \cup T) \ldots$ (2)

From (1) and (2),

$$L(S \cup T) = L(S) + L(T)$$

(C) Let $L(S) = S$.

Since $L(S)$ is a subspace of V. we get S is a subspace $V(F)$.

Conversely let S is a subspace $V(F)$.

We know that $S \subseteq L(S) \ldots$ (3).

Let $v \in L(S)$. Then $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots .. + \alpha_n v_n$ where $v_1, v_2, \dots, v_n \in S$ and

$\alpha_1, \alpha_2, \dots, \alpha_n \in F$

Since S is a subspace of $V, \alpha_1 v_1 + \alpha_2 v_2 + \cdots, + \alpha_n v_n \in S$

i.e., $v \in S$

$v \in L(S) \Rightarrow v \in S$

$\therefore L(S) \subseteq S \dots (4)$

From (3) and (4), we get

Hence $L(S) = S$.

Corollary 1.15: $L[L(S)] = L(S)$

Proof: If S is a subspace of V, then $L(S) = S \dots (1)$

Since $L(S)$ is a subspace of V, then $L[L(S)] = L(S) = S$[From (1)]

$\therefore L[L(S)] = L(S)$

Example 46. Let $S = \{(1,2), (2,1)\}; V = R^2$. Prove that V is a linear span of S.

Sol: We know that $L(S) \subseteq V \dots (1)$

Let us consider $(x, y) \in V$

$$(x, y) = \alpha_1(1,2) + \alpha_2(2,1) \dots (2)$$

$$= (\alpha_1, 2\alpha_1) + (2\alpha_2, \alpha_2)$$

$$(x, y) = (\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2)$$

$$\alpha_1 + 2\alpha_2 = x \dots (3)$$

$$2\alpha_1 + \alpha_2 = y \dots (4)$$

(3) × 2 => $2a_1 + 4a_2 = 2x$

(4) => $2a_1 + a_2 = y$

$$3a_2 = 2x - y$$

$$a_2 = \frac{2x-y}{3}$$

From equation (4)

$$2a_1 = y - a_2$$

$$2a_1 = y - \left(\frac{2x-y}{3}\right)$$

$$= \frac{3y-2x+y}{3}$$

$$2a_1 = \frac{4y-2x}{3}$$

$$a_1 = \frac{2y-x}{3}$$

Sabtitute the values of a_4 and a_2 in (2), we get

$$x(x,y) = \left(\frac{2y-x}{3}\right)(1,2) + \left(\frac{2x-y}{3}\right)(2,1)$$

Hence (x,y) is a lincar combination of S

$$s(x,y) \in L(S)$$

We have $(x,y) \in V \Rightarrow (x,y) \in L(S)$

$$\therefore V \subset L(S) - (5)$$

From (1) and (5), we get

$$L(S) = V$$

Therefore S gonerates V.

Example 47. Prove that in $V_2(R), (3,7)$ belongs to the linear space $((1,2),(0,1))$

sol: Let $S = ((1,2),(0,1))$

$$v_1 = (1,2), v_2 = (0,1)$$

Lat $v = (x, y) \in L(S)$

$v = a_1 v_1 + a_z v_2$

$(x, y) = \alpha_1(1,2) + a_2(0,1) \ldots (1)$

$$=(a_1, 2a_2 + a_2)$$

$$a_1 = x$$

$$2a_1 + a_2 = y$$

$$2x + a_2 = y$$

$$a_2 = y - 2x$$

(1)⇒ $(x, y) = x(1,2) + (y - 2x)(0,1)$

we check(3,7) $\in L(S)$

Here $x = 3, y = 7$

(1) ⇒ $(3,7) = 3(1,2) + (7 - 6)(0,1)$

$= (3,6) + (0,1)$

$= (3,7)$

which is true.

$(3,7) \in L(S$am

Example48. Prove that the vectors $(1,1,0),(1,0,1),(0,1,1)$ generates R^3.

Sol:Let $S = \{(1,1,0),(1,0,1),(0,1,1)\}$

We know that $L(S) \subseteq R^3$… (1)

Let v $\in R^3$. Then $v = (a,b,c)$

Let $v = a_1(1,1,0) + \alpha_2(1,0,1) + \alpha_3(0,1,1)$

$(a,b,c) = (\alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3)$

$\alpha_1 + \alpha_2 = a$ ... (1)

$\alpha_1 + \alpha_3 = b$ ... (2)

$\alpha_2 + \alpha_3 = c$ ... (3)

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$[A,B] = \begin{bmatrix} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{bmatrix}$$

$$[A,B] \begin{bmatrix} 1 & 1 & 0 & a \\ 0 & -1 & 1 \mid & b-a \\ 0 & 1 & 1 & c \end{bmatrix} R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 & & 0 \\ 0 & -1 & 1 & \\ 0 & 0 & 2 & b-a \\ & & & b-a+c \end{bmatrix} R_3 \to R_3 + R_2$$

$$2\alpha_3 = b - a + c$$

$$\alpha_3 = \frac{1}{2}(b - a + c)$$

$$-\alpha_2 + \alpha_3 = b - a$$

$$-\alpha_2 = b - a - \alpha_3$$

$$= b - a - \frac{1}{2}(b - a + c)$$

$$= \frac{1}{2}(2b - 2a - b + a - c)$$

$$= \frac{1}{2}(b - a + c)$$

$$\alpha_2 = \frac{1}{2}(a - b + c)$$

$$\alpha_1 + \alpha_2 = a$$

$$\alpha_1 = a - \alpha_2$$

$$\alpha_1 = a - \frac{1}{2}(a - b + c)$$

$$= \frac{1}{2}(2a - a + b - c)$$

$$= \frac{1}{2}(a + b - c)$$

Substitute the values of $\alpha_1, \alpha_2, \alpha_3$ in (1), we get

$$v = \frac{1}{2}(a + b - c)(1,1,0) + \frac{1}{2}(a - b + c)(1,0,1) + \frac{1}{2}(b - a + c)(0,1,1)$$

$$\therefore v \in L(S)$$

$$\therefore R^3 \subseteq L(S) \ldots (5)$$

From (1) and (5), we get

$$L(S) = R^3$$

Therefore S generates R^3.

Example 49. Prove that the polynomials $x^2 + 3x - 2, 2x^2 + 5x - 3$ and $-x^2 -$

$4x + 4$ generates $P_2(R)$

Let $p(x) = x^2 + 3x - 2, q(x) = 2x^2 + 5x - 3$ and $r(x) = -x^2 - 4x + 4$

Let $S = \{p(x), q(x), r(x)\}$. Then

$$L(S) \subseteq P_2(R) \ldots (1)$$

Let $t(x) \in P_2(R)$. Then

$$t(x) = ax^2 + bx + c; a, b, c \in R$$

Let $t(x) = \alpha_1 p(x) + \alpha_2 q(x) + \alpha_3 r(x)$

$$= \alpha_1(x^3 + 3x - 2) + \alpha_2(2x^2 + 5x - 3) + \alpha_3(-x^2 - 4x + 4) \ldots.(1)$$

$ax^2 + bx + c$

$$= (\alpha_1 + 2\alpha_2 - \alpha_3)x^2 + (3\alpha_1 + 5\alpha_2 - 4\alpha_3)x + (-2\alpha_1 - 3\alpha_2 + 4\alpha_3)$$

$$\alpha_1 + 2\alpha_2 - \alpha_3 = a \ldots (2)$$

$$3\alpha_1 + 5\alpha_2 - 4\alpha_3 = b \ldots (3)$$

$$-2\alpha_1 - 3\alpha_2 + 4\alpha_3 = c \ldots (4)$$

$$(A, B) \sim \left(\begin{array}{cccc|} 1 & 2 & -1 & a \\ 3 & 5 & -4 & b \\ -2 & -3 & & 4 \end{array}\right)$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & a \\ 0 & -1 & -1 & b - 3a \\ 0 & 1 & 2 & c + 2a \end{pmatrix} R_2 \to R_2 - 3R_1, R_3 \to R_3 + 2R_1$$

$$\sim \begin{pmatrix} 1 & 2 & & -1 \\ 0 & -1 & & -1 \\ 0 & 0 & 1 & b - 3a \\ & & & c + b - a \end{pmatrix} R_3 \to R_3 + R_2$$

$$\alpha_3 = c + b - a$$

$$-\alpha_2 - \alpha_3 = b - 3a$$

$$-\alpha_2 = b - 3a + \alpha_3$$

$$= b - 3a + c + b - a$$

$$= 2b - 4a + c$$

$$\alpha_2 = 4a - 2b - c$$

$$\alpha_1 + 2\alpha_2 - \alpha_3 = a$$

$$\alpha_1 = a - 2\alpha_2 + \alpha_3$$

$$= a - 2(4a - 2b - c) + (c + b - a)$$

$$= -8a + 5b + 3c$$

Substitute the values of $\alpha_1, \alpha_2, \alpha_3$ in (1), we get

$$t(x) = (-8a + 5b + 3c)(x^3 + 3x - 2) + (4a - 2b - c)(2x^2 + 5x - 3)$$

$$+(c + b - a)(-x^2 - 4x + 4) \in L(S)$$

$\therefore P_2(R) \subseteq L(S)$ ... (5)

From (1) and (5), we get

$$L(S) = P_2(R)$$