TAYLORS AND LAURENTS SERIES

Taylor's Series

If $f\ (z)$ is analytic inside and on a circle C with centre at point 'a' and radius 'R' then at each point Z inside C,

$$f\ (z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \cdots$$

(OR)

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^n(a)$$

This is known as Taylor's series of $f(z)$ about $z\ = a$.

Note: 1 Putting $a = 0$ in the Taylor's series we get

$f\ (z) = f(0) + (z-0)\frac{f'(0)}{1!} + (z-0)^2\frac{f''(0)}{1!} + \cdots$ this series is called Maclaurin's Series.

Note: 2 The Maclaurin's for some elementary functions are

1) $(1-z)^{-1} = 1 + z + z^2 + z^3\ + \cdots$, when $|z| < 1$
2) $(1+z)^{-1} = 1 - z + z^2 - z^3\ + \cdots$, when $|z| < 1$
3) $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3\ + \cdots$, when $|z| < 1$
4) $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3\ + \cdots$, when $|z| < 1$
5) $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$ when $|z| < \infty$
6) $e^z = 1 - \frac{z}{1!} + \frac{z^2}{2!} + \cdots$ when $|z| < \infty$
7) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$ when $|z| < \infty$
8) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$ when $|z| < \infty$

LAURENTS SERIES

If $c_1\ and\ c_2$ are two concentric circles with centre at $z = a$ and radii $r_1\ and\ r_2\ (r_1 < r_2)$ and if $f(z)$ is analytic inside on the circles and within the annulus between $c_1\ and\ c_2$ then for any z in the annulus, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}\ \ldots (1)$$

Where $a_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{(z-a)^{n+1}}\ dz$ and $b_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{(z-a)^{1-n}}\ dz$ and the integration being taken in positive direction. This series (1) is called Laurent series of $f(z)$ about the point $z = a$

Example: Expand $f(z) = cosz$ as a Taylor's series about $z = \frac{\pi}{4}$.

Solution:

Function	Value of function at $z = \frac{\pi}{4}$
$f(z) = cosz$	$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
$f'(z) = -sinz$	$f'\left(\frac{\pi}{4}\right) = -sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f''(z) = -cosz$	$f''\left(\frac{\pi}{4}\right) = -cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f'''(z) = sinz$	$f''\left(\frac{\pi}{4}\right) = sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

The Taylor series of $f(z)$ about $z = \frac{\pi}{4}$ is $f(z) = f\left(\frac{\pi}{4}\right) + \left(z - \frac{\pi}{4}\right)\frac{f'\left(\frac{\pi}{4}\right)}{1!} + \left(z - \frac{\pi}{4}\right)^2 \frac{f''\left(\frac{\pi}{4}\right)}{2!} + \cdots$

$$cosz = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{-\frac{1}{\sqrt{2}}}{1!} + \left(z - \frac{\pi}{4}\right)^2 \frac{-\frac{1}{\sqrt{2}}}{2!} + \cdots$$

Example: Expand $f(z) = log\,(1+z)$ as a Taylor's series about $z = 0$.

Solution:

Function	Value of function at $z = 0$
$f(z) = log\,(1+z)$	$f(0) = log\,(1+0) = 0$
$f'(z) = \frac{1}{1+z}$	$f'(0) = \frac{1}{1+0} = 1$
$f''(z) = \frac{-1}{(1+z)^2}$	$f''(0) = \frac{-1}{(1+0)^2} = -1$
$f'''(z) = \frac{2}{(1+z)^3}$	$f'''(0) = \frac{2}{(1+0)^3} = 2$

The Taylor series of $f(z)$ about $z = 0$ is

$$f(z) = f(0) + (z-0)\frac{f'(0)}{1!} + (z-0)^2\frac{f''(0)}{2!} + \cdots$$

$$log\,(1+z) = 0 + (z)\frac{1}{1!} + (z)^2\frac{-1}{2!} + \cdots$$

$$log\,(1+z) = (z)\frac{1}{1!} - (z)^2\frac{1}{2!} + \cdots$$

Example: Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ as a Laurent's series if (i) $|z| < 2$ (ii) $|z| > 3$ (iii) $2 < |z| < 3$

Solution:

Given $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ is an improper fraction. Since degree of numerator and degree of denominator of $f(z)$ are same

∴ Apply division process

$$\begin{array}{r|l} & 1 \\ z^2+5z+6 & z^2-1 \\ & z^2+5z+6 \\ \hline & -5z-7 \end{array}$$

$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 - \frac{5z+7}{(z+2)(z+3)} \quad \ldots (1)$

Consider $\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$\Rightarrow 5z + 7 = A(z+3) + B(z+2)$

Put $z = -2$, we get $-10 + 7 = A\,(1)$

$\Rightarrow A = -3$

Put $z = -3$, we get $-15 + 7 = B(-1)$

$\Rightarrow B = 8$

$$\therefore \frac{5z+7}{(z+2)(z+3)} = \frac{-3}{z+2} + \frac{8}{z+3}$$

$$\therefore (1) \Rightarrow 1 - \frac{3}{z+2} - \frac{8}{z+3}$$

(i) Given $|z| < 2$

$$\therefore f(z) = 1 + \frac{3}{2(1+{}^{z}/_{2})} - \frac{8}{3(1+{}^{z}/_{3})}$$

$$= 1 + \frac{3}{2}\left(1 + {}^{z}/_{2}\right)^{-1} - \frac{8}{3}\left(1 + {}^{z}/_{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left[\frac{z}{2}\right]^2 + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left[\frac{z}{3}\right]^2 + \cdots\right]$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z}{2}\right]^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z}{3}\right]^n$$

(ii) Given $|z| > 3$

$$\therefore f(z) = 1 + \frac{3}{z(1+{}^{2}/_{z})} - \frac{8}{z(1+{}^{3}/_{z})}$$

$$= 1 + \frac{3}{z}\left(1 + {}^{2}/_{z}\right)^{-1} - \frac{8}{z}\left(1 + {}^{3}/_{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left[\frac{2}{z}\right]^2 + \cdots\right] - \frac{8}{z}\left[1 - \frac{3}{z} + \left[\frac{3}{z}\right]^2 \ldots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left[\frac{2}{z}\right]^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{z}\right]^n$$

(iii) Given $2 < |z| < 3$

$$\therefore\ f(z) = 1 + \frac{3}{z(1+2/z)} - \frac{8}{3(1+z/3)}$$

$$= 1 + \frac{3}{z}\left(1 + 2/z\right)^{-1} - \frac{8}{3}\left(1 + z/3\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left[\frac{2}{z}\right]^2 + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left[\frac{z}{3}\right]^2 \ldots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left[\frac{2}{z}\right]^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z}{3}\right]^n$$

Example: Find the Laurent's series expansion of $\boldsymbol{f(z) = \frac{7z-2}{z(z-2)(z+1)}}$ **in** $\boldsymbol{1 < |z+1| < 3}$.

Also find the residue of $\boldsymbol{f(a)}$ **at** $\boldsymbol{z = -1}$

Solution:

Given $f(z) = \frac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z - 2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Put $z = 2$, we get $14 - 2 = B\ (2)\ (2+1)$

$$\Rightarrow 12 = 6B$$

$$\Rightarrow B = 2$$

Put $z = -1$, we get $-7 - 2 = C(-1)(-1-2)$

$$\Rightarrow -9 = 3C$$

$$\Rightarrow C = -3$$

Put $z = 0$ we get $-2 = A(-2)$

$$\Rightarrow A = 1$$

$$\therefore f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Given region is $1 < |z+1| < 3$

Let $u = z + 1 \Rightarrow z = u - 1$

$(i.e)\ 1 < |u| < 3$

Now $f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$

$$= \frac{1}{u(1-1/u)} + \frac{2}{-3(1-u/3)} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1 - 1/u\right)^{-1} - \frac{2}{3}\left(1 - u/3\right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left[1 + \frac{1}{u} + \left[\frac{1}{u}\right]^2 + \cdots\right] - \frac{2}{3}\left[1 + \frac{u}{3} + \left[\frac{u}{3}\right]^2 + \cdots\right] - \frac{3}{u}$$

$$= \frac{1}{z+1}\left[1 + \frac{1}{z+1} + \left[\frac{1}{z+1}\right]^2 + \cdots\right] - \frac{2}{3}\left[1 + \frac{z+1}{3} + \left[\frac{z+1}{3}\right]^2 + \cdots\right] - \frac{3}{z+1}$$

$$= \frac{1}{z+1}\sum_{n=0}^{\infty}\left[\frac{1}{z+1}\right]^n - \frac{2}{3}\sum_{n=0}^{\infty}\left[\frac{1}{\frac{z+1}{3}}\right]^n - \frac{3}{z+1}$$

Also $Res[f(z), z = -1]$ =coefficient of $\frac{1}{z+1} = -2$

Example: Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in a Laurent's series valid in the region

(i)$|z-1| > 1$(ii) $0 < |z-2| < 1$ (iii) $|z| > 2$ (iv) $0 < |z-1| < 1$

Solution:

Given $f(z) = \frac{1}{(z-1)(z-2)}$

Consider $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

$$\Rightarrow 1 = A(z-2) + B(z-1)$$

Put $z = 2$, we get $1 = B(1)$

$$\Rightarrow B = 1$$

Put $z = 1$we get $1 = A(1-2)$

$$\Rightarrow A = -1$$

$$\therefore f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$$

(i) Given region is $|z-1| > 1$

Let $u = z - 1 \Rightarrow z = u + 1$

$(i.e)\ |u| > 1$

Now $f(z) = -\frac{1}{u} + \frac{1}{u-1}$

$$= \frac{-1}{u} + \frac{1}{u(1-{}^{1}/_{u})}$$

$$= \frac{-1}{u} + \frac{1}{u}(1 - {}^{1}/_{u})^{-1}$$

$$= \frac{-1}{u} + \frac{1}{u}\left[1 + \frac{1}{u} + \left[\frac{1}{u}\right]^2 + \cdots\right]$$

$$= \frac{-1}{z+1} + \frac{1}{z+1}\left[1 + \frac{1}{z+1} + \left[\frac{1}{z+1}\right]^2 + \cdots\right]$$

$$= \frac{-1}{z+1} + \frac{1}{z+1}\sum_{n=0}^{\infty}\left[\frac{1}{z+1}\right]^n$$

(ii) Given $0 < |z-2| < 1$

Let $u = z - 2 \Rightarrow z = u + 2$

$(i.e)\ 0 < |u| < 1$

Now $f(z) = -\frac{1}{u+1} + \frac{1}{u}$

$$= -(1+u)^{-1} + \frac{1}{u}$$

$$= -[1 - u + [u]^2 + \cdots] + \frac{1}{u}$$

$$= -[1 - (z-2) + [z-2]^2 + \cdots] + \frac{1}{z-2}$$

$$= -\sum_{n=0}^{\infty}[-1]^n[z-2]^n + \frac{1}{z-2}$$

(iii) Given $|z| > 2$

Now $f(z) = -\frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{z\left(1-\frac{2}{z}\right)}$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{z}\left[1 + \frac{1}{z} + \left[\frac{1}{z}\right]^2 + \cdots\right] + \frac{1}{z}\left[1 + \frac{2}{z} + \left[\frac{2}{z}\right]^2 + \cdots\right]$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left[\frac{1}{z}\right]^n + \frac{1}{z}\sum_{n=0}^{\infty}\left[\frac{2}{z}\right]^n$$

(iv) Given $0 < |z-1| < 1$

Let $u = z - 1 \Rightarrow z = u + 1$

$(i.e)\ 0 < |u| < 1$

Now $f(z) = -\frac{1}{u} + \frac{1}{u-1}$

$$= -\frac{1}{u} + \frac{1}{-1\,[1-u]}$$

$$= -\frac{1}{u} - (1-u)^{-1}$$

$$= -\frac{1}{u} - [1 + u + [u]^2 + \cdots]$$

$$= -\frac{1}{z-1} - [1 + z - 1 + [z-1]^2 + \cdots]$$

$$= -\frac{1}{z-1} - \sum_{n=0}^{\infty}[z-1]^n$$

Example: Expand $f(z) = \frac{z}{(z+1)(z-2)}$ in a Laurent's series about (i) $z = -1$ (ii) $z = 2$

Solution:

Consider $\frac{z}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$

$$\Rightarrow z = A(z-2) + B(z+1)$$

Put $z = 2$, we get $2 = B(3)$

$$\Rightarrow B = \frac{2}{3}$$

Put $z = -1$we get $-1 = A(-3)$

$$\Rightarrow A = \frac{1}{3}$$

$$\therefore f(z) = \frac{1}{3(z+1)} + \frac{2}{3(z-2)}$$

(i)To expand $f(z)$ about $z = -1$

(or) $|z-1| < 1$

Put $z+1 = u \Rightarrow z = u-1$

$\Rightarrow |z-1| < 1 \Rightarrow |u| < 1$

Now $f(z) = \frac{1}{3u} + \frac{2}{3(u-3)}$

$= \frac{1}{3u} + \frac{2}{3\left((-3)(1-{}^{u}/_{3})\right)}$

$= \frac{1}{3u} - \frac{2}{9}\left(1-{}^{u}/_{3}\right)^{-1}$

$= \frac{1}{3u} - \frac{2}{9}\left[1+\frac{u}{3}+\left[\frac{u}{3}\right]^2+\cdots\right]$

$= \frac{1}{3(z+1)} - \frac{2}{9}\left[1+\frac{(z+1)}{3}+\left[\frac{(z+1)}{3}\right]^2+\cdots\right]$

$= \frac{1}{3(z+1)} - \frac{2}{9}\sum_{n=0}^{\infty}\left[\frac{(z+1)}{3}\right]^n$

(ii) To expand $f(z)$ about $z = 2$

(or) $|z-2| < 1$

Put $z-2 = u \Rightarrow z = u+2$

$\Rightarrow |z-2| < 1 \Rightarrow |u| < 1$

Now $f(z) = \frac{1}{3(u+3)} + \frac{2}{3(u)}$

$= \frac{1}{3\left((3)(1+{}^{u}/_{3})\right)} + \frac{2}{3(u)}$

$= \frac{1}{9}\left(1+{}^{u}/_{3}\right)^{-1} + \frac{2}{3(u)}$

$= \frac{1}{9}\left[1-\frac{u}{3}+\left[\frac{u}{3}\right]^2+\cdots\right] + \frac{2}{3(u)}$

$= \frac{1}{9}\left[1-\frac{(z-2)}{3}+\left[\frac{(z-2)}{3}\right]^2+\cdots\right] + \frac{2}{3(z-2)}$

$= \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n\left[\frac{(z-2)}{3}\right]^n + \frac{2}{3(z-2)}$

Example: Expand the Laurent's series about for $f(z) = \frac{6z+5}{z(z-2)(z+1)}$ in the region $1 < |z+1| < 3$

Solution:

Consider $\frac{6z+5}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{c}{z+1}$

$\Rightarrow 6z+5 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$

Put $z = 0$, we get $5 = A(-2)(1)$

$$\Rightarrow A = \frac{-5}{2}$$

Put $z = -1$we get $-11 = C(-1)(-3)$

$$\Rightarrow C = -\frac{11}{3}$$

Put $z = 2$we get $17 = B(2)(3)$

$$\Rightarrow B = \frac{17}{6}$$

$$\therefore f(z) = \frac{-5}{2z} + \frac{17}{6(z-2)} - \frac{11}{3(z+1)}$$

Given region $1 < |z+1| < 3$

Put $z + 1 = u \Rightarrow z = u - 1$

$(i.e)\ 1 < |u| < 3$

Now $f(z) = \frac{-5}{2(u-1)} + \frac{17}{6(u-3)} - \frac{11}{3u}$

$$= \frac{-5}{2u(1-\frac{1}{u})} + \frac{17}{6(-3)(1-\frac{u}{3})} - \frac{11}{3u}$$

$$= \frac{-5}{2u}\left[1 - \frac{1}{u}\right]^{-1} - \frac{17}{18}\left[1 - \frac{u}{3}\right]^{-1} - \frac{11}{3u}$$

$$= \frac{-5}{2u}\left[1 + \frac{1}{u} + \left[\frac{1}{u}\right]^2 + \cdots\right] - \frac{17}{18}\left[1 + \frac{u}{3} + \left[\frac{u}{3}\right]^2 + \cdots\right] - \frac{11}{3u}$$

$$= \frac{-5}{2(z+1)}\left[1 + \frac{1}{(z+1)} + \left[\frac{1}{(z+1)}\right]^2 + \cdots\right] - \frac{17}{18}\left[1 + \frac{(z+1)}{3} + \left[\frac{(z+1)}{3}\right]^2 + \cdots\right] - \frac{11}{3(z+1)}$$

$$= \frac{-5}{2(z+1)}\sum_{n=0}^{\infty}\left[\frac{1}{(z+1)}\right]^n - \frac{17}{18}\sum_{n=0}^{\infty}\left[\frac{(z+1)}{3}\right]^n - \frac{11}{3(z+1)}$$