

#### 4.4 Approximation of derivatives using interpolation

Newton's forward difference formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h} \left[ \Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \dots \right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2} \left[ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 + \dots \right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3} \left[ \Delta^3 y_0 - \frac{3}{2} \Delta^4 y_0 + \dots \right]$$

Newton's backward difference formula

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h} \left[ \nabla y_n + \frac{1}{2} \nabla^2 y_n + \frac{1}{3} \nabla^3 y_n + \frac{1}{4} \nabla^4 y_n + \dots \right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2} \left[ \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \dots \right]$$

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3} \left[ \nabla^3 y_n + \frac{3}{2} \nabla^4 y_n + \dots \right]$$

1. Find the first, second and third derivatives of  $f(x)$  at  $x = 1.5$  if

$x$	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	3.375	7.000	13.625	24.000	38.875	59.000

Solution

$x$	$y$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	3.375				
2.0	7.000	3.625			
2.5	13.625	6.625	3		
3.0	24.000	10.375	3.75	0.75	0
3.5	38.875	14.875	4.5	0.75	0
4.0	59.000	20.125	5.25	0.75	

$$\begin{aligned} \left(\frac{dy}{dx}\right)_{x=x_0} &= \frac{1}{h} \left[ \Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \dots \right] \\ &= \frac{1}{0.5} \left[ 3.625 - \frac{1}{2} * 3 + \frac{1}{3} * 0.75 - \frac{1}{4} * 0 \right] \\ &= 4.75 \end{aligned}$$

$$\begin{aligned} \left(\frac{d^2y}{dx^2}\right)_{x=x_0} &= \frac{1}{h^2} \left[ \Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 + \dots \right] \\ &= \frac{1}{(0.5)^2} \left[ 3 - 0.75 + \frac{11}{12} * 0 \right] = 9 \end{aligned}$$

$$\begin{aligned} \left(\frac{d^3y}{dx^3}\right)_{x=x_0} &= \frac{1}{h^3} \left[ \Delta^3 y_0 - \frac{3}{2} \Delta^4 y_0 + \dots \right] \\ &= \frac{1}{(0.5)^3} [0.75] = 6 \end{aligned}$$

2. Compute  $f'(0)$  and  $f''(4)$  from the data

$x$	0	1	2	3	4
$y$	1	2.718	7.381	20.086	54.598

Solution

$x$	$y$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
1	2.718	1.718	2.945	5.097	
2	7.381	4.663	8.042	13.765	8.668
3	20.086	12.705	21.807		
4	54.598	34.512			

$$\begin{aligned} \left(\frac{dy}{dx}\right)_{x=x_0} &= \frac{1}{h} \left[ \Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \dots \right] \\ &= \frac{1}{1} \left[ 1.718 - \frac{1}{2} * 2.945 + \frac{1}{3} * 5.097 - \frac{1}{4} * 8.668 \right] \end{aligned}$$

$$= 1.718 - 1.4725 + 1.699 - 2.167 = -0.2225$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2} \left[ \nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \dots \right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{1} \left[ 21.807 + 13.765 + \frac{11}{12} * 8.668. \right] = 43.5177$$

