UNIT-IV

FOURIER TRANSFORMS

Convolution Theorem and Parseval's identity.

The convolution of two functions f(x) and g(x) is defined as

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, g(x-t).\, dt.$$

Convolution Theorem for Fourier Transforms.

The Fourier Transform of the convolution of f(x) and g(x) is the product of theirFourier Transforms,

i.e, F{f(x) * g(x)} = F{f(x).F{g(x)}.

Proof:

F{f(x) * g(x)} = F{(f*g)x)}

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (f * g)(x).\, e^{isx}\, .\, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, g(x-t).\, dt \right. \; e^{isx}\, dx\, .$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x-t).\, e^{isx}\, dx \,.\right\} dt.$$

(by changing the order of integration).

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).F\{g(x-t)\}.\, dt.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, e^{its}.G(s).\, dt. \quad \text{(by shifting property)}$$

$$= G(s).\, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, e^{ist}\, dt.$$

$$= F(s).G(s).$$

Hence, F{f(x) * g(x)} = F{f(x).F{g(x)}.

Parseval's identity for Fourier Transforms

If F(s) is the F.T of f(x), then

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |F(s)|^2\, ds.$$

Proof:

By convolution theorem, we have

F{f(x) * g(x)} = F(s).G(s).

Therefore, $(f*g)(x) = F^{-1}\{F(s).G(s)\}$.

$$\text{i.e, } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, g(x-t).\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s).G(s).e^{-isx}\, ds \text{-----------------(1)}$$

(by using the inversion formula)

Putting x = 0 in (1) , we get

$$\int_{-\infty}^{\infty} f(t).\, g(-t).\, dt = \int_{-\infty}^{\infty} F(s).G(s).ds \text{------------------ (2)}$$

$-\infty \qquad\qquad -\infty$

Since (2) is true for all g(t), take $g(t) = \overline{f(-t)}$ and hence $\overline{g(-t)} = f(t)$-------------------- (3)

Also, G(s) = F{g(t)}

$= F\{\overline{f(-t)}\}$

$= \overline{F(s)}$ --------------------- (4) (by the property of F.T).

Using (3) & (4) in (2), we have

$$\int_{-\infty}^{\infty} f(t).\overline{f(t)}.\,dt = \int_{-\infty}^{\infty} F(s).\overline{F(s)}.ds.$$

$$\Rightarrow \int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |F(s)|^2\,ds.$$

$$\text{i.e, } \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds.$$

Example 6

Find the F.T of $f(x) = 1-|x|$ for $|x| < 1$.

$= 0 \quad$ for $|x| > 1$

and hence find the value $\int_0^{\infty} \frac{\sin^4 t}{t^4}\,dt$.

$$\text{Here, } F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)e^{isx}\,dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} (1-|x|)(\cos sx + i \sin sx)\,dx.$$

1 1 i 1

$$= \frac{\quad}{\sqrt{2\pi}} \int_{-1} (1-|x|) \cos sx \, dx. + \frac{\quad}{\sqrt{2\pi}} \int_{-1} (1-|x|) \sin sx \, dx.$$

$$= \frac{1}{\sqrt{2\pi}} \, 2\int_0^1 (1-x)\cos sx \, dx.$$ by the property of definite integral.

$$= \sqrt{(2/\pi)} \int_0^1 (1-x)\, d\left(\frac{\sin sx}{s}\right)$$

$$= \sqrt{(2/\pi)} \left\{ (1-x)\left(\frac{\sin sx}{s}\right) - (-1) - \left(\frac{\cos sx}{s^2}\right) \right\}_0^1$$

$$= \sqrt{(2/\pi)} \left(\frac{1-\cos s}{s^2}\right)$$

Using Parseval‟s identity, we get

$$\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{(1-\cos s)^2}{s^4}\, ds = \int_{-1}^{1} (1-|x|)^2\, dx.$$

$$\Rightarrow \frac{4}{\pi}\int_0^{\infty} \frac{(1-\cos s)^2}{s^4}\, ds = 2\int_0^1 (1-x)^2\, dx = 2/3.$$

$$\text{i.e., } \frac{16}{\pi}\int_0^{\infty} \frac{\sin^4(s/2)}{s^4}\, ds = 2/3.$$

Setting s/2 = x , we get

$$\frac{16}{\pi}\int_0^{\infty} \frac{\sin^4 x}{16x^4}\, 2.dx. = 2/3.$$

$$\Rightarrow \int_0^{\infty} \frac{\sin^4 x}{x^4}\, dx. = \pi/3.$$

Example 7

Find the F.T of f(x) if

$$f(x) = \begin{cases} 1 & \text{for } |x|<a \\ 0 & \text{for } |x|>a>0. \end{cases}$$

Using Parseval‟s identity, prove $\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt. = \pi/2.$

Here,

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} .(1) .dx .$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{isx}}{is}\right)_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{isa} - e^{isa}}{is}$$

$$= (\sqrt{2/\pi}) \frac{\sin as}{s}$$

i.e., $F(s) = (\sqrt{2/\pi}) \dfrac{\sin as}{s}$.

Using Parseval‟s identity

$$\int_{-\infty}^{\infty} |\, f (x) \,|^2 dx = \int_{-\infty}^{\infty} |\, F(s) \,|^2 ds,$$

we have

$$\int_{-a}^{a} 1 . \; dx = \int_{-\infty}^{\infty} (2/\pi) \left(\frac{\sin as}{s}\right)^2 ds.$$

$$2a = (2/\pi) \int_{-\infty}^{\infty} \left(\frac{\sin as}{s}\right)^2 ds.$$

Setting as = t, we get

$$(2/\pi) \int_{-\infty}^{\infty} \left(\frac{\sin t}{(t/a)} \right)^2 dt/a = 2a$$

i.e., $$\int_{-\infty}^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \pi$$

$$\Rightarrow \quad 2 \int_{0}^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \pi$$

Hence, $$\int^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \pi / 2.$$