INVERSE LAPLACE TRANSFORM

Inverse Laplace transform of elementary functions

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
$L[1] = \frac{1}{s}$	$L^{-1}\left[\frac{1}{s}\right] = 1$
$L[t] = \frac{1}{s^2}$	$L^{-1}\left[\frac{1}{s^2}\right] = t$
$L[t^n] = \frac{n!}{s^{n+1}}$ *if n is an integer*	$L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$ $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$
$L[e^{at}] = \frac{1}{s-a}$	$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
$L[e^{-at}] = \frac{1}{s+a}$	$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
$L[sinat] = \frac{a}{s^2+a^2}$	$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{sinat}{a}$
$L[cosat] = \frac{s}{s^2+a^2}$	$L^{-1}\left[\frac{s}{s^2+a^2}\right] = cosat$
$L[sinhat] = \frac{a}{s^2-a^2}$	$L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{sinhat}{a}$

$L[cosat] = \frac{s}{s^2 - a^2}$	$L^{-1}\left[\frac{s}{s^2 - a^2}\right] = coshat$

Result on inverse Laplace transform

Result: 1 Linear property

$L[f(t)] = F(s)$ **and** $L[g(t)] = G(s)$ **,then** $L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

Where a and b are constants.

Proof:

We know that $L[aF(s) \pm bG(s)] = aL[F(s)] \pm bL[G(s)]$

$$= a\,F(s) \pm b\,G(s)$$

$(i.e.)\, a\,F(s) \pm b\,G(s) = L[af(t) \pm bg(t)]$

Operating L^{-1} on both sides, we get

$L^{-1}[aF(s) \pm bG(s)] = af(t) \pm bg(t)$

$\because f(t) = L^{-1}[F(s)]$

$\because g(t) = L^{-1}[G(s)]$

$L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

Result: 2 First shifting property

(i) $L^{-1}\left[F(s+a) = e^{-at}L^{-1}[F(s)]\right]$

(ii) $L^{-1}\left[F(s-a) = e^{at}L^{-1}[F(s)]\right]$

Proof:

Let $L[e^{-at}f(t)] = F[s+a]$

Operating L^{-1} on both sides, we get

$e^{-at}f(t) = L^{-1}\left[F[s+a]\right]$

$L^{-1}\left[F[s+a]\right] = e^{-at}L^{-1}[F(s)]$

Result: 3 Multiplication by s.

If $L^{-1}[F(s)] = f(t)$ **and** $f(0) = 0$**, then** $L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) = sF(s)$

Operating L^{-1} on both sides, we get

$f'(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}f(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}L^{-1}[F(s)] = L^{-1}[sF(s)]$

$\therefore L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$

Result: 4 Division by s.

$\boldsymbol{L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t} L^{-1}[F(s)]dt$

Proof:

We know that $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)] = \frac{1}{s}F(s)$

Operating L^{-1} on both sides ,we get

$\int_0^t f(t)dt = L^{-1}\left[\frac{1}{s}F(s)\right]$

$\int_0^t L^{-1}[F(s)]\,dt = L^{-1}\left[\frac{1}{s}F(s)\right]$

$\therefore L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$

Result: 5 Inverse Laplace transform of derivative

$\boldsymbol{L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]}$

Proof:

We know that $L[tf(t)] = \frac{-d}{ds}L[f(t)] = \frac{-d}{ds}F(s)$

Operating L^{-1} on both sides ,we get

$tf(t) = -L^{-1}\left[\frac{d}{ds}F(s)\right]$

$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$

$f(t) = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$

$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$

Result: 6 Inverse Laplace transform of integral

$\boldsymbol{L^{-1}[F(s)] = t}L^{-1}\left[\int_s^\infty F(s)ds\right]$

Proof:

We know that $L\left[\frac{f(t)}{t}\right] = \int_s^\infty L(f(t))\,ds$

$= \int_s^\infty F(s)\,ds$

Operating L^{-1} on both sides, we get

$\frac{f(t)}{t} = L^{-1}\left[\int_s^\infty F(s)\,ds\right]$

$f(t) = tL^{-1}\left[\int_s^\infty F(s)\,ds\right]$

$$L^{-1}[F(s)] = tL^{-1}\left[\int_s^{\infty} F(s)\,ds\right]$$

Problems under inverse Laplace transform of elementary functions

Example: Find the inverse Laplace for the following

(i) $\frac{1}{2s+3}$ **(ii)** $\frac{1}{4s^2+9}$ **(iii)** $\frac{s^3-3s^2+7}{s^4}$ **(iv)** $\frac{3s+5}{s^2+36}$

Solution:

(i) $L^{-1}\left[\frac{1}{2s+3}\right] = L^{-1}\left[\frac{1}{2\left[s+\frac{3}{2}\right]}\right]$

$= \frac{1}{2}e^{-\frac{-3t}{2}}$

(ii) $L^{-1}\left[\frac{1}{4s^2+9}\right] = L^{-1}\left[\frac{1}{4\left[s^2+\frac{9}{4}\right]}\right]$

$= \frac{1}{4}L^{-1}\left[\frac{1}{\left[s^2+\frac{9}{4}\right]}\right]$

$= \frac{1}{4}\frac{1}{3/2}sin\frac{3}{2}t$

$= \frac{1}{6}sin\frac{3}{2}t$

(iii) $L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right] = L^{-1}\left[\frac{s^3}{s^4} - \frac{3s^2}{s^4} + \frac{7}{s^4}\right]$

$= L^{-1}\left[\frac{1}{s}\right] - 3L^{-1}\left[\frac{1}{s^2}\right] + 7L^{-1}\left[\frac{1}{s^4}\right]$

$L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right] = 1 - 3t + \frac{7t^3}{3!}$

(iv) $L^{-1}\left[\frac{3s+5}{s^2+36}\right] = 3L^{-1}\left[\frac{s}{s^2+36}\right] + 5L^{-1}\left[\frac{1}{s^2+36}\right]$

$L^{-1}\left[\frac{3s+5}{s^2+36}\right] = 3cos6t + \frac{5sin6t}{6}$

Inverse Laplace transform using First shifting theorem

$$\boldsymbol{L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]}$$

Example: 5.40 Find the inverse Laplace transform for the following:

(i) $\frac{1}{(s+2)^2}$ **(ii)** $\frac{1}{(s-3)^4}$ **(iii)** $\frac{1}{(s+3)^2+9}$ **(iv)** $\frac{1}{s^2-2s+2}$

Solution:

(i) $L^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}L^{-1}\left[\frac{1}{s^2}\right] = e^{-2t}t$

(ii) $L^{-1}\left[\frac{1}{(s-3)^4}\right] = e^{3t}L^{-1}\left[\frac{1}{s^4}\right] = e^{-2t}\,\frac{t^3}{3!}$

(iii) $L^{-1}\left[\frac{1}{(s+3)^2+9}\right] = e^{-3t}L^{-1}\left[\frac{1}{s^2+9}\right] = e^{-3t}\,\frac{sin3t}{3}$

(iv) $L^{-1}\left[\frac{1}{s^2-2s+2}\right] = L^{-1}\left[\frac{1}{(s-1)^2+1}\right] = e^t L^{-1}\left[\frac{1}{s^2+1}\right] = e^t sint$

Inverse using the formula

$$\boldsymbol{L^{-1}[F(s)] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}F(s)\right]}$$

Note: This formula is used when $F(s)$ is $\cot^{-1}\emptyset(s)$ or $\tan^{-1}\emptyset(s)$ or $log\emptyset(s)$

Example: 5.41 Find the inverse Laplace transform for the following

(i) $cot^{-1}\left(\frac{s}{a}\right)$ (ii) $tan^{-1}\left(\frac{a}{s}\right)$ (iii) $cot^{-1}as$

Solution:

(i) $L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(cot^{-1}\left(\frac{s}{a}\right)\right)\right]$

$= \frac{-1}{t}L^{-1}\left[\frac{-1}{1+\frac{s^2}{a^2}}\left(\frac{1}{a}\right)\right] = \frac{1}{t}L^{-1}\left[\frac{-1}{\frac{a^2+s^2}{a^2}}\left(\frac{1}{a}\right)\right]$

$= \frac{1}{t}L^{-1}\left[\frac{a}{s^2+a^2}\right]$

$L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{1}{t}sinat$

(ii) $L^{-1}\left[tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(tan^{-1}\left(\frac{a}{s}\right)\right)\right]$

$= \frac{-1}{t}L^{-1}\left[\frac{1}{1+\left(\frac{a}{s}\right)^2}\left(\frac{-a}{s^2}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{1}{\frac{s^2+a^2}{s^2}}\left(\frac{-a}{s^2}\right)\right]$

$= \frac{1}{t}L^{-1}\left[\frac{a}{s^2+a^2}\right]$

$L^{-1}\left[tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{1}{t}sinat$

(iii) $L^{-1}[cot^{-1}as] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}(cot^{-1}(as))\right]$

$= \frac{-1}{t}L^{-1}\left[\frac{-1}{1+a^2s^2}(a)\right] = \frac{1}{t}L^{-1}\left[\frac{a}{a^2\left(s^2+\frac{1}{a^2}\right)}\right]$

$= \frac{1}{at}L^{-1}\left[\frac{1}{s^2+\frac{1}{a^2}}\right] = \frac{1}{at}\left[\frac{sin\frac{1}{a}t}{\frac{1}{a}}\right]$

$L^{-1}[cot^{-1}as] = \frac{1}{t}sin\frac{t}{a}$

Inverse using the formula

$$\boldsymbol{L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]}$$

Example: Find $L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$

Solution:

$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{d}{dt}L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] \cdots (1)$

$$L^{-1}\left[log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = L^{-1}\frac{d}{ds}\left[log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{d}{ds}(\log(s^2+a^2)-\log(s^2+b^2))\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{s^2+a^2}2s - \frac{1}{s^2+b^2}2s\right]$$

$$= \frac{-2}{t}L^{-1}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$$

$$= \frac{-2}{t}[cosat - cosbt]$$

$$= \frac{2}{t}[cosbt - cosat]$$

Substituting in (1), we get

$$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{d}{dt}\left[\frac{2}{t}[cosbt - cosat]\right]$$

$$= 2\left[\frac{t(-bsinbt+asinat)-(cosbt-cosat)}{t^2}\right]$$

$$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = 2\left[\frac{t(-bsinbt+asinat)-(cosbt-cosat)}{t^2}\right]$$

Inverse using the formula

$$\boldsymbol{L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]dt}$$

This formula is used when $F(s) = \frac{one\ term}{s(another\ term)}$

Example: Find $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s^2+a^2)}\right]dt$$

$$= \int_0^t \left[\frac{sinat}{a}\right]dt$$

$$= \frac{1}{a}\left[\frac{-cosat}{a}\right]_0^t$$

$$= \frac{-1}{a^2}[cosat]_0^t$$

$$= \frac{-1}{a^2}(cosat - cos0) = \frac{-1}{a^2}(cosat - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1-cosat}{a^2}$$

Example: Find $L^{-1}\left[\frac{1}{s(s^2-a^2)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s^2-a^2)}\right]dt$$

$$= \int_0^t \left[\frac{sinhat}{a}\right]dt$$

$$= \frac{1}{a}\left[\frac{coshat}{a}\right]_0^t$$

$$= \frac{1}{a^2}[coshat]_0^t$$

$$= \frac{1}{a^2}(coshat - cosh0) = \frac{1}{a^2}(coshat - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2-a^2)}\right] = \frac{coshat-1}{a^2}$$

Example: Find $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s+a)}\right]dt$$

$$= \int_0^t e^{-at}dt$$

$$= \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= \frac{-1}{a}(e^{-at} - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s+a)}\right] = \frac{1-e^{-at}}{a}$$