## 4.4 POWER SPECTRAL DENSITY

**DEFINITION :**

Let {X(t)} be a stationary random process. Then the power spectral density of {X(T)} is the Fourier transform of its auto correlation function. It is denoted by $S_{XX}(\omega)\ or\ s(\omega)$

$$S_{XX}(\omega) = F[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau}\, d\tau$$

Given the power spectral density $S(\omega)$, the auto correlation function $R_{XX}(\tau)$ is given by the Fourier inverse transform.

(i.e)

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{-i\omega\tau}\, d\omega$$

**State any two uses of Spectral density**

Soln:

i) The Power spectrum of a signal has important application in Electronic communication systems like Radio, Radars, Microwave communication and so on.

ii) It is used in colorimetry. It is helpful in analyzing the color characteristics of a particular light source.

## PROBLEMS UNDER POWER SPECTRAL DENSITY

**1. A WSS process {X(t)} has ACF $R_{XX}(\tau) = \rho e^{-3|\tau|}$. Where $\rho$ is a constant. Find the PSD OF {x(t)}.**

Soln:

Given $R_{XX}(\tau) = \rho e^{-3|\tau|}$

The PSD of {X(t)} is given by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau}\, d\tau = \int_{-\infty}^{\infty} \rho e^{-3|\tau|} e^{-i\omega\tau}\, d\tau$$

$$= \rho \int_{-\infty}^{\infty} e^{-3|\tau|} e^{-i\omega\tau}\, d\tau$$

$$== \rho \int_{-\infty}^{\infty} e^{-3|\tau|} (cos\omega\tau - isin\omega\tau)\, d\tau$$

$$= \rho \int_{-\infty}^{\infty} e^{-3|\tau|} cos\omega\tau\, d\tau - i\rho \int_{-\infty}^{\infty} e^{-3|\tau|} sin\omega\tau\, d\tau$$

$$= 2\rho \int_{0}^{\infty} e^{-3|\tau|} cos\omega\tau\, d\tau - i\rho\, (0)$$

$$= 2\rho \int_{0}^{\infty} e^{-3|\tau|} cos\omega\tau\, d\tau \qquad \because \int_{0}^{\infty} e^{-at} cosbt\, dt = \frac{a}{a^2+b^2}$$

$$= 2\rho \left(\frac{3}{3^2+\omega^2}\right)$$

$$=\left(\frac{6\,\rho}{3^2+\omega^2}\right)$$

**2. Find the PSD for X(t) if $R(\tau) = \begin{cases} 1 - |\tau|, |\tau| \le 1 \\ 0, elsewhere \end{cases}$**

Solution:

The PSD of the process is given by $S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$

$$= \int_{-1}^{1} (1 - |\tau|) e^{-i\omega\tau} d\tau$$

$$= \int_{-1}^{1} (1 - |\tau|)(cos\omega\tau - isin\ \omega\tau) d\tau$$

$$= \int_{-1}^{1} (1 - |\tau|) cos\omega\tau d\tau - i \int_{-1}^{1} (1 - |\tau|) sin\omega\tau d\tau$$

$$= \int_{-1}^{1} (1 - |\tau|) cos\omega\tau d\tau - i\ (0)$$

$$= 2 \int_{0}^{1} (1 - |\tau|) cos\omega\tau d\tau$$

$$= 2 \left[ (1 - \tau) \left( \frac{sin\omega\tau}{\omega} \right) - (-1) \left( -\frac{cos\omega\tau}{\omega^2} \right) \right]_0^1$$

$$= 2 \left[ 0 - \frac{cos\omega}{\omega^2} - 0 + \frac{1}{\omega^2} \right]$$

$$= 2 \left[ \frac{1 - cos\omega}{\omega^2} \right]$$

**3. The ACF of the Poisson increment process is given by**

$\boldsymbol{R(\tau) = \begin{cases} \lambda^2, |\tau| > \epsilon \\ \lambda^2 + \frac{\lambda}{\epsilon}\left(1 - \frac{|\tau|}{\epsilon}\right), |\tau| \le \epsilon \end{cases}}$ **Prove that its spectral density is given by**

$\boldsymbol{S(\omega) = 2\pi\lambda^2\delta(\omega) + \frac{4\lambda}{\omega^2\epsilon^2} sin^2 \frac{\omega\epsilon}{2}}$

Solution:

Given $R(\tau) = \begin{cases} \lambda^2, |\tau| > \epsilon \\ \lambda^2 + \frac{\lambda}{\epsilon}\left(1 - \frac{|\tau|}{\epsilon}\right), |\tau| \le \epsilon \end{cases}$

The PSD of the process is given by $S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau$

$$= \int_{-\infty}^{-\epsilon} \lambda^2 e^{-i\omega\tau}d\tau + \int_{-\epsilon}^{\epsilon}\left[\lambda^2 + \frac{\lambda}{\epsilon}\left(1 - \frac{|\tau|}{\epsilon}\right)\right]e^{-i\omega\tau}d\tau + \int_{\epsilon}^{\infty} \lambda^2 e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{-\epsilon} \lambda^2 e^{-i\omega\tau}d\tau + \int_{-\epsilon}^{\epsilon} \lambda^2 e^{-i\omega\tau}d\tau + \int_{-\epsilon}^{\epsilon} \frac{\lambda}{\epsilon}\left(1 - \frac{|\tau|}{\epsilon}\right)e^{-i\omega\tau}d\tau + \int_{\epsilon}^{\infty} \lambda^2 e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{-\epsilon} \lambda^2 e^{-i\omega\tau}d\tau + \int_{-\epsilon}^{\epsilon} \lambda^2 e^{-i\omega\tau}d\tau + \int_{\epsilon}^{\infty} \lambda^2 e^{-i\omega\tau}d\tau + \frac{\lambda}{\epsilon}\int_{-\epsilon}^{\epsilon}\left(1 - \frac{|\tau|}{\epsilon}\right)e^{-i\omega\tau}d\tau$$

$$= \int_{-\infty}^{\infty} \lambda^2 e^{-i\omega\tau}d\tau + \frac{\lambda}{\epsilon}\int_{-\epsilon}^{\epsilon}\left(1 - \frac{|\tau|}{\epsilon}\right)e^{-i\omega\tau}d\tau$$

$$= \lambda^2 \int_{-\infty}^{\infty} e^{-i\omega\tau}d\tau + \frac{\lambda}{\epsilon}\int_{-\epsilon}^{\epsilon}\left(1 - \frac{|\tau|}{\epsilon}\right)(cos\omega\tau - isin\omega\tau)d\tau$$

$$=\lambda^2 2\pi\delta(\omega) + \frac{\lambda}{\varepsilon}\left[\int_{-\varepsilon}^{\varepsilon}(1-\frac{|\tau|}{\varepsilon})cos\omega\tau d\tau - i\int_{-\varepsilon}^{\varepsilon}(1-\frac{|\tau|}{\varepsilon})sin\omega\tau d\tau\right]=$$

$$= \lambda^2 2\pi\delta(\omega) + 2\frac{\lambda}{\varepsilon}\left[\int_0^{\varepsilon}(1-\frac{|\tau|}{\varepsilon})cos\omega\tau d\tau - i(0)\right]$$

$$= \lambda^2 2\pi\delta(\omega) + \frac{\lambda}{\varepsilon}\left[\int \quad (1-\frac{|\tau|}{\varepsilon})\frac{sin\omega\tau}{\omega} - (-\frac{1}{\varepsilon})\left(\frac{-cos\omega\tau}{\omega^2}\right)\right]_0^{\varepsilon}$$

$$= 2\pi\lambda^2\delta(\omega) + \frac{2\lambda}{\omega}\left[0 - \frac{1}{\epsilon}\frac{cos\omega\epsilon}{\omega^2} - 0 + \frac{1}{\epsilon\omega^2}\right]$$

$$= 2\pi\lambda^2\delta(\omega) + \frac{2\lambda}{\epsilon^2\omega^2}[1 - cos\omega\epsilon]$$

$$S(\omega) = 2\pi\lambda^2\delta(\omega) + \frac{4\lambda}{\omega^2\epsilon^2}sin^2\frac{\omega\epsilon}{2}$$

**4. Find the PSD for the random telegraph signal process.**

SOLUTION:

The ACF of the random telegraph signal process is $R(\tau) = e^{-2\lambda|\tau|}$

The PSD of the process is given by

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|}e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|}\,(cos\omega\tau - isin\omega\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|}\, cos\omega\tau\, d\tau - i\int_{-\infty}^{\infty} e^{-2\lambda|\tau|}\, sin\omega\tau\, d\tau$$

$= \int_{-\infty}^{\infty} e^{-2\lambda|\tau|} \cos\omega\tau \, d\tau - i(0)$

$= 2\int_{0}^{\infty} e^{-2\lambda\tau} \cos\omega\tau \, d\tau$

$= 2\left[\frac{2\lambda}{4\lambda^2+\omega^2}\right]$

$S(\omega) = 2\left[\frac{2\lambda}{4\lambda^2+\omega^2}\right]$

5. **Find the Power spectral density of random process , if its ACF is given by $R(\tau) = e^{-a|\tau|}\cos\beta\tau$.**

Solution :

Given $R(\tau) = e^{-a|\tau|}\cos\beta\tau$.

The PSD of the process is given by

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a|\tau|}\cos\beta\tau e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a|\tau|}\cos\beta\tau(\cos\omega\tau - i\sin\omega\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-a|\tau|}\cos\beta\tau\cos\omega\tau\, d\tau - i\int_{-\infty}^{\infty} e^{-a|\tau|}\cos\beta\tau\, \sin\omega\tau\, d\tau$$

$$= 2\int_{0}^{\infty} e^{-\alpha\tau} cos\beta\tau cos\omega\tau \, d\tau - i(0)$$

$$= 2\int_{0}^{\infty} e^{-\alpha\tau} cos\beta\tau cos\omega\tau \, d\tau$$

$$= 2\int_{0}^{\infty} e^{-\alpha\tau} \left[\frac{\cos(\beta+\omega)\,\tau + \cos(\beta-\omega)\tau}{2}\right] d\tau$$

$=\int_0^{\infty} e^{-\alpha\tau} \cos(\beta+\omega)\,\tau \, d\tau + \int_0^{\infty} e^{-\alpha\tau}\cos(\beta-\omega) \, d\tau$

$S(\omega) = \frac{\alpha}{\alpha^2+(\beta+\omega)^2} + \frac{\alpha}{\alpha^2+(\beta-\omega)^2}$

6. **Find the spectral density of the random process {X(t)} whose ACF is given by** $\boldsymbol{R(\tau) = \begin{cases} -1 & -2 < \tau < 2 \\ 0 & otherwise \end{cases}}$

Solution:

The PSD of the process is given by

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} \, d\tau$$

$$= \int_{-2}^{2} (-1) e^{-i\omega\tau} \, d\tau$$

$$=$$

$$= -\int_{-2}^{2} (cos\omega\tau - isin\omega\tau) \, d\tau$$

$$= -\int_{-2}^{2} cos\omega\tau \, d\tau + i\int_{-2}^{2} sin\omega\tau \ d\tau$$

$$= \ -2\int_{0}^{2} cos\omega\tau \, d\tau + i(0)$$

$= -2\left(\frac{sin\omega\tau}{\omega}\right)_0^2$

$$S(\omega) = \ \frac{-2sin2\omega}{\omega}$$

7. **Find the PSD of the Random process whose auto correlation function is** $R(\tau) = \begin{cases} 1 - \frac{|\tau|}{T} & |\tau| < T \\ 0 & elsewhere \end{cases}$

Solution:

The PSD of the process is given by

$$S(\omega) = \ \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau} \, d\tau$$

$= \ \int_{-T}^{T} 1 - \frac{|\tau|}{T} e^{-i\omega\tau} \, d\tau$

$$= \int_{-T}^{T} 1 - \frac{|\tau|}{T}(cos\omega\tau - isin\omega\tau)\ d\tau$$

$$= \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) cos\omega\tau\ d\tau - i \int_{-T}^{T} \left(1 - \frac{|\tau|}{T}\right) sin\omega\tau\ \ d\tau$$

$$= 2 \int_{0}^{T} \left(1 - \frac{\tau}{T}\right) cos\omega\tau\ d\tau - i(0)$$

$$= 2 \int_{0}^{T} \left(1 - \frac{\tau}{T}\right) cos\omega\tau\ d\tau$$

$$= 2 \left[\left(1 - \frac{\tau}{T}\right) \frac{sin\omega\tau}{\omega} - (-\frac{1}{T}) \left(\frac{-cos\omega\tau}{\omega^2}\right)\right]_0^T$$

$$= 2 \left[\frac{-cos\omega T}{T\omega^2} + 0 + \frac{1}{T\omega^2}\right]$$

$= \frac{2(1-cos\omega T)}{T\omega^2}$ $\quad (1 - cos\theta = 2\ sin^2 \frac{\theta}{2})$

$= \frac{2}{T\omega^2}\left[2sin^2 \frac{\omega T}{2}\right]$

$$S(\omega) = \frac{4}{T\omega^2} sin^2 \frac{\omega T}{2}$$

**NOTE :**

$$\int_{-\infty}^{\infty} e^{-i\omega\tau}\, d\tau = 2\pi\delta(\omega)$$

**8. If {X(t)} is a constant random process with $\boldsymbol{R(\tau) = m^2}$ for all , where m is constant , show that the spectral density of the process is $\boldsymbol{S(\omega) = 2\pi m^2 \delta(\omega)}$**

Solution :

Given $R(\tau) = m^2$ for all $\tau$

The PSD of the process is given by

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} m^2 e^{-i\omega\tau}\, d\tau$$

$$= m^2 \int_{-\infty}^{\infty} e^{-i\omega\tau}\, d\tau$$

$$S(\omega) = 2\pi m^2 \delta(\omega)$$

**9. Find the PSD of random process { X(t)} if E[X(t)] = 1 and $R_{XX}(\tau) = 1 + e^{-\alpha|\tau|}$**

SOLUTION :

The PSD of the process is given by

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau}\, d\tau$$

$= \int_{-\infty}^{\infty} \left(1 + e^{-\alpha|\tau|}\right) e^{-i\omega\tau}\, d\tau$

$$= \int_{-\infty}^{\infty} e^{-i\omega\tau}\, d\tau + \int_{-\infty}^{\infty} \left(e^{-\alpha|\tau|}\right) e^{-i\omega\tau}\, d\tau$$

$$= 2\pi\delta(\omega) + \int_{-\infty}^{\infty} \left(e^{-\alpha|\tau|}\right)(cos\omega\tau - isin\omega\tau)\, d\tau$$

$$= 2\pi\delta(\omega) + \int_{-\infty}^{\infty} \left(e^{-\alpha|\tau|}\right) cos\omega\tau\, d\tau - i \int_{-\infty}^{\infty} \left(e^{-\alpha|\tau|}\right) sin\omega\tau\, d\tau$$

$$= 2\pi\delta(\omega) + 2\int_{0}^{\infty} \left(e^{-\alpha\tau}\right) cos\omega\tau\, d\tau - i(0)$$

$$= 2\pi\delta(\omega) + 2\int_{0}^{\infty} (e^{-\alpha\tau}) cos\omega\tau \, d\tau$$

$$= 2\pi\delta(\omega) + 2\frac{\alpha}{\alpha^2 + \omega^2}$$

$$S(\omega) = 2\pi\delta(\omega) + \frac{2\alpha}{\alpha^2 + \omega^2}$$

**10.Find the spectral density function of whose ACF is given by**

$$\boldsymbol{R_{XX}(\tau) = \frac{A^2}{2} cos\omega_0\tau}$$

Solution:

The PSD of the process is given by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau} \, d\tau$$

$$= \int_{-\infty}^{\infty} \frac{A^2}{2} cos\omega_0\tau \, e^{-i\omega\tau} \, d\tau$$

$$= \frac{A^2}{2} \int_{-\infty}^{\infty} \left(\frac{e^{i\omega_0\tau} + e^{-i\omega_0\tau}}{2}\right) e^{-i\omega\tau} \, d\tau$$

$$= \frac{A^2}{4}\left[ \int_{-\infty}^{\infty} e^{i\omega_0\tau} e^{-i\omega\tau} \, d\tau + = \int_{-\infty}^{\infty} e^{-i\omega_0\tau} e^{-i\omega\tau} \, d\tau \right]$$

$$= \frac{A^2}{4}\left[ \int_{-\infty}^{\infty} e^{-i(\omega-\omega_0)\tau}\ d\tau + \int_{-\infty}^{\infty} e^{-i(\omega+\omega_0)\tau}\ d\tau \right]$$

$$= \frac{A^2}{4}[2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)]$$

$$S_{XX}(\omega) = \frac{\pi A^2}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

**11. Let $X(t) = a\cos(bt + y)$, where y is a RV uniform in ( $, 2\pi$) and $b, \omega$ are constants. Find the PSD of { X(t)}**

SOLUTION :

Given $X(t) = a\cos(bt + y)$, where y is a RV uniform in ( $0, 2\pi$)

$$f(y) = \frac{1}{2\pi}; 0 < y < 2\pi$$

Since { X(t)} is given , first find the ACF $R_{XX}(\tau)$

The ACF of the process is given by

$$R(\tau) = E[a\cos(bt + y)\, a\cos(bt + y)]$$

$$= a^2 E[\cos(bt_1 + y)\cos(bt_2 + y]$$

$$= \frac{a^2}{2} E[\cos(bt_1 + y + bt_2 + y) + \cos(bt_1 + y - bt_2 - y)]$$

$$= \frac{a^2}{2} E[\cos(bt_1 + bt_2 + 2y) + \cos(bt_1 - bt_2)]$$

$= \frac{a^2}{2} E[\cos(bt_1 + bt_2 + 2y) + \cos b(t_1 - t_2)]$

$= \frac{a^2}{2} E[\cos(bt_1 + bt_2 + 2y)] + \frac{a^2}{2} cosb\tau$

$= \frac{a^2}{2} \int_0^{2\pi} \cos(bt_1 + bt_2 + 2y)\, f(y)dy + \frac{a^2}{2} cosb\tau$

$= \frac{a^2}{2} \int_0^{2\pi} \cos(bt_1 + bt_2 + 2y) \frac{1}{2\pi} dy + \frac{a^2}{2} cosb\tau$

$= \frac{a^2}{4\pi} \left[\frac{\sin(bt_1 + bt_2 + 2y)}{2}\right]_0^{2\pi} + \frac{a^2}{2} cosb\tau$

$= \frac{a^2}{8\pi} [\sin(bt_1 + bt_2 + 4\pi) - \sin(bt_1 + bt_2)] + \frac{a^2}{2} cosb\tau$

$= \frac{a^2}{8\pi} [\sin(bt_1 + bt_2) - \sin(bt_1 + bt_2)] + \frac{a^2}{2} cosb\tau$

$= \frac{a^2}{8\pi} (0) + \frac{a^2}{2} cosb\tau$

$$R(\tau) = \frac{a^2}{2} cosb\tau$$

The PSD of the process is given by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau}\, d\tau = \int_{-\infty}^{\infty} \frac{a^2}{2} cosb\tau e^{-i\omega\tau}\, d\tau$$

$$= \frac{a^2}{2} \int_{-\infty}^{\infty} \left(\frac{e^{ib\tau} + e^{-ib\tau}}{2}\right) e^{-i\omega\tau}\, d\tau$$

$$= \frac{a^2}{4} \left[ \int_{-\infty}^{\infty} e^{ib\tau} e^{-i\omega\tau}\, d\tau + \int_{-\infty}^{\infty} e^{-ib\tau} e^{-i\omega\tau}\, d\tau \right]$$

$$= \frac{a^2}{4}\left[\int_{-\infty}^{\infty} e^{-i(\omega-b)\tau}\,d\tau + \int_{-\infty}^{\infty} e^{-i(\omega+b)\tau}\,d\tau\right]$$

$$= \frac{a^2}{4}[2\pi\delta(\omega-b) + 2\pi\delta(\omega+b)]$$

$$= \frac{\pi a^2}{2}[\delta(\omega-b) + \delta(\omega+b)]$$

$$S_{XX}(\omega) = \frac{\pi a^2}{2}[\delta(\omega-b) + \delta(\omega+b)]$$

**12. Find the PSD of a process $\boldsymbol{Y(t) = X(t)\cos(2\pi\omega_0 t + \theta)}$, where {X(t)} is a WSS random process and $\boldsymbol{\theta}$ is uniformly distributed over $\boldsymbol{(0, 2\pi)}$ which is independent of X(t).**

Solution:

Given $Y(t) = X(t)\cos(2\pi\omega_0 t + \theta)$, where $\theta$ is uniformly distributed over $(0,2\pi)$

$f(\theta) = \frac{1}{2\pi}, 0 < \theta < 2\pi$

Also { X(t)} is a WSS process

$$\therefore (i) E[X(t)] \textit{ is a constant}$$

$$(ii) R_{XX}(t_1, t_2) \text{ is a function of } \tau$$

$$\therefore R_{XX}(t_1, t_2) = R_{XX}(\tau)$$

To find $S_{YY}(\omega)$ first we find $R_{YY}(\tau)$

The ACF of the process is given by

$$R_{YY}(\tau) = E[Y(t_1)Y(t_2)]$$

$$= E[X(t_1)\cos(2\pi\omega_0 t_1 + \theta)\cos(2\pi\omega_0 t_2 + \theta)]$$

$$= E[X(t_1)X(t_2)]\text{E}[\cos(2\pi\omega_0 t_1 + \theta)\cos(2\pi\omega_0 t_2 + \theta)]$$

$$= \frac{R_{XX}(\tau)}{2}\ \text{E}[\cos(\ 2\pi\omega_0 t_1 + \theta + 2\pi\omega_0 t_2 + \theta) + \cos(2\pi\omega_0 t_1 + \theta - 2\pi\omega_0 t_2 - \theta)]$$

- $= \frac{R_{XX}(\tau)}{2}\ \text{E}[\cos(\ 2\pi\omega_0 t_1 + 2\pi\omega_0 t_2 + 2\theta) + \cos(2\pi\omega_0 t_1 + 2\pi\omega_0 t_2)$
- $= \frac{R_{XX}(\tau)}{2}\int_0^{2\pi} \cos(\ 2\pi\omega_0 t_1 + 2\pi\omega_0 t_2 + 2\theta)\ f(\theta) d\theta +$

$\frac{R_{XX}(\tau)}{2}\cos(2\pi\omega_0(t_1 - t_2)$

$$= \frac{R_{XX}(\tau)}{2}\int_0^{2\pi} \cos(\ 2\pi\omega_0 t_1 + 2\pi\omega_0 t_2 + 2\theta)\frac{1}{2\pi} d\theta$$

$$\frac{R_{XX}(\tau)}{2}\cos(2\pi\omega_0\tau) = \frac{R_{XX}(\tau)}{4\pi}\left[\frac{\sin(2\pi\omega_0 t_1 +)2\pi\omega_0 t_2 + 2\theta)}{2}\right]_0^{2\pi} + \frac{R_{XX}(\tau)}{2}\cos(2\pi\omega_0\tau)$$

$$= \frac{R_{XX}(\tau)}{8\pi}[\sin(2\pi\omega_0 t_1 +)\ 2\pi\omega_0 t_2 + 4\pi)] - \sin(2\pi\omega_0 t_1 +)\ 2\pi\omega_0 t_2) +$$

$$\frac{R_{XX}(\tau)}{2}\cos(2\pi\omega_0\tau)$$

$$= \frac{R_{XX}(\tau)}{8\pi}[\sin(2\pi\omega_0 t_1 +)\ 2\pi\omega_0 t_2)] - \sin(2\pi\omega_0 t_1 +)\ 2\pi\omega_0 t_2) +$$

$$\frac{R_{XX}(\tau)}{2}\cos(2\pi\omega_0\tau)$$

$$= \frac{R_{XX}(\tau)}{8\pi}(0) + \frac{R_{XX}(\tau)}{2}\cos(2\pi\omega_0\tau)$$

$$R_{YY}(\tau) = \frac{R_{XX}(\tau)}{2}\cos(2\pi\omega_0\tau)$$

The PSD of y(t) is given by

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau)e^{-i\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty} \frac{R_{XX}(\tau)}{2}\cos(2\pi\omega_0\tau)\,e^{-i\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty} \frac{R_{XX}(\tau)}{2}\left(\frac{e^{i2\pi\omega_0\tau} + e^{-i2\pi\omega_0\tau}}{2}\right)e^{-i\omega\tau}\,d\tau$$

$$= \frac{1}{4}\left[\int_{-\infty}^{\infty} R_{XX}(\tau)e^{i2\pi\omega_0\tau}e^{-i\omega\tau}d\tau + R_{XX}(\tau)e^{\,-i2\pi\omega_0\tau}\;e^{-i\omega\tau}d\tau\right]$$

$$= \frac{1}{4}\left[\int_{-\infty}^{\infty} R_{XX}(\tau)e^{i2\pi\omega_0\tau}e^{-i\omega\tau}d\tau + R_{XX}(\tau)e^{\,-i2\pi\omega_0\tau}\;e^{-i\omega\tau}d\tau\right]$$

$$= \frac{1}{4}\left[\int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i(\omega-2\pi\omega_0)}d\tau + R_{XX}(\tau)e^{-i(\omega+2\pi\omega_0)}d\tau\right]$$

$$= \frac{1}{4}\left[S_{XX}(\omega-2\pi\omega_0) + S_{XX}(\omega+2\pi\omega_0)\right]$$

$$S_{YY}(\omega) = \frac{1}{4}[S_{XX}(\omega - 2\pi\omega_0) + S_{XX}(\omega + 2\pi\omega_0)]$$

**13.Show that the PSD of $e^{-\alpha|\tau|}(1 + \alpha|\tau|)$ is $\frac{4\alpha^3}{(\alpha^2+\omega^2)^2}$**

Solution:

Given $R(\tau) = e^{-\alpha|\tau|}(1 + \alpha|\tau|)$

The PSD of the process is given by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\alpha|\tau|}(1 + \alpha|\tau|)\ e^{-i\omega\tau}\, d\tau \qquad \because |\tau| = \begin{cases} -\tau; & -\infty < \tau < 0 \\ \tau; & 0 < \tau < \infty \end{cases}$$

$$= \int_{-\infty}^{0} e^{\alpha\tau}(1 - \alpha\tau)\ e^{-i\omega\tau}\, d\tau + \int_{-\infty}^{\infty} e^{-\alpha\tau}(1 + \alpha\tau)\ e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{0}(1 - \alpha\tau)\ e^{(\alpha-i\omega)\tau}\, d\tau + \int_{-\infty}^{\infty}(1 + \alpha\tau)\ e^{-(\alpha+i\omega)\tau}\, d\tau$$

$$= \left[(1 - \alpha\tau)\frac{e^{(\alpha-i\omega)\tau}}{\alpha-i\omega} + \frac{\alpha e^{(\alpha-i\omega)\tau}}{(\alpha-i\omega)^2}\right]_{-\infty}^{0} + \left[(1 + \alpha\tau)\frac{e^{-(\alpha+i\omega)\tau}}{-(\alpha+i\omega)} - \frac{\alpha e^{-(\alpha+i\omega)\tau}}{(\alpha+i\omega)^2}\right]_{0}^{\infty}$$

$$= \frac{1}{\alpha-i\omega} + \frac{\alpha}{(\alpha-i\omega)^2} + \frac{1}{\alpha+i\omega} + \frac{\alpha}{(\alpha+i\omega)^2}$$

$$= \frac{1}{\alpha-i\omega} + \frac{1}{\alpha+i\omega} + \frac{\alpha}{(\alpha-i\omega)^2} + \frac{\alpha}{(\alpha+i\omega)^2}$$

$$= \frac{\alpha+i\omega+\alpha-i\omega}{(\alpha-i\omega)(\alpha+i\omega)} + \frac{\alpha(\alpha+i\omega)^2+\alpha(\alpha-i\omega)^2}{(\alpha+i\omega)^2(\alpha-i\omega)^2}$$

$$= \frac{2\alpha}{\alpha^2+\omega^2} + \frac{\alpha[\alpha^2-\omega^2+2\alpha i\omega+\alpha^2-\omega^2-2\alpha i\omega]}{(\alpha+i\omega)^2(\alpha-i\omega)^2}$$

$$= \frac{2\alpha}{\alpha^2+\omega^2} + \frac{\alpha(2\alpha^2-2\omega^2)}{(\alpha^2+\omega^2)^2}$$

$$= \frac{2\alpha(\alpha^2+\omega^2)+\alpha(2\alpha^2-2\omega^2)}{(\alpha^2+\omega^2)^2}$$

$$= \frac{2\alpha^3+2\alpha\omega^2+2\alpha^3-2\alpha\omega^2}{(\alpha^2+\omega^2)^2}$$

$$S(\omega) = \frac{4\alpha^3}{(\alpha^2+\omega^2)^2}$$

**14. Find the PSD of random binary process whose ACF is $R(\tau) = e^{-\alpha\tau^2}$**

Solution:

The PSD is given by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-i\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\alpha\tau^2}e^{-i\omega\tau}\,d\tau$$

$$= \int_{-\infty}^{\infty} e^{-(\alpha\tau^2+i\omega\tau)}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\left[\left(\sqrt{\alpha}\tau\right)^2+i\omega\tau\right]}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-[A^2+2AB]}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-[(A+B)^2-B^2]}\, d\tau$$

$\left(\left(\because where\ A = \sqrt{\alpha}\tau; 2AB = i\omega\tau\right.\right.$ and $\left. B = \frac{i\omega\tau}{2A} = \frac{i\omega\tau}{2\sqrt{\alpha}\tau} = \frac{i\omega}{2\sqrt{\alpha}}\right)$

$$= \int_{-\infty}^{\infty} e^{-\left\{\left[\sqrt{\alpha}\tau+\frac{i\omega}{2\sqrt{\alpha}}\right]^2-\left(\frac{i\omega}{2\sqrt{\alpha}}\right)^2\right\}}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\left[\sqrt{\alpha}\tau+\frac{i\omega}{2\sqrt{\alpha}}\right]^2+\left(\frac{i\omega}{2\sqrt{\alpha}}\right)^2}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\left[\sqrt{\alpha}\tau+\frac{i\omega}{2\sqrt{\alpha}}\right]^2} e^{\left(\frac{i\omega}{2\sqrt{\alpha}}\right)^2} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\left[\sqrt{\alpha}\tau+\frac{i\omega}{2\sqrt{\alpha}}\right]^2} e^{\frac{-\omega^2}{4\alpha}} d\tau$$

$$= e^{\frac{-\omega^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-\left[\sqrt{\alpha}\tau+\frac{i\omega}{2\sqrt{\alpha}}\right]^2} d\tau$$

$$= e^{\frac{-\omega^2}{4\alpha}} \int_{-\infty}^{\infty} e^{-x^2} \frac{dx}{\sqrt{\alpha}} \quad (\because Let\ \sqrt{\alpha}\tau + \frac{i\omega}{2\sqrt{\alpha}} = x\ ,\ \sqrt{\alpha}\ d\tau = dx\ \Rightarrow d\tau = \frac{dx}{\sqrt{\alpha}}\ )$$

$$= \frac{e^{\frac{-\omega^2}{4\alpha}}}{\sqrt{\alpha}} \int_{-\infty}^{\infty} e^{-x^2} dx$$

$$= \frac{e^{\frac{-\omega^2}{4\alpha}}}{\sqrt{\alpha}} \sqrt{\pi} \qquad (\because \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\ )$$

$$S(\omega) = \sqrt{\frac{\pi}{\alpha}}\, e^{\frac{-\omega^2}{4\alpha}}$$

**15. Find the power spectral density of the random process if its ACF is**

$$\boldsymbol{R(\tau) = e^{-\frac{\alpha^2\tau^2}{2}}}$$

Solution:

The PSD is given by

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\frac{\alpha^2\tau^2}{2}} e^{-i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\frac{\alpha^2\tau^2}{2} - i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\frac{\alpha^2\tau^2 - 2i\omega\tau}{2}}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\left[\frac{\alpha^2\tau^2 + 2i\omega\tau}{2}\right]}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}[A^2 + 2AB]}\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}[(A+B)^2 - B^2]}\, d\tau$$

$$((\because where\ A = \alpha\tau; 2AB = 2\, i\omega\tau \text{ and } \quad B = \frac{2\, i\omega\tau}{2A} = \frac{i\omega\tau}{\alpha\tau} = \frac{i\omega}{\alpha})$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left\{\left[\alpha\tau + \frac{i\omega}{\alpha}\right]^2 - \left(\frac{i\omega}{\alpha}\right)^2\right\}} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\alpha\tau + \frac{i\omega}{\alpha}\right]^2 + \frac{1}{2}\left(\frac{i\omega}{\alpha}\right)^2} d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\alpha\tau + \frac{i\omega}{\alpha}\right]^2}\ e^{\frac{1}{2}\left(\frac{i\omega}{\alpha}\right)^2} d\tau$$

$$= e^{\frac{1}{2}\left(\frac{i\omega}{\alpha}\right)^2} \int_{-\infty}^{\infty} e^{-\left[\frac{1}{\sqrt{2}}\left(\alpha\tau + \frac{i\omega}{\alpha}\right)\right]^2} d\tau$$

$$= e^{\frac{-\omega^2}{2\alpha^2}} \int_{-\infty}^{\infty} e^{-x^2} \frac{\sqrt{2}dx}{\alpha} \quad (\because Let\ \frac{1}{\sqrt{2}}\left(\alpha\tau + \frac{i\omega}{\alpha}\right) = x\ ,\ \frac{1}{\sqrt{2}}\alpha d\tau = dx\ \Rightarrow d\tau = \frac{\sqrt{2}dx}{\alpha}\ )$$

$$= e^{\frac{-\omega^2}{2\alpha^2}} \frac{\sqrt{2}}{\alpha} \int_{-\infty}^{\infty} e^{-x^2} dx = e^{\frac{-\omega^2}{2\alpha^2}} \frac{\sqrt{2}}{\alpha}\sqrt{\pi} \quad (\because \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}\ )$$

$$S(\omega) = e^{\frac{-\omega^2}{2\alpha^2}} \frac{\sqrt{2\pi}}{\alpha}$$

**Problems to compute ACF if PSD of a process is given**

$$\boldsymbol{R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}\,d\omega\,; \quad \int_{-\infty}^{\infty} \delta(\omega)e^{i\omega\tau}\,d\omega = 1\,; \quad \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}d\omega}{\omega^2+\alpha^2} = \frac{\pi}{\alpha}e^{-\alpha|\tau|}}$$

1. **The PSD of a RP is given by $S(\omega) = \begin{cases} \pi; & |\omega| \le 1 \\ 0; & else \end{cases}$. Find the ACF of the process.**

SOLUTION :

Given $S(\omega) = \begin{cases} \pi; & |\omega| \le 1 \\ 0; & else \end{cases}$

The ACF of the process is given by

$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{-1}^{1} \pi e^{i\omega\tau}\,d\omega$

$= \frac{1}{2}\int_{-1}^{1}(cos\omega\tau + isin\omega\tau)\,d\omega$

$= \frac{1}{2}\left[\int_{-1}^{1} cos\omega\tau\,d\omega + i\int_{-1}^{1} sin\omega\tau\,d\omega\right]$

$= \frac{1}{2}\left[2\int_0^1 cos\omega\tau\, d\omega + i(0)\right]$

$$R(\tau) = \left[\frac{sin\omega\tau}{\tau}\right]_0^1$$

$$R(\tau) = \frac{sin\tau}{\tau}$$

2. **The PSD function of zero mean WSS process {X(t)} is given by $S(\omega) = \begin{cases} 1; & |\omega| < \omega_0 \\ 0; & else \end{cases}$. Find $R(\tau)$ and show that X(t) and $X(t+\frac{\pi}{\omega_0})$ are uncorrelated.**

Solution:

Given $S(\omega) = \begin{cases} 1; & -\omega_0 < \omega < \omega_0 \\ 0; & else \end{cases}$

The ACF of the process is given by

$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}(1)e^{i\omega\tau}\, d\omega$

$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0}(cos\omega\tau + isin\omega\tau)\, d\omega$

$$= \frac{1}{2\pi}\left[\int_{-\omega_0}^{\omega_0} cos\omega\tau\, d\omega + i\int_{-\omega_0}^{\omega_0} sin\omega\tau\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[2\int_0^{\omega_0} cos\omega\tau\, d\omega + i(0)\right]$$

$$R(\tau) = \frac{1}{\pi}\left[\frac{sin\omega\tau}{\tau}\right]_0^{\omega_0}$$

$$R(\tau) = \frac{sin\omega_0\tau}{\tau\pi}$$

(i.e) $E[X(t)X(t+\tau)] = \frac{sin\omega_0\tau}{\tau\pi}$

Put $\tau = \frac{\pi}{\omega_0}$ we get

$$E\left[X(t)X(t+\frac{\pi}{\omega_0})\right] = \frac{sin\left(\omega_0\frac{\pi}{\omega_0}\right)}{\frac{\pi}{\omega_0}\pi}$$

$$= \frac{sin\pi}{\frac{\pi^2}{\omega_0}} \quad \because\ sin\pi = 0$$

$$\because E\left[X(t)X(t+\frac{\pi}{\omega_0})\right] = 0 \ldots\ldots\ldots\ldots\ldots..(1)$$

Given mean of X(t) is zero

(i.e) $E[X(t)] = 0$

$$\Rightarrow E[X(t)]E\left[X(t+\frac{\pi}{\omega_0})\right] = 0 \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2), we get

$$E\left[X(t)X(t+\frac{\pi}{\omega_0})\right] = E[X(t)]E\left[X(t+\frac{\pi}{\omega_0})\right]$$

$\therefore X(t)\ and\ X(t+\frac{\pi}{\omega_0})$ are uncorrelated.

3. **If the PSD OF WSS process is given by** $\boldsymbol{S(\omega) = \begin{cases} \frac{b}{a}(a-|\omega|); & |\omega| \le a \\ 0; & |\omega| > a \end{cases}}$**.**

**Find auto correlation function**

Solution :

$$\text{Given } S(\omega) = \begin{cases} \frac{b}{a}(a-|\omega|); & -a \le \omega \le a \\ 0; & else \end{cases}$$

The ACF of the process is given by

$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}\,d\omega = \frac{1}{2\pi}\int_{-a}^{a}\frac{b}{a}(a-|\omega|)e^{i\omega\tau}\,d\omega$

$= \frac{1}{2\pi}\frac{b}{a}\int_{-a}^{a}(a-|\omega|)e^{i\omega\tau}\,d\omega$

$= \frac{1}{2\pi}\frac{b}{a}\int_{-a}^{a}(a-|\omega|)(cos\omega\tau + isin\omega\tau)\,d\omega$

$= \frac{1}{2\pi}\frac{b}{a}\left[\int_{-a}^{a}(a-|\omega|)cos\omega\tau\,d\omega + i\int_{-a}^{a}(a-|\omega|)sin\omega\tau\,d\omega\right]$

$= \frac{b}{2\pi a}\left[\int_{-a}^{a}(a-|\omega|)cos\omega\tau\,d\omega + i(0)\right]$

$= \frac{b}{2\pi a}\ 2\int_0^a (a-\omega) cos\omega\tau\, d\omega$

$= \frac{b}{\pi a}\left[(a-\omega)\frac{sin\omega\tau}{\tau} - \frac{cos\omega\tau}{\tau^2}\right]_0^a$

$= \frac{b}{\pi a}\left[\frac{-cosa\tau}{\tau^2} + \frac{1}{\tau^2}\right]$

$= \frac{b}{\pi a}\left[\frac{1-cosa\tau}{\tau^2}\right]$

$R(\tau) = \frac{2b}{\pi a\tau^2} sin^2\frac{a\tau}{2}$ $\quad\left(\because 1 - cos\theta = 2sin^2\frac{\theta}{2}\right)$

**AVERAGE POWER OF A PROCESS**

**DEFINITION :**

The average power of the process is defined as $P_{XX} =$

$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E[X^2\ (t)]dt;\ if\ \{X(t)\}\ is\ given$

NOTE: The average power of a process is nothing but the mean square value of the process.

1. Find the average power of a process whose PSD is $S(\omega) = \frac{1}{\omega^2+4}$

Solution:

First compute the ACF of the process

The ACF of the process is given by

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\omega^2 + 4} e^{i\omega\tau} \, d\omega$$

$$= \frac{1}{2\pi} \times \frac{\pi}{2} e^{-2|\tau|}$$

$$= \frac{1}{4} e^{-2|\tau|}$$

Average power of the process = R(0)

$$\therefore R(0) = \frac{1}{4} e^{0} = \frac{1}{4}$$

Average power = $\frac{1}{4}$

**2. Find the average power of a process {X(t)} if its PSD is given by**

$$\boldsymbol{S(\omega) = \frac{10\omega^2 + 35}{(\omega^2 + 4)(\omega^2 + 9)}}$$

**Solution :**

Given $S(\omega) = \frac{10\omega^2+35}{(\omega^2+4)(\omega^2+9)}$

Put $\omega^2 = x, we\ get\ S(\omega) = \frac{10x+35}{(x+4)(x+9)}$

$$\frac{10x+35}{(x+4)(x+9)} = \frac{A}{(x+4)} + \frac{B}{(x+9)}$$

$$A(x+9) + B(x+4) = 10x + 35 \ldots\ldots..(1)$$

Put $x = -9 \; in \; (1), we \; get$

$$A(-9+9) + B(-9+4) = -90 + 35$$

$$\Rightarrow B = 11$$

Put $x = -4 \; in \; (1), we \; get$

$$A(-4+9) + B(-4+4) = -40 + 35$$

$$\Rightarrow B = -1$$

Substitute the values of A,B in (1), we get

$$\frac{10x+35}{(x+4)(x+9)} = \frac{-1}{(x+4)} + \frac{11}{(x+9)}$$

$$S(\omega) = \frac{-1}{(\omega^2+4)} + \frac{11}{(\omega^2+9)}$$

The ACF of the process is given by

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)e^{i\omega\tau}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{11}{(\omega^2+9)}-\frac{1}{(\omega^2+4)}\right\}e^{i\omega\tau}\,d\omega$$

$$= \frac{1}{2\pi}\left[11\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}}{(\omega^2+9)}d\omega-\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}}{(\omega^2+4)}d\omega\right]$$

$$= \frac{1}{2\pi}\left[11\ \times\frac{\pi}{3}e^{-3|\tau|}-\frac{\pi}{2}e^{-2|\tau|}\right]$$

Average power of the process = R(0)

$$\therefore R(0)=\frac{11}{6}-\frac{1}{4}=\frac{19}{12}$$

Average power = $\frac{19}{12}$

**3. Given the power spectral density of a continuous process are $S(\omega)=\frac{\omega^2+9}{\omega^4+5\omega^2+4}$. Find the mean square value of the process.**

Solution:

Given $S(\omega)=\frac{\omega^2+9}{\omega^4+5\omega^2+4}=\frac{\omega^2+9}{(\omega^2+1)(\omega^2+4)}$

Put $\omega^2=x, we\ get\ S(\omega)=\frac{x+9}{(x+1)(x+4)}$

$$\frac{x+9}{(x+1)(x+4)} = \frac{A}{(x+1)} + \frac{B}{(x+4)} \ldots\ldots\ldots (1)$$

$$A(x+4) + B(x+1) = x+9$$

Put $x = -4 \ in \ (1), we \ get$

$$A(-4+4) + B(-4+1) = -4+9$$

$$\Rightarrow B = \frac{-5}{3}$$

Put $x = -1 \ in \ (1), we \ get$

$$A(-1+4) + B(-1+1) = -1+9$$

$$\Rightarrow A = \frac{8}{3}$$

Substitute the values of A,B in (1), we get

$$\frac{x+9}{(x+1)(x+4)} = \frac{8}{3(x+1)} - \frac{5}{3(x+4)}$$

$$S(\omega) = \frac{8}{3(\omega^2+1)} - \frac{5}{3(\omega^2+4)}$$

The ACF of the process is given by

$$R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} \, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{8}{3(\omega^2+1)} - \frac{5}{3(\omega^2+4)}\right\} e^{i\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{8}{3}\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}}{(\omega^2+1)}d\omega - \frac{5}{3}\int_{-\infty}^{\infty}\frac{e^{i\omega\tau}}{(\omega^2+4)}d\omega\right]$$

$$= \frac{1}{2\pi}\left[\frac{8}{3} \times \pi e^{-|\tau|} - \frac{5}{3}\frac{\pi}{2}e^{-2|\tau|}\right]$$

$$= \frac{1}{2\pi} \times \frac{\pi}{3}\left[8e^{-|\tau|} - \frac{5}{2}e^{-2|\tau|}\right]$$

$$R(\tau) = \frac{1}{6}\left[8e^{-|\tau|} - \frac{5}{2}e^{-2|\tau|}\right]$$

Average power of the process = R(0)

$$\therefore R(0) = \frac{1}{6}[8 - \frac{5}{2}] = \frac{11}{12}$$

Average power = $\frac{11}{12}$