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DEPARTMENT OF MATHEMATICS

NAME OF THE SUBJECT : TRANSFORMS & PARTIAL DIFFERENTIAL EQUATION

SUBJECT CODE : MA8353

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UNIT – V
Z – TRANSFORM & DIFFERENCE EQUATION

Z-TRANSFORMS AND DEFFERENCE EQUATION

CLASS NOTES

Z-Transform of some basic functions:	
1.	$Z\left[a^n\right]=\dfrac{z}{z-a}$; $Z[1]=\dfrac{z}{z-1}$; $Z\left[(-a)^n\right]=\dfrac{z}{z+a}$
2.	$Z[n]=\dfrac{z}{(z-1)^2}$
3.	$Z\left[\dfrac{1}{n}\right]=\log\left(\dfrac{z}{z-1}\right)$
4.	$Z\left[\dfrac{1}{n+1}\right]=z\log\left(\dfrac{z}{z-1}\right)$
5.	$Z\left[\dfrac{1}{n-1}\right]=\dfrac{1}{z}\log\left(\dfrac{z}{z-1}\right)$
6.	$Z\left[\dfrac{1}{n!}\right]=e^{\frac{1}{z}}$
7.	$Z[\cos n\theta]=\dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
8.	$Z[\sin n\theta]=\dfrac{z\sin\theta}{z^2-2z\cos\theta+1}$
Inverse Z-Transforms: The inverse Z-transform of $Z[f(n)]=F(z)$ is defined as $f(n)=Z^{-1}[F(z)]$.	
The inverse Z-Transform of some basic functions:	
1.	$Z^{-1}\left[\dfrac{z}{z-1}\right]=1$; $Z^{-1}\left[\dfrac{z}{z+1}\right]=(-1)^n$
2.	$Z^{-1}\left[\dfrac{z}{z-a}\right]=a^n$; $Z^{-1}\left[\dfrac{z}{z+a}\right]=(-a)^n$; $Z^{-1}\left[\dfrac{1}{z+a}\right]=a^{n-1}$
3.	$Z^{-1}\left[\dfrac{z^2}{(z-a)^2}\right]=(n+1)a^n$ For Eg. 1) $Z^{-1}\left[\dfrac{z}{(z-a)^2}\right]=(n-1+1)a^{n-1}=na^{n-1}$ 2) $Z^{-1}\left[\dfrac{1}{(z-a)^2}\right]=(n-2+1)a^{n-2}=(n-1)a^{n-2}$ 3) $Z^{-1}\left[\dfrac{z^2}{(z-1)^2}\right]=(n+1)1^n=n+1$ 4) $Z^{-1}\left[\dfrac{z}{(z-1)^2}\right]=(n-1+1)1^n=n$ 5) $Z^{-1}\left[\dfrac{1}{(z-1)^2}\right]=(n-2+1)1^n=n-1$
4.	$Z^{-1}\left[\dfrac{z^2}{z^2+a^2}\right]=a^n\cos\dfrac{n\pi}{2}$

5.	$Z^{-1}\left[\dfrac{z}{z^2+a^2}\right]=a^n\cos(n-1)\dfrac{\pi}{2}=a^n\cos\left(\dfrac{\pi}{2}-\dfrac{n\pi}{2}\right)=a^n\sin\dfrac{n\pi}{2}$

Finding Inverse Z-transform by method of **Partial Fractions**:

Rules of Partial Fractions:

1. Denominator containing Linear factors:

$$\frac{f(z)}{(z-a)(z-b)(z-c)...}=\frac{A}{(z-a)}+\frac{B}{(z-b)}+\frac{C}{(z-c)}+...$$

2. Denominator containing factors $(z-a)^n$:

$$\frac{f(z)}{(z-a)^n}=\frac{A}{(z-a)}+\frac{B}{(z-a)^2}+\frac{C}{(z-a)^3}+...+\frac{D}{(z-a)^n}$$

3. Denominator contains a quadratic factor of the form az^2+bz+c (where a,b,c are constants):

$$\frac{f(z)}{az^2+bz+c}=\frac{A}{az^2+bz+c}+\frac{Bz}{az^2+bz+c}$$

(Or) $\dfrac{f(z)}{az^2+bz+c}=\dfrac{Az+B}{az^2+bz+c}$

1. **Find $Z^{-1}\left[\dfrac{z}{(z+1)(z-1)^2}\right]$ using the method partial fraction.**

Solution:

$$F(z)=\frac{z}{(z+1)(z-1)^2}$$

$$\frac{F(z)}{z}=\frac{1}{(z+1)(z-1)^2}\quad -----(1)$$

Now,

$$\frac{1}{(z+1)(z-1)^2}=\frac{A}{z+1}+\frac{B}{z-1}+\frac{C}{(z-1)^2}$$

$$1=A(z-1)^2+B(z+1)(z-1)+C(z+1)$$

Put $z=1\Rightarrow 1=2C\ \Rightarrow \boxed{C=\frac{1}{2}}$

Put $z=-1,\ \Rightarrow\ 1=4A\ \Rightarrow\ \boxed{A=\frac{1}{4}}$

Put $z=0\ \Rightarrow\ 1=A-B+C\ \Rightarrow B=\frac{1}{4}+\frac{1}{2}-1\ \Rightarrow B=\frac{1+2-4}{4}\ \Rightarrow \boxed{B=\frac{-1}{4}}$

$$\frac{1}{(z+1)(z-1)^2}=\frac{\frac{1}{4}}{z+1}+\frac{\frac{-1}{4}}{z-1}+\frac{\frac{1}{2}}{(z-1)^2}$$

$$(1)\ \Rightarrow F(z)=\frac{1}{4}\frac{z}{z+1}-\frac{1}{4}\frac{z}{z-1}+\frac{1}{2}\frac{z}{(z-1)^2}$$

Taking Z^{-1} on both sides

$$(1)\ \Rightarrow Z^{-1}\left[F(z)\right]=\frac{1}{4}Z^{-1}\left[\frac{z}{z+1}\right]-\frac{1}{4}Z^{-1}\left[\frac{z}{z-1}\right]+\frac{1}{2}Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$\boxed{f(n)=\frac{1}{4}(-1)^n-\frac{1}{4}(1)+\frac{1}{2}n}$$

2. **Find** $Z^{-1}\left[\dfrac{z^{-2}}{\left(1+z^{-1}\right)^{2}\left(1-z^{-1}\right)}\right]$.

Solution:

$$F(z)=\frac{z^{-2}}{\left(1+z^{-1}\right)^{2}\left(1-z^{-1}\right)}=\frac{1}{\cancel{z^{2}}\dfrac{(z+1)^{2}}{\cancel{z^{2}}}\left(\dfrac{z-1}{z}\right)}$$

$$F(z)=\frac{z}{(z+1)^{2}(z-1)}$$

$$\frac{F(z)}{z}=\frac{1}{(z+1)^{2}(z-1)}\qquad ------(1)$$

$$\frac{1}{(z-1)(z+1)^{2}}=\frac{A}{z-1}+\frac{B}{z+1}+\frac{C}{(z+1)^{2}}$$

$$1=A(z+1)^{2}+B(z-1)(z+1)+C(z-1)$$

Put $z=1,\quad 1=4A\quad\Rightarrow\ \boxed{A=\frac{1}{4}}$

Put $z=-1,\ \Rightarrow\ 1=-2c\ \Rightarrow\ \boxed{c=-\frac{1}{2}}$

Equating co-efficients of $z^{2}\quad\Rightarrow\quad 0=A+B\ \Rightarrow\boxed{B=-\frac{1}{4}}$

$$(1)\ \Rightarrow\frac{F(z)}{z}=\frac{1}{4}\frac{1}{z-1}+\frac{-1}{4}\frac{1}{z+1}-\frac{1}{2}\frac{1}{(z+1)^{2}}$$

$$(1)\ \Rightarrow Z^{-1}\left[F(z)\right]=\frac{1}{4}Z^{-1}\left[\frac{z}{z-1}\right]-\frac{1}{4}Z^{-1}\left[\frac{z}{z+1}\right]-\frac{1}{2}Z^{-1}\left[\frac{z}{(z+1)^{2}}\right]$$

$$f(n)=\frac{1}{4}(1)^{n}-\frac{1}{4}(-1)^{n}+\frac{1}{2}n(-1)^{n}$$

$$\boxed{f(n)=\frac{1}{4}-\frac{1}{4}(-1)^{n}+\frac{1}{2}n(-1)^{n}}$$

3. **Find** $Z^{-1}\left[\dfrac{z^{-2}}{\left(1-z^{-1}\right)\left(1-2z^{-1}\right)\left(1-3z^{-1}\right)}\right]$.

Solution:

$$F(z)=\left[\frac{z^{-2}}{\left(1-z^{-1}\right)\left(1-2z^{-1}\right)\left(1-3z^{-1}\right)}\right]=\frac{\dfrac{1}{z^{2}}}{\left(1-\dfrac{1}{z}\right)\left(1-\dfrac{2}{z}\right)\left(1-\dfrac{3}{z}\right)}$$

$$=\frac{1}{\cancel{z^{2}}\left(\dfrac{z-1}{\cancel{z}}\right)\left(\dfrac{z-2}{\cancel{z}}\right)\left(\dfrac{z-3}{z}\right)}$$

$$F(z)=\frac{z}{(z-1)(z-2)(z-3)}$$

$$\frac{F(z)}{z}=\frac{1}{(z-1)(z-2)(z-3)}\quad ------(1)$$

	Now by Partial Fraction, $\frac{1}{(z-1)(z-2)(z-3)}=\frac{A}{z-1}+\frac{B}{z-2}+\frac{C}{z-3}$ $1=A(z-2)(z-3)+B(z-1)(z-3)+C(z-1)(z-2)$ Put $z=2, \quad \Rightarrow \quad 1=-B\Rightarrow \boxed{B=-1}$ Put $z=1, \quad \Rightarrow \quad 1=2A\Rightarrow \boxed{A=\frac{1}{2}}$ Put $z=3, \quad \Rightarrow \quad 1=2C\Rightarrow \boxed{C=\frac{1}{2}}$ $(1)\Rightarrow F(z)=\frac{1}{2}\frac{z}{z-1}-\frac{z}{z-2}+\frac{1}{2}\frac{z}{z-3}$ $(1)\Rightarrow Z^{-1}[F(z)]=\frac{1}{2}Z^{-1}\left[\frac{z}{z-1}\right]-Z^{-1}\left[\frac{z}{z-2}\right]+\frac{1}{2}Z^{-1}\left[\frac{z}{z-3}\right]$ $f(n)=\frac{1}{2}(1)^n-(2)^n+\frac{1}{2}(3)^n$ $\boxed{f(n)=\frac{1}{2}-2^n+\frac{1}{2}3^n}$
4.	**Find the Z-transform of $\frac{z^2+z}{(z-1)(z^2+1)}$ using partial fraction.** **Solution:** $F(z)=\frac{z^2+z}{(z-1)(z^2+1)}$ $\frac{F(z)}{z}=\frac{z+1}{(z-1)(z^2+1)}$ $\frac{z+1}{(z-1)(z^2+1)}=\frac{A}{(z-1)}+\frac{B}{(z^2+1)}+\frac{Cz}{(z^2+1)}$ $z+1=A(z^2+1)+B(z-1)+Cz(z-1)$ Put $z=1, \quad \Rightarrow \quad 2=2A\Rightarrow \boxed{A=1}$ Equating co-efficients of $z^2 \quad \Rightarrow 0=A+C \Rightarrow \boxed{C=-1}$ Put $z=0, \quad \Rightarrow \quad 1=A-B\Rightarrow B=A-1=1-1=0\boxed{B=0}$ $\frac{F(z)}{z}=\frac{1}{(z-1)}+\frac{0}{(z^2+1)}+\frac{-z}{(z^2+1)}$ $F(z)=\frac{z}{(z-1)}-\frac{z^2}{(z^2+1)}$ Put Z^{-1} on both sides $Z^{-1}[F(z)]=Z^{-1}\left[\frac{z}{z-1}\right]-Z^{-1}\left[\frac{z^2}{z^2+1}\right]$ $\boxed{f(n)=1-\cos\frac{n\pi}{2}} \qquad \because Z^{-1}\left[\frac{z^2}{z^2+a^2}\right]=\cos\frac{n\pi}{2}$

Finding Inverse Z-transform by Residue Method:

By Inverse Z-Transforms $Z^{-1}[F(z)]=f(n)$

Procedure:

1. write $F(z)$ from given expression and write $F(z)z^{n-1}$

2. Find the poles by equating denominator to zero in $F(z)z^{n-1}$

3. Write the order of poles

4. Find the residue at these poles

Case i: If $z = a$ is pole of order 1 (or) simple pole then

$$\left[\operatorname{Res} F(z)z^{n-1}\right]_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$$

Case ii: If $z = a$ is pole of order ***m*** then $\left[\operatorname{Res} F(z)z^{n-1}\right]_{z=a} = \frac{1}{\lfloor m-1}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m F(z)z^{n-1}$

5. $f(n) = \text{sum of residues of } F(z)z^{n-1}$

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1 **Find $Z^{-1}\left[\frac{2z}{(z-2)(z^2+1)}\right]$ by the method of residues.**

Solution:

Let $F(z) = \frac{2z}{(z-1)(z^2+1)}$

$$F(z)z^{n-1} = \frac{2zz^{n-1}}{(z-1)(z^2+1)}$$

$$F(z)z^{n-1} = \frac{2z^n}{(z-1)(z+i)(z-i)} \quad ------(1)$$

Here $z=1$, $z=i$ and $z=-i$ are poles of order 1.

1) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = \lim_{z\to 1}(z-1)\frac{2z^n}{(z-1)(z+i)(z-i)}$$

$$= \lim_{z\to 1}\frac{2z^n}{(z+i)(z-i)}$$

$$= \frac{2(1)^n}{(1+i)(1-i)}$$

$$= \frac{2}{2} \qquad \because (1+i)(1-i) = 1^2 - i^2 = 1-(-1) = 1+1 = 2$$

$$\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = 1}$$

2) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=i} = \lim_{z\to a}(z-i)F(z)z^{n-1}$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=i} = \lim_{z\to i}(z-i)\frac{2z^n}{(z-1)(z-i)(z+i)}$$

$$= \lim_{z\to i}\frac{2z^n}{(z-1)(z+i)}$$

$$= \frac{2(i)^n}{(i-1)(i+i)}$$

$$= \frac{2(i)^n}{2i(i-1)}$$

$$= \frac{(i)^n}{i(i-1)} = \frac{(i)^n}{(i^2-i)} = \frac{(i)^n}{(-1-i)}$$

$$\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=i} = \frac{-(i)^n}{(1+i)}}$$

	3) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-i} = \lim_{z\to -i}(z+i)F(z)z^{n-1}$ $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-i} = \lim_{z\to -i} \cancel{(z+i)} \frac{2z^n}{(z-1)\cancel{(z+i)}(z-i)}$ $= \lim_{z\to -i} \frac{2z^n}{(z-1)(z-i)}$ $= \frac{2(-i)^n}{(-i-1)(-i-i)} = \frac{2(-i)^n}{(1+i)(2i)}$ $= \frac{(-i)^n}{(1+i)(i)} = \frac{(-i)^n}{(i+i^2)} = \frac{(-i)^n}{(i-1)}$ $\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-i} = \frac{(-i)^n}{(i-1)}}$ $f(n) = \text{sum of residues of } F(z)z^{n-1}$ $\boxed{f(n) = 1 - \frac{(i)^n}{(1+i)} + \frac{(-i)^n}{(i-1)}}$
2.	**Find the inverse Z-Transform of** $\frac{z(z+1)}{(z-1)^3}$ **by residue method**. **Solution:** Let $F(z) = \frac{z(z+1)}{(z-1)^3}$ $F(z)z^{n-1} = \frac{zz^{n-1}(z+1)}{(z-1)^3}$ $F(z)z^{n-1} = \frac{z^n(z+1)}{(z-1)^3} \quad -----(1)$ $z=1$ is a pole of order 3 $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m F(z)z^{n-1}$ $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = \frac{1}{(3-1)!}\lim_{z\to 1}\frac{d^2}{dz^2}\cancel{(z-1)^3}\frac{z^n(z+1)}{\cancel{(z-1)^3}}$ $= \frac{1}{2}\lim_{z\to 1}\frac{d^2}{dz^2}\left[z^{n+1}+z^n\right]$ $= \frac{1}{2}\lim_{z\to 1}\frac{d}{dz}\left[(n+1)z^n + nz^{n-1}\right]$ $= \frac{1}{2}\lim_{z\to 1}\left[(n+1)nz^{n-1} + n(n-1)z^{n-2}\right]$ $= \frac{1}{2}\lim_{z\to 1}\left[(n^2+n)(1)^{n-1} + (n^2-n)1^{n-2}\right]$ $= \frac{1}{2}\left[n^2+n+n^2-n\right]$ $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = \frac{1}{2}\left[2n^2\right]$ $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = n^2$

	$\boxed{f(n) = \text{sum of residues of } F(z)z^{n-1} = n^2}$
3.	**Find the inverse Z-transform of the function $\frac{z}{z^2+7z+10}$ by the method of residues.** **Solution:** $Z^{-1}\left[\frac{z}{z^2+7z+10}\right] = ?$ $F(z) = \frac{z}{z^2+7z+10} = \frac{z}{(z+2)(z+5)}$ $F(z)z^{n-1} = \frac{zz^{n-1}}{(z+2)(z+5)}$ $F(z)z^{n-1} = \frac{z^n}{(z+2)(z+5)} \quad ------(1)$ Here *z=-2* and *z=-5* are pole of order 1 1) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$ $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-2} = \lim_{z\to -2} \cancel{(z+2)} \frac{z^n}{\cancel{(z+2)}(z+5)}$ $= \frac{(-2)^n}{(-2+5)} = \frac{(-2)^n}{3}$ $\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-2} = \frac{(-2)^n}{3}}$ 2) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-5} = \lim_{z\to -5} \cancel{(z+5)} \frac{z^n}{(z+2)\cancel{(z+5)}}$ $= \frac{(-5)^n}{(-5+2)} = \frac{(-5)^n}{-3}$ $\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-5} = \frac{-(-5)^n}{3}}$ $f(n) = \text{sum of residues of } F(z)z^{n-1}$ $\boxed{f(n) = \frac{(-2)^n}{3} - \frac{(-5)^n}{3} = \frac{1}{3}\left[(-2)^n - (-5)^n\right]}$
4.	**Find $Z^{-1}\left[\frac{z^{-2}}{\left(1+z^{-1}\right)^2\left(1-z^{-1}\right)}\right]$ by using residue method.** **Solution:** $F(z) = \frac{z^{-2}}{\left(1+z^{-1}\right)^2\left(1-z^{-1}\right)} = \frac{1}{z^2\left(\frac{z+1}{z}\right)^2\left(\frac{z-1}{z}\right)}$ $F(z) = \frac{z}{\left(z+1\right)^2\left(z-1\right)}$ $F(z)z^{n-1} = \frac{zz^{n-1}}{\left(z+1\right)^2\left(z-1\right)}$

$$F(z)z^{n-1} = \frac{z^n}{(z+1)^2(z-1)} \quad -----(1)$$

Here $z=-1$ is pole of order 2, and $z=1$ is pole of order 1

1) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m F(z)z^{n-1}$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-1} = \frac{1}{(2-1)!}\lim_{z\to -1}\frac{d^2}{dz^2}\cancel{(z+1)^2}\frac{z^n}{\cancel{(z+1)^2}(z-1)}$$

$$= \frac{1}{1!}\lim_{z\to -1}\frac{d}{dz}\left[\frac{z^n}{z-1}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-1)nz^{n-1} - z^n(1-0)}{(z-1)^2}\right]$$

$$= \frac{(-1-1)n(-1)^{n-1} - (-1)^n}{(-1-1)^2} = \frac{-2n(-1)^{n-1} - (-1)^n}{4} = \frac{(-1)^n}{4}\left[2n-1\right]$$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-1} = \frac{(-1)^n}{4}\left[2n-1\right]$$

2) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = \lim_{z\to 1}\cancel{(z-1)}\frac{z^n}{(z+1)^2\cancel{(z-1)}}$$

$$= \lim_{z\to 1}\frac{z^n}{(z+1)^2} = \frac{1^n}{(1+1)^2} = \frac{1}{2}$$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = \frac{1}{2}$$

$f(n) = \text{sum of residues of } F(z)z^{n-1}$

$$\boxed{f(n) = \frac{(-1)^n}{4}\left[2n-1\right] + \frac{1}{2}}$$

5. **Using complex residue theorem evaluate** $Z^{-1}\left[\frac{9z^3}{(3z-1)^2(z-2)}\right]$.

Solution:

$$Z^{-1}\left[\frac{9z^3}{(3z-1)^2(z-2)}\right] = Z^{-1}\left[\frac{9z^3}{9(z-\frac{1}{3})^2(z-2)}\right] = Z^{-1}\left[\frac{z^3}{(z-\frac{1}{3})^2(z-2)}\right]$$

$$F(z) = \frac{z^3}{(z-\frac{1}{3})(z-2)}$$

$$F(z)z^{n-1} = \frac{z^3 z^{n-1}}{(z-\frac{1}{3})^2(z-2)}$$

$$F(z)z^{n-1} = \frac{z^{n+2}}{(z-\frac{1}{3})^2(z-2)}$$

Here $z=\frac{1}{3}$ are pole of order 2 and $z=2$ is simple pole.

1) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m F(z)z^{n-1}$ here $m=2$

$$\text{Res}\left[F(z)z^{n-1}\right]_{z=\frac{1}{3}} = \lim_{z\to\frac{1}{3}} \frac{d}{dz}\left[(z-\tfrac{1}{3})^2 \frac{z^{n+2}}{(z-\tfrac{1}{3})^2 (z-2)} \right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{d}{dz}\left[\frac{z^{n+2}}{z-2} \right]$$

$$= \lim_{z\to\frac{1}{3}} \left[\frac{(z-2)(n+2)z^{n+1} - z^{n+2}(1)}{(z-2)^2} \right]$$

$$= \lim_{z\to\frac{1}{3}} \left\{ \frac{z^{n+1}\left[(z-2)(n+2)-z\right]}{(z-2)^2} \right\}$$

$$= \left[\frac{\left(\frac{1}{3}\right)^{n+1}\left[\left(\frac{1}{3}-2\right)(n+2)-\frac{1}{3}\right]}{\left(\frac{1}{3}-2\right)^2} \right]$$

$$\text{Res}\left[F(z)z^{n-1}\right]_{z=\frac{1}{3}} = \frac{\left(\frac{1}{3}\right)^{n+1}\left[\left(\frac{-5(n+2)}{3}\right)-\frac{1}{3}\right]}{\left(\frac{-5}{3}\right)^2} = \frac{\left(\frac{1}{3}\right)^{n+1}\left(\frac{-5n-10-1}{3}\right)}{\frac{25}{9}}$$

$$= \frac{9}{25}\left(\frac{1}{3}\right)^n\left(\frac{1}{3}\right)\left(\frac{-5n-11}{3}\right) = \frac{-1}{25}\left(\frac{1}{3}\right)^n (5n+11)$$

$$\boxed{\text{Res}\left[F(z)z^{n-1}\right]_{z=\frac{1}{3}} = \frac{-1}{25}\left(\frac{1}{3}\right)^n (5n+11)}$$

2) $$\text{Res}\left[F(z)z^{n-1}\right]_{z=2} = \lim_{z\to 2}\left((z-2) \frac{z^{n+2}}{(z-2)(z-\frac{1}{3})^2} \right)$$

$$\text{Res}\left[F(z)z^{n-1}\right]_{z=2} = \frac{2^{n+2}}{\left(2-\frac{1}{3}\right)^2} = \frac{9}{25}2^{n+2}$$

$$\boxed{\text{Res}\left[F(z)z^{n-1}\right]_{z=2} = \frac{9}{25}2^{n+2}}$$

$f(n) = \text{sum of residues of } F(z)z^{n-1}$

$$\boxed{f(n) = f(n) = \frac{9}{25}2^{n+2} + \frac{-1}{25}\left(\frac{1}{3}\right)^n (5n+11)}$$

Finding Inverse Z-transform by Convolution theorem:

Convolution of two sequences:

If $\{f(n)\}$ and $\{g(n)\}$ are any two sequences then its convolution is defined by

$$f(n)*g(n) = \sum_{k=0}^{n} f(k)g(n-k)$$

Convolution Theorem:

If $Z\left[f(n)\right] = F(z)$ and $Z\left[g(n)\right] = G(z)$ then $Z\left[f(n)*g(n)\right] = Z\left[f(n)\right]\cdot Z\left[g(n)\right] = F(z)\cdot G(z)$

Note:

1) $Z\left[f(n)*g(n)\right] = F(z)\cdot G(z)$

$f(n)*g(n) = Z^{-1}\left[F(z)\cdot G(z)\right]$

$Z^{-1}\left[F(z)\right]*Z^{-1}\left[G(z)\right] = Z^{-1}\left[F(z)\cdot G(z)\right] \quad \because Z^{-1}\left[F(z)\right] = f(n) \,\&\, Z^{-1}\left[G(z)\right] = g(n)$

$\boxed{Z^{-1}\left[F(z)\cdot G(z)\right]=Z^{-1}\left[F(z)\right]*Z^{-1}\left[G(z)\right]}$ 2) $1+a+a^2+a^3+...+a^n=\frac{a^{n+1}-1}{a-1}$	

1. **Find inverse Z-transform of $\frac{z^2}{(z-a)^2}$ by using convolution theorem.**

Solution:

Given $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]=?$

By convolution theorem

$$Z^{-1}\left[F(z)\cdot G(z)\right]=Z^{-1}\left[F(z)\right]*Z^{-1}\left[G(z)\right]$$

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]=Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-a}\right]$$

$$=Z^{-1}\left[\frac{z}{z-a}\right]*Z^{-1}\left[\frac{z}{z-a}\right]$$

$$=a^n*a^n$$

$$=\sum_{k=0}^{n}a^k a^{n-k} \qquad \because f(n)*g(n)=\sum_{k=0}^{n}f(k)g(n-k)$$

$$=\sum_{k=0}^{n}\cancel{a^k}\,\cancel{a^{-k}}\,a^n$$

$$=a^n\sum_{k=0}^{n}1$$

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]=a^n(n+1)\cdot 1=(n+1)a^n$$

$$\boxed{Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]=(n+1)a^n}$$

2. **By using convolution theorem, show that the inverse Z-transform of $\frac{z^2}{(z+a)(z+b)}$ is**

$\frac{(-1)^n}{b-a}\left[b^{n+1}-a^{n+1}\right]$

Solution:

Given $Z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right]=?$

By convolution theorem

$$Z^{-1}\left[F(z)\cdot G(z)\right]=Z^{-1}\left[F(z)\right]*Z^{-1}\left[G(z)\right]$$

$$Z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right]=Z^{-1}\left[\frac{z}{z+a}\cdot\frac{z}{z+b}\right]$$

$$=Z^{-1}\left[\frac{z}{z+a}\right]*Z^{-1}\left[\frac{z}{z+b}\right]$$

$$=(-a)^n*(-b)^n$$

$$=\sum_{k=0}^{n}(-a)^k(-b)^{n-k} \qquad \because f(n)*g(n)=\sum_{k=0}^{n}f(k)g(n-k)$$

$$= (-1)^n \sum_{k=0}^{n} a^k b^{-k} b^n$$

$$= (-1)^n b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$

$$= (-1)^n b^n \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^3 + \ldots + \left(\frac{a}{b}\right)^n\right]$$

$$= (-1)^n b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1}\right] = b^n \left[\frac{\frac{a^{n+1}}{b^{n+1}} - 1}{\frac{a}{b} - 1}\right] = b^n \left[\frac{\frac{a^{n+1} - b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}}\right]$$

$$= (-1)^n b^n \left[\frac{a^{n+1} - b^{n+1}}{b^{n+1}} \times \frac{b}{a-b}\right] = (-1)^n \cancel{b^n} \left[\frac{a^{n+1} - b^{n+1}}{\cancel{b^n}\,\cancel{b}} \times \frac{\cancel{b}}{a-b}\right]$$

$$= (-1)^n \left[\frac{a^{n+1} - b^{n+1}}{a-b}\right]$$

$$\boxed{Z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right] = \frac{(-1)^n}{b-a}\left[b^{n+1} - a^{n+1}\right]}$$

3. **Find** $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$ **using convolution theorem.**

Solution:

Given $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = ?$

By convolution theorem

$$Z^{-1}\left[F(z)\cdot G(z)\right] = Z^{-1}\left[F(z)\right] * Z^{-1}\left[G(z)\right]$$

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$$

$$= (a)^n * (b)^n$$

$$= \sum_{k=0}^{n} (a)^k (b)^{n-k} \qquad \because f(n) * g(n) = \sum_{k=0}^{n} f(k) g(n-k)$$

$$= \sum_{k=0}^{n} a^k b^{-k} b^n$$

$$= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$

$$= b^n \left[1 + \left(\frac{a}{b}\right) + \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^3 + \ldots + \left(\frac{a}{b}\right)^n\right]$$

	$$=b^n\left[\frac{\left(\frac{a}{b}\right)^{n+1}-1}{\frac{a}{b}-1}\right]=b^n\left[\frac{\frac{a^{n+1}}{b^{n+1}}-1}{\frac{a}{b}-1}\right]=b^n\left[\frac{\frac{a^{n+1}-b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}}\right]$$ $$=b^n\left[\frac{a^{n+1}-b^{n+1}}{b^{n+1}}\times\frac{b}{a-b}\right]=(-1)^n\,b^n\left[\frac{a^{n+1}-b^{n+1}}{b^n\,b}\times\frac{b}{a-b}\right]$$ $$=\frac{a^{n+1}-b^{n+1}}{a-b}$$ $$\boxed{Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]=\frac{a^{n+1}-b^{n+1}}{a-b}}$$
4.	**Using convolution theorem, find** $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$ Solution: Given $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]=?$ By convolution theorem $Z^{-1}[F(z)\cdot G(z)]=Z^{-1}[F(z)]*Z^{-1}[G(z)]$ $$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]=Z^{-1}\left[\frac{8z^2}{2\left(z-\frac{1}{2}\right)4\left(z+\frac{1}{4}\right)}\right]=Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\cdot\frac{z}{\left(z-\frac{1}{4}\right)}\right]$$ $$=Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\right]*Z^{-1}\left[\frac{z}{\left(z-\frac{1}{4}\right)}\right]$$ $$=\left(\frac{1}{2}\right)^n*\left(\frac{1}{4}\right)^n$$ $$=\sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{1}{4}\right)^{n-k}\qquad\because f(n)*g(n)=\sum_{k=0}^{n}f(k)g(n-k)$$ $$=\sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{1}{4}\right)^n\left(\frac{1}{4}\right)^{-k}$$ $$=\left(\frac{1}{4}\right)^n\sum_{k=0}^{n}\left(\frac{1}{2}\right)^k(4)^k=\left(\frac{1}{4}\right)^n\sum_{k=0}^{n}\left(\frac{4}{2}\right)^k=\left(\frac{1}{4}\right)^n\sum_{k=0}^{n}(2)^k$$ $$=\left(\frac{1}{4}\right)^n\left[1+2+2^2+2^3+\ldots+2^n\right]$$ $$=\left(\frac{1}{4}\right)^n\left[\frac{2^{n+1}-1}{2-1}\right]\qquad\because 1+a+a^2+a^3+\ldots+a^n=\frac{a^{n+1}-1}{a-1}$$ $$\boxed{Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]=\left(\frac{1}{4}\right)^n\left[2^{n+1}-1\right]}$$
5.	**Using convolution theorem find** $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$

	Solution: Given $Z^{-1}\left[\dfrac{z^2}{(z-1)(z-3)}\right]=?$ By convolution theorem $Z^{-1}\left[F(z)\cdot G(z)\right]=Z^{-1}\left[F(z)\right]*Z^{-1}\left[G(z)\right]$ $Z^{-1}\left[\dfrac{z^2}{(z-1)(z-3)}\right]=Z^{-1}\left[\dfrac{z}{z-1}\cdot\dfrac{z}{z-3}\right]$ $=Z^{-1}\left[\dfrac{z}{z-1}\right]*Z^{-1}\left[\dfrac{z}{z-3}\right]$ $=(1)^n*(3)^n$ $=\sum_{k=0}^{n}(1)^k(3)^{n-k} \qquad \because f(n)*g(n)=\sum_{k=0}^{n}f(k)g(n-k)$ $=\sum_{k=0}^{n}1^k3^{-k}3^n$ $=3^n\sum_{k=0}^{n}\left(\dfrac{1}{3}\right)^k$ $=3^n\left[1+\left(\dfrac{1}{3}\right)+\left(\dfrac{1}{3}\right)^2+\left(\dfrac{1}{3}\right)^3+\ldots+\left(\dfrac{1}{3}\right)^n\right]$ $=3^n\left[\dfrac{\left(\dfrac{1}{3}\right)^{n+1}-1}{\dfrac{1}{3}-1}\right]=3^n\left[\dfrac{\dfrac{1^{n+1}}{3^{n+1}}-1}{\dfrac{1}{3}-1}\right]=3^n\left[\dfrac{\dfrac{1^{n+1}-3^{n+1}}{3^{n+1}}}{\dfrac{1-3}{3}}\right]$ $=3^n\left[\dfrac{1-3^{n+1}}{3^{n+1}}\times\dfrac{3}{1-3}\right]=\cancel{3^n}\left[\dfrac{3^{n+1}-3^{n+1}}{\cancel{3^n}\,\cancel{3}}\times\dfrac{\cancel{3}}{-2}\right]$ $=\dfrac{-1}{2}\left[1-3^{n+1}\right]$ $\boxed{Z^{-1}\left[\dfrac{z^2}{(z-1)(z-3)}\right]==\dfrac{-1}{2}\left[1-3^{n+1}\right]}$
Formation of Difference Equation:	
1.	**Derive the difference equation from** $y_n=(A+Bn)2^n$ Solution: Given $y_n=(A+Bn)2^n$ $y_n=A2^n+Bn2^n \quad ------(1)$ Replace n by $n+1$ in (1) $y_{n+1}=A2^{n+1}+B(n+1)2^{n+1}$ $y_{n+1}=2A2^n+2(n+1)B2^n \quad -----(2)$ Replace n by $n+2$ in (1) $y_{n+2}=A2^{n+2}+(n+2)B2^{n+2}$ $y_{n+2}=4A2^n+4(n+2)B2^n \quad ----(3)$ From (1), (2) and (3)

$$\begin{vmatrix} y_n & 1 & n \\ y_{n+1} & 2 & 2(n+1) \\ y_{n+2} & 4 & 4(n+2) \end{vmatrix} = 0$$

$$y_n\left[8(n+2)-8(n+1)\right]-1\left[4(n+2)y_{n+1}-2(n+1)y_{n+2}\right]+n\left[4y_{n+1}-2y_{n+2}\right]=0$$

$$y_n\left[(8n+16-8n-8\right]-1\left[(4n+8)y_{n+1}+(-2n-2)y_{n+2}\right]+4ny_{n+1}-2ny_{n+2}=0$$

$$8y_n \cancel{-4ny_{n+1}} -8y_{n+1} \cancel{+2ny_{n+2}} +2y_{n+2} \cancel{+4ny_{n+1}} \cancel{-2ny_{n+2}} = 0$$

$$2y_{n+2}-8y_{n+1}+8y_n=0$$

$$\boxed{y_{n+2}-4y_{n+1}+4y_n=0}$$

2. **Derive the difference equation from** $u_n = a + b3^n$

Solution: $u_n = a+b3^n \quad ----(1)$

Replace n by $n+1$ in (1)

$$u_{n+1} = a + b3^{n+1}$$

$$u_{n+1} = a+3b3^n \quad ----(2)$$

Replace n by $n+2$ in (1)

$$u_{n+2} = a+b3^{n+2}$$

$$u_{n+2} = a+9b3^n \quad ----(3)$$

From (1), (2) and (3)

$$\begin{vmatrix} u_n & 1 & 1 \\ u_{n+1} & 1 & 3 \\ u_{n+2} & 1 & 9 \end{vmatrix} = 0$$

$$u_n(9-3)-1(3u_{n+2}-9u_{n+1})+1(u_{n+1}-u_{n+2})=0$$

$$6u_n-3u_{n+2}+9u_{n+1})+u_{n+1}-u_{n+2}=0$$

$$-4u_{n+2}+10u_{n+1}+6u_n=0$$

$$\div(-2) \quad \Rightarrow \boxed{2u_{n+2}-5u_{n+1}-3u_n=0}$$

3. **Form the difference equation** $y_n = \cos\left(\frac{n\pi}{2}\right)$

Solution:

Given $y_n = \cos\left(\frac{n\pi}{2}\right) \quad ----(1)$

Replace n by $n+1$ in (1)

$$y_{n+1} = \cos\left(\frac{(n+1)\pi}{2}\right) = \cos\left(\frac{\pi}{2}+\frac{n\pi}{2}\right) = -\sin\left(\frac{n\pi}{2}\right) \quad ----(2)$$

Replace n by $n+2$ in (1)

$$y_{n+2} = \cos\left(\frac{(n+2)\pi}{2}\right) = \cos\left(\frac{2\pi}{2}+\frac{n\pi}{2}\right)$$

$$y_{n+2} = \cos\left(\pi+\frac{n\pi}{2}\right) = -\cos\left(\frac{n\pi}{2}\right)$$

$$y_{n+2} = -y_n \qquad \text{from}\,(1)$$

$$\Rightarrow \boxed{y_{n+2}+y_n=0}$$

Solutions of difference equation using Z-Transforms.

1. $Z\left[y_n\right] = Z\left[y(n)\right] = y(z)$

2. $Z\left[y_{n+1}\right]=Z\left[y(n+1)\right]=zy(z)-zy(0)$ 3. $Z\left[y_{n+2}\right]=Z\left[y(n+2)\right]=z^2y(z)-z^2y(0)-zy(1)$ 4. $Z\left[y_{n+3}\right]=Z\left[y(n+3)\right]=z^3y(z)-z^3y(0)-z^2y(1)-zy(2)$		

1. **Solve using Z-transforms technique the difference equation** $y_{n+2}+4y_{n+1}+3y_n=3^n$ **with** $y_0=0, y_1=1$**.**

Solution:

$$y_{n+2}+4y_{n+1}+3y_n=3^n \ .$$

Taking Z-transform on both sides

$$Z\left[y_{n+2}\right]+4Z\left[y_{n+1}\right]+3Z\left[y_n\right]=Z\left[3^n\right]$$

$$\left[z^2y(z)-z^2y(0)-zy(1)\right]+4\left[zy(z)-zy(0)\right]+3y(z)=\frac{z}{z-3}$$

Given $y_0=y(0)=0, y_1=y(1)=1$

$$z^2y(z)-z+4zy(z)+3y(z)=\frac{z}{z-3}$$

$$(z^2+4z+3)y(z)=\frac{z}{z-3}+z$$

$$(z^2+4z+3)y(z)=\frac{z+z^2-3z}{z-3}$$

$$y(z)=\frac{z^2-2z}{(z-3)(z^2+4z+3)}$$

$$y(z)=\frac{z(z-2)}{(z-3)(z+1)(z+3)}$$

By Partial Fraction,

$$\frac{y(z)}{z}=\frac{(z-2)}{(z-3)(z+1)(z+3)} \quad ------(1)$$

Now $\frac{(z-2)}{(z-3)(z+1)(z+3)}=\frac{A}{(z-3)}+\frac{B}{(z+1)}+\frac{C}{(z+3)}$

$$z-2=A(z+1)(z+3)+B(z-3)(z+3)+C(z+1)(z-3)$$

Put z=3 $\Rightarrow 1=24A \Rightarrow A=\frac{1}{24}$

Put z= -1 $\Rightarrow -3=-8B \Rightarrow B=\frac{3}{8}$

Put z = -3 $\Rightarrow -5=12C \Rightarrow C=\frac{-5}{12}$

$$\frac{(z-2)}{(z-3)(z+1)(z+3)}=\frac{1/24}{(z-3)}+\frac{3/8}{(z+1)}+\frac{-5/12}{(z+3)}$$

$$(1)\Rightarrow \frac{y(z)}{z}=\frac{1/24}{(z-3)}+\frac{3/8}{(z+1)}+\frac{-5/12}{(z+3)}$$

$$y(z)=\frac{1}{24}\frac{z}{(z-3)}+\frac{3}{8}\frac{z}{(z+1)}-\frac{5}{12}\frac{z}{(z+3)}$$

Taking Z^{-1} on both sides

$$Z^{-1}\left[y(z)\right]=\frac{1}{24}Z^{-1}\left[\frac{z}{z-3}\right]+\frac{3}{8}Z^{-1}\left[\frac{z}{z+1}\right]-\frac{5}{12}Z^{-1}\left[\frac{z}{z+3}\right]$$

	$y(\mathrm{n})=\frac{1}{24}(3)^n+\frac{3}{8}(-1)^n-\frac{5}{12}(-3)^n$ $\quad\because Z^{-1}\left[\frac{z}{z-a}\right]=a^n$
2.	**Solve $y_{n+2} - 3y_{n+1} - 10y_n = 0$, given $y_0 = 1$, $y_1 = 0$.** **Solution:** $y_{n+2}-3y_{n+1}-10y_n=0$. Taking Z-transform on both sides $Z[y_{n+2}]-3Z[y_{n+1}]-10Z[y_n]=Z[0]$ $\left[z^2y(z)-z^2y(0)-zy(1)\right]-3[zy(z)-zy(0)]-10y(z)=0$ Given $y_0=y(0)=1, y_1=y(1)=0$ $z^2y(z)-z^2-3zy(z)+3z-10y(z)=0$ $\left(z^2-3z-10\right)y(z)=z^2-3z$ $y(z)=\frac{z^2-3z}{\left(z^2-3z-10\right)}$ $y(z)=\frac{z(z-3)}{(z+2)(z-5)}$ By Partial Fraction, $\frac{y(z)}{z}=\frac{(z-3)}{(z+2)(z-5)} \quad -----(1)$ Now $\frac{(z-3)}{(z+2)(z-5)}=\frac{A}{(z+2)}+\frac{B}{(z-5)}$ $z-3=A(z-5)+B(z+2)$ <u>Put z= - 2</u> $\Rightarrow -5=-7A \Rightarrow A=\frac{5}{7}$ <u>Put z= 5</u> $\Rightarrow 2=7B \Rightarrow B=\frac{2}{7}$ $\frac{(z-3)}{(z+2)(z-5)}=\frac{\frac{5}{7}}{(z+2)}+\frac{\frac{2}{7}}{(z-5)}$ $(1)\Rightarrow \frac{y(z)}{z}=\frac{\frac{5}{7}}{z+2}+\frac{\frac{2}{7}}{z-5}$ $y(z)=\frac{5}{7}\frac{z}{z+2}+\frac{2}{7}\frac{z}{z-5}$ Taking Z^{-1} on both sides $Z^{-1}[y(z)]=\frac{5}{7}Z^{-1}\left[\frac{z}{z+2}\right]+\frac{2}{7}Z^{-1}\left[\frac{z}{z-5}\right]$ $y(\mathrm{n})=\frac{5}{7}(-2)^n-\frac{2}{7}5^n$ $\quad\because Z^{-1}\left[\frac{z}{z-a}\right]=a^n$
3.	**Solve the equation** $y(n+3)-3y(n+1)+2y(n)=0$ **given that** $y(0)=4$**,** $y(1)=0$ **and** $y(2)=8$**.** **Solution:** $Z[y(n+3)]-3Z[y(n+1)]+2Z[y(n)]=Z[0]$ $\left[z^3y(z)-z^3y(0)-z^2y(1)-zy(2)\right]-3[zy(z)-zy(0)]+2y(z)=0$ Given that $y(0)=4$, $y(1)=0$

$$z^3 y(z) - 4z^3 - 8z - 3zy(z) + 12z + 2y(z) = 0$$

$$\left[z^3 - 3z + 2\right] y(z) = 4z^3 - 4z$$

$$y(z) = \frac{4z^3 - 4z}{z^3 - 3z + 2}$$

$$y(z) = \frac{4z(z^2 - 1)}{(z-1)^2 (z+2)}$$

$$y(z) = \frac{4z\,\cancel{(z-1)}\,(z+1)}{(z-1)^{\cancel{2}}\,(z+2)} \qquad \because a^2 - b^2 = (a+b)(a-b)$$

$$y(z) = \frac{4z(z+1)}{(z-1)(z+2)}$$

By Partial Fraction

$$\frac{y(z)}{z} = \frac{4(z+1)}{(z-1)(z+2)} \quad -----(1)$$

$$\frac{4(z+1)}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2}$$

$$4(z+1) = A(z+2) + B(z-1)$$

Put $z = 1 \Rightarrow 8 = 3A \Rightarrow A = \frac{8}{3}$

Put $z = -2 \Rightarrow -4 = -3B \Rightarrow B = \frac{4}{3}$

$$\frac{y(z)}{z} = \frac{8/3}{z-1} + \frac{4/3}{z+2}$$

$$Z^{-1}\left[y(z)\right] = \frac{8}{3} Z^{-1}\left[\frac{z}{z-1}\right] + \frac{4}{3} Z^{-1}\left[\frac{z}{z+2}\right]$$

$$\boxed{y(n) = \frac{8}{3} + \frac{4}{3}(-2)^n} \qquad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

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4. **Using Z-transform solve** $y(n) + 3y(n-1) - 4y(n-2) = 0, n \geq 2$ **given that** $y(0) = 3$ and $y(1) = -2$

Solution:

Given $y(n) + 3y(n-1) - 4y(n-2) = 0, n \geq 2$

Replace n by $n+2$, we get

$$y(n+2) + 3y(n+1) - 4y(n) = 0$$

Taking Z transforms on both sides

$$Z\left[y(n+2)\right] + 3Z\left[y(n+1)\right] - 4Z\left[y(n)\right] = Z\left[0\right]$$

$$\left[z^2 y(z) - z^2 y(0) - zy(1)\right] + 3\left[zy(z) - zy(0)\right] - 4y(z) = 0$$

Given that $y(0) = 3$ and $y(1) = -2$

$$\left[z^2 y(z) - 3z^2 + 2z\right] + 3\left[zy(z) - 3z\right] - 4y(z) = 0$$

$$\left[z^2 + 3z - 4\right] y(z) - 3z^2 + 2z - 9z = 0$$

$$\left[z^2 + 3z - 4\right] y(z) = 3z^2 + 7z$$

$$y(z) = \frac{3z^2 + 7z}{z^2 + 3z - 4}$$

By Partial Fraction

$$\frac{y(z)}{z} = \frac{3z + 7}{z^2 + 3z - 4} = \frac{3z + 7}{(z+4)(z-1)}$$

	Now, $\frac{3z+7}{(z+4)(z-1)} = \frac{A}{z+4} + \frac{B}{z-1}$ $3z+7 = A(z-1)+B(z+4)$ Put $z=1 \Rightarrow 10=5B \Rightarrow B=2$ Put $z=-4 \Rightarrow -5=-5A \Rightarrow A=1$ $\frac{y(z)}{z} = \frac{1}{z+4} + \frac{2}{z-1}$ $y(z) = \frac{z}{z+4} + 2\frac{z}{z-1}$ $Z^{-1}\left[y(z)\right] = Z^{-1}\left[\frac{z}{z+4}\right] + 2Z^{-1}\left[\frac{z}{z-1}\right]$ $\boxed{y(n) = (-4)^n + 2(1)^n = 2+(-4)^n}$ $\qquad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$
5.	**Solve using Z-transforms technique the difference equation** $u_{n+2}+6u_{n+1}+9u_n = 2^n$ **with** $u_0 = u_1 = 0$. Solution: $u_{n+2}+6u_{n+1}+9u_n = 2^n$ Assume u=y $y_{n+2}+6y_{n+1}+9y_n = 2^n \quad ; y_0 = y_1 = 0$ Taking Z-transform on both sides $Z\left[y_{n+2}\right]+6Z\left[y_{n+1}\right]+9Z\left[y_n\right] = Z\left[2^n\right]$ $\left[z^2y(z)-z^2y(0)-zy(1)\right]+6\left[zy(z)-zy(0)\right]+9y(z) = \frac{z}{z-2}$ Given $y_0 = y(0) = 0$; $y_1 = y(1) = 0$ $z^2y(z)+6zy(z)+9y(z) = \frac{z}{z-2}$ $(z^2+6z+9)y(z) = \frac{z}{z-2}$ $y(z) = \frac{z}{(z-2)(z^2+6z+9)}$ $y(z) = \frac{z}{(z-2)(z+3)^2}$ By Partial Fraction, $\frac{y(z)}{z} = \frac{1}{(z-2)(z+3)^2} \quad -----(1)$ Now $\frac{1}{(z-2)(z+3)^2} = \frac{A}{(z-2)} + \frac{B}{(z+3)} + \frac{C}{(z+3)^2}$ $1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$ Put $z=2 \Rightarrow 1=25A \Rightarrow A = \frac{1}{25}$ Put $z=-3 \Rightarrow 1=-5C \Rightarrow C = \frac{-1}{5}$ Equating co-efft. of z^2 on both sides $\Rightarrow A+B=0 \Rightarrow B=-A \Rightarrow B = -\frac{1}{25}$

	$$\frac{y(z)}{z}=\frac{\frac{1}{25}}{(z-2)}+\frac{\frac{-1}{25}}{(z+3)}+\frac{\frac{-1}{5}}{(z+3)^2}$$ Taking Z^{-1} on both sides $$Z^{-1}[y(z)]=\frac{1}{25}Z^{-1}\left[\frac{z}{z-2}\right]-\frac{1}{25}Z^{-1}\left[\frac{z}{z+3}\right]-\frac{1}{5}Z^{-1}\left[\frac{z}{(z+3)^2}\right]$$ $$\boxed{y(\mathrm{n})=\frac{1}{25}(2)^n-\frac{1}{25}(-3)^n-\frac{1}{5}n(-3)^{n-1}}\quad \because Z^{-1}\left[\frac{z}{(z-a)^2}\right]=na^{n-1}\ \&\ Z^{-1}\left[\frac{z}{z-a}\right]=a^n$$ $$\boxed{u(\mathrm{n})=\frac{1}{25}(2)^n-\frac{1}{25}(-3)^n-\frac{1}{5}n(-3)^{n-1}}\quad \because u=y$$
6.	**Using Z-transform method solve** $y(k+2)+y(k)=2$ **given that** $y_0=y_1=0$. **Solution:** Given $y(k+2)+y(k)=2$; $y_0=y_1=0$. Assume $k=n$ $y(n+2)+y(n)=2$ Taking Z-transform on both sides $Z[y(n+2)]+Z[y(n)]=2Z[1]$ $\left[z^2y(z)-z^2y(0)-zy(1)\right]+y(z)=2\frac{z}{z-1}$ Given that $y_0=y_1=0$. $(z^2+1)y(z)=\frac{2z}{z-1}$ $y(z)=\frac{2z}{(z-1)(z^2+1)}$ $\frac{y(z)}{z}=\frac{2}{(z-1)(z^2+1)}\ -----(1)$ By partial fraction Now, $\frac{2}{(z-1)(z^2+1)}=\frac{A}{z-1}+\frac{B}{z^2+1}+\frac{Cz}{z^2+1}$ $2=A(z^2+1)+B(z-1)+Cz(z-1)$ Put $z=1\ \Rightarrow\ 2=2A\Rightarrow A=1$ Put $z=0\ \Rightarrow\ 2=A-B\Rightarrow B=A-2\Rightarrow B=-1$ Equating co-efft. of z^2 on both sides $\Rightarrow\ 0=A+C\Rightarrow C=-A\Rightarrow C=-1$ $(1)\Rightarrow\frac{y(z)}{z}=\frac{1}{z-1}+\frac{-1}{z^2+1}+\frac{-z}{z^2+1}$ $y(z)=\frac{z}{z-1}-\frac{z}{z^2+1}-\frac{z^2}{z^2+1}$ Taking Z^{-1} on both sides $Z^{-1}[y(z)]=Z^{-1}\left[\frac{z}{z-1}\right]-Z^{-1}\left[\frac{z}{z^2+1}\right]-Z^{-1}\left[\frac{z^2}{z^2+1}\right]$ $y(n)=(1)^n-1^n\sin\frac{n\pi}{2}-1^n\cos\frac{n\pi}{2}$ $\boxed{y(n)=1-\sin\frac{n\pi}{2}-\cos\frac{n\pi}{2}}$

	$y(k)=1-\sin\frac{k\pi}{2}-\cos\frac{k\pi}{2}$ $\because Z^{-1}\left[\frac{z}{z^2+a^2}\right]=a^n\sin\frac{n\pi}{2}$ $\& Z^{-1}\left[\frac{z^2}{z^2+a^2}\right]=a^n\cos\frac{n\pi}{2}$ here $a=1$
Problems based on Z-Transforms:	

1. **Find** $Z[\cos n\theta]$ **,** $Z[\sin n\theta]$ **and hence find i)** $Z\left[\cos\frac{n\pi}{2}\right]$, ii) $Z\left[\sin\frac{n\pi}{2}\right]$

iii) $Z\left[r^n\cos n\theta\right]$ iv) $Z\left[r^n\sin n\theta\right]$

Solution:

We know that $e^{in\theta}=\cos n\theta+\sin n\theta$

$\cos n\theta=$real part of $e^{in\theta}$ & $\sin n\theta=$imaginary part of $e^{in\theta}$

and $Z\left[a^n\right]=\frac{z}{z-a}$

$$Z\left[e^{in\theta}\right]=Z\left[\left(e^{i\theta}\right)^n\right]=\frac{z}{z-e^{i\theta}}$$

$$=\frac{z}{z-(\cos\theta+i\sin\theta)}$$

$$=\frac{z}{(z-\cos\theta)-i\sin\theta}\times\frac{(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$Z\left[e^{in\theta}\right]=\frac{z(z-\cos\theta)+i\sin\theta}{(z-\cos\theta)^2-i^2\sin^2\theta}\qquad\because(a+b)(a-b)=a^2-b^2$$

$$Z\left[\cos n\theta+i\sin n\theta\right]=\frac{z(z-\cos\theta)+iz\sin\theta}{z^2-2z\cos\theta+\cos^2\theta+\sin^2\theta}\qquad\because i^2=-1$$

$$Z\left[\cos n\theta\right]+iZ\left[\sin n\theta\right]=\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}+i\frac{z\sin\theta}{z^2-2z\cos\theta+1}\qquad\because\cos^2\theta+\sin^2\theta=1$$

Equating co-efft. Of real and img parts on both sides

$$Z\left[\cos n\theta\right]=\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\quad;\ Z\left[\sin n\theta\right]=\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

Deduction:

We know that

$$Z\left[\cos n\theta\right]=\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

i) $Z\left[\cos\frac{n\pi}{2}\right]=Z\left[\cos n\theta\right]_{\theta\to\frac{\pi}{2}}=\dfrac{z\left(z-\cos\frac{\pi}{2}\right)}{z^2-2z\cos\frac{\pi}{2}+1}$

$$Z\left[\cos\frac{n\pi}{2}\right]=\frac{z^2}{z^2+1}\qquad\because\cos\frac{\pi}{2}=0$$

$$Z\left[\sin n\theta\right]=\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

ii) $Z\left[\sin\frac{n\pi}{2}\right]=Z\left[\sin n\theta\right]_{\theta\to\frac{\pi}{2}}=\dfrac{z\sin\frac{\pi}{2}}{z^2-2z\cos\frac{\pi}{2}+1}$

$$\therefore Z\left[\sin\frac{n\pi}{2}\right]=\frac{z}{z^2+1}\qquad\because\cos\frac{\pi}{2}=0\ \&\sin\frac{\pi}{2}=1$$

We know that

	$Z\left[a^n f(n)\right] = Z\left[f(n)\right]_{z\to\frac{z}{a}}$ iii) $Z\left[r^n \cos n\theta\right] = \left[Z\left[\cos n\theta\right]\right]_{z\to\frac{z}{r}}$ $=\left[\dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right]_{z\to\frac{z}{r}}$ $=\left[\dfrac{\frac{z}{r}\left(\frac{z}{r}-\cos\theta\right)}{\frac{z}{r^2}^2-\frac{2z}{r}\cos\theta+1}\right]$ $=\dfrac{\frac{z}{r}\left(\frac{z-r\cos\theta}{r}\right)}{\frac{z^2-2zr\cos\theta+r^2}{r^2}}$ $Z\left[r^n\cos n\theta\right]=\dfrac{z(z-r\cos\theta)}{z^2-2zr\cos\theta+r^2}$ iv) $Z\left[r^n \sin n\theta\right]=\left[Z\left\{\sin n\theta\right\}\right]_{z\to\frac{z}{r}} = \dfrac{\frac{z}{r}\sin\theta}{\frac{z^2}{r^2}-2\frac{z}{r}\cos\theta+r^2} = \dfrac{\frac{z}{r}\sin\theta}{\frac{z^2-2zr\cos\theta+r^2}{r^2}}$ $Z\left[r^n\sin n\theta\right]=\dfrac{zr\sin\theta}{z^2-2zr\cos\theta+r^2}$
2.	**Find the Z-transform of** $\dfrac{1}{n(n+1)}, \, for\, n\ge 1$ Solution $Z\left[\dfrac{1}{n(n+1)}\right]=?$ By partial Fraction: $\dfrac{1}{n(n+1)}=\dfrac{A}{n}+\dfrac{B}{n+1}$ $1=A(n+1)+Bn$ Put $n=-1;\quad 1=-B \Rightarrow B=-1$ Put $n=0;\quad A=1$ $\dfrac{1}{n(n+1)}=\dfrac{1}{n}-\dfrac{1}{n+1}$ $Z\left[\dfrac{1}{n(n+1)}\right]=Z\left[\dfrac{1}{n}\right]-Z\left[\dfrac{1}{n+1}\right] ------(1)$ Now, we know that $Z\left[f(n)\right]=\sum_{n=0}^{\infty} f(n)z^{-n}$ $Z\left[\dfrac{1}{n}\right]=\sum_{n=1}^{\infty}\dfrac{1}{n}\left(\dfrac{1}{z}\right)^n \quad \because n>0$ $=\dfrac{1}{z}+\dfrac{1}{2}\left(\dfrac{1}{z}\right)^2+\dfrac{1}{3}\left(\dfrac{1}{z}\right)^3+....$

	$= x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots.$ here $\frac{1}{z} = x$ $= -\log(1-x)$ $Z\left[\frac{1}{n}\right] = -log\left(1-\frac{1}{z}\right) = -\log\left(\frac{z-1}{z}\right) = \log\left(\frac{z}{z-1}\right)$ $Z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)$ $Z\left(\frac{1}{n+1}\right) = \sum_{n=0}^{\infty}\frac{1}{n+1}\left(\frac{1}{z}\right)^n$ $= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \ldots.$ $= z\left[\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + ..\right]$ $= z\left[-\log\left(1-\frac{1}{z}\right)\right] = -z\log\left(\frac{z-1}{z}\right)$ $Z\left(\frac{1}{n+1}\right) = z\log\left(\frac{z}{z-1}\right)$ $(1) \Rightarrow \quad Z\left[\frac{1}{n(n+1)}\right] = \log\left(\frac{z}{z-1}\right) + z\log\left(\frac{z}{z-1}\right)$ $\therefore Z\left[\frac{1}{n(n+1)}\right] = (z+1)\log\left(\frac{z}{z-1}\right)$
3.	**Find** $Z\left[n(n-1)(n-2)\right]$ **.** **Solution:** $Z\left[n(n-1)(n-2)\right] = Z\left[(n^2-n)(n-2)\right] = Z\left[n^3-2n^2-n^2+2n\right] = Z\left[n^3-3n^2+2n\right]$ $Z\left[n(n-1)(n-2)\right] = Z\left[n^3\right] - 3Z\left[n^2\right] + 2Z\left[n\right] \quad ----(1)$ We know that $Z\left[f(n)\right] = \sum_{n=0}^{\infty} f(n)z^{-n}$ $Z\left[n\right] = \sum_{n=0}^{\infty} n\left(\frac{1}{z}\right)^n$ $= 0 + 1\left(\frac{1}{z}\right)^1 + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + \ldots..$ $= x + 2x^2 + 3x^3 + \ldots..$ $= x(1+2x+3x^2+\ldots..) = x(1-x)^{-2} = \frac{1}{z}\left(1-\frac{1}{z}\right)^{-2}$ $= \frac{1}{z}\left(\frac{z-1}{z}\right)^{-2} = \frac{1}{z}\left(\frac{z}{z-1}\right)^2 = \frac{1}{z}\left(\frac{z^2}{(z-1)^2}\right)$ $Z\left[n\right] = \frac{z}{(z-1)^2}$ We know that $Z\left[nf(n)\right] = -z\frac{d}{dz}\left\{Z\left[f(n)\right]\right\}$

	$Z\left[n^2\right]=-z\frac{d}{dz}\{Z[n]\}$ $=-z\frac{d}{dz}\left\{\frac{z}{(z-1)^2}\right\}$ $=-z\left\{\frac{(z-1)^2(1)-z[2(z-1)]}{(z-1)^4}\right\}$ $=-z\left\{\frac{(z-1)(z-1-2z)}{(z-1)^4}\right\}$ $=-z\left\{\frac{-1-z}{(z-1)^3}\right\}$ $Z\left[n^2\right]=\frac{z+z^2}{(z-1)^3}$ $Z\left[n^3\right]=Z\left[n n^2\right]=-z\frac{d}{dz}\left\{Z\left[n^2\right]\right\}$ $=-z\frac{d}{dz}\left\{\frac{z+z^2}{(z-1)^3}\right\}$ $=-z\left[\frac{(z-1)^3(2z+1)-(z^2+z)3(z-1)^2(1-0)}{(z-1)^6}\right]$ $=-z\left[\frac{(z-1)^2\left[(z-1)(2z+1)-3(z^2+z)\right]}{(z-1)^6}\right]$ $=-z\left[\frac{2z^2-2z+z-1-3z^2-3z}{(z-1)^4}\right]$ $=-z\left[\frac{-z^2-4z-1}{(z-1)^4}\right]$ $Z\left[n^3\right]=\frac{z(z^2+4z+1)}{(z-1)^4}$ $(1)\Rightarrow\ Z\left[n(n-1)(n-2)\right]=\frac{z(z^2+4z+1)}{(z-1)^4}-3\frac{z+z^2}{(z-1)^3}+2\frac{z}{(z-1)^2}$
4.	**If** $U(z)=\frac{2z^2+5z+14}{(z-1)^4}$ **, evaluate** u_2 and u_3. **Solution:** **Given** $U(z)=F(z)=\frac{2z^2+5z+14}{(z-1)^4}$ We know that $u_0=f(0)=\lim_{z\to\infty}F(z)=\lim_{z\to\infty}\frac{2z^2+5z+14}{(z-1)^4}=\lim_{z\to\infty}\frac{z^2\left(2+\frac{5}{z}+\frac{14}{z^2}\right)}{z^4\left(1-\frac{1}{z}\right)^2}$ $u_0=f(0)=0\quad\because\frac{1}{\infty}=0$

	$u_1 = f(1) = \lim_{z\to\infty}\left[zF(z) - zf(0)\right]$ $= \lim_{z\to\infty}\left[\frac{z(2z^2+5z+14)}{(z-1)^4} - z(0)\right]$ $= \lim_{z\to\infty}\left[\frac{z^3\left(2+\frac{5}{z}+\frac{14}{z^2}\right)}{z^4\left(1-\frac{1}{z}\right)^4} - 0\right]$ $u_1 = f(1) = 0 \quad \because \frac{1}{\infty} = 0$ $u_2 = f(2) = \lim_{z\to\infty}\left[z^2F(z) - z^2 f(0) - zf(1)\right]$ $= \lim_{z\to\infty}\left[\frac{z^2(2z^2+5z+14)}{(z-1)^4} - z^2(0) - z(0)\right]$ $= \lim_{z\to\infty}\left[\frac{z^4\left(2+\frac{5}{z}+\frac{14}{z^2}\right)}{z^4\left(1-\frac{1}{z}\right)^4}\right] = \frac{2+0+0}{(1-0)^4} = 2$ $u_2 = f(2) = 2$ $u_3 = f(3) = \lim_{z\to\infty}\left[z^3F(z) - z^3 f(0) - z^2 f(1) - zf(2)\right]$ $= \lim_{z\to\infty}\left[\frac{z^3(2z^2+5z+14)}{(z-1)^4} - z^3(0) - z^2(0) - z(2)\right]$ $= \lim_{z\to\infty}\left[\frac{z^3(2z^2+5z+14)}{(z-1)^4} - 2z\right]$ $= \lim_{z\to\infty} z^3\left(\frac{(2z^2+5z+14)}{(z-1)^4} - \frac{2}{z^2}\right)$ $= \lim_{z\to\infty} z^3\left(\frac{z^2(2z^2+5z+14) - 2(z-1)^4}{z^2(z-1)^4}\right) \because (a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^3$ $= \lim_{z\to\infty} z^3\left(\frac{(2z^4+5z^3+14z^2) - 2(z^4-4z^3+6z^2-4z+1)}{z^2(z-1)^4}\right)$ $= \lim_{z\to\infty} z^3\left(\frac{2z^4+5z^3+14z^2-2z^4+8z^3-12z^2+8z-2}{z^2(z-1)^4}\right)$ $= \lim_{z\to\infty} z^3\left(\frac{13z^3+2z^2+8z-2}{z^2(z-1)^4}\right)$ $= \lim_{z\to\infty}\frac{z^6\left(13+\frac{2}{z}+\frac{8}{z^2}-\frac{2}{z^3}\right)}{z^6\left(1-\frac{1}{z}\right)^4} = \lim_{z\to\infty}\frac{\left(13+\frac{2}{z}+\frac{8}{z^2}-\frac{2}{z^3}\right)}{\left(1-\frac{1}{z}\right)^4} = \frac{13+0+0-0}{(1-0)^4}$ $u_3 = f(3) = 13$
5.	**State and prove initial and final value theorem of Z-transform.** **Initial value theorem:** **If** $Z\left[f(n)\right] = F(z)$ **then** $f(0) = \lim_{z\to\infty} F(z)$ **Proof:**

	We know that $Z\left[f(n)\right]=\sum_{n=0}^{\infty} f(n)z^{-n}$ $\lim_{z\to\infty} F(z)=\lim_{z\to\infty}\sum_{n=0}^{\infty} f(n)\left(\frac{1}{z}\right)^{n}$ $=\lim_{z\to\infty}\left[f(0)\left(\frac{1}{z}\right)^{0}+f(1)\left(\frac{1}{z}\right)^{1}+f(2)\left(\frac{1}{z}\right)^{2}+\ldots\right]$ $\lim_{z\to\infty} F(z)=f(0) \quad \because \frac{1}{\infty}=0$ **Final value theorem:** **If** $Z\left[f(n)\right]=F(z)$ **then** $\lim_{n\to\infty} f(n)=\lim_{z\to 1}(z-1)F(z)$ **Proof:** $Z\left[f(n)\right]=\sum_{n=0}^{\infty} f(n)z^{-n} \quad -----(1)$ $Z\left[f(n+1)\right]=\sum_{n=0}^{\infty} f(n+1)z^{-n} \quad -----(2)$ $(1)-(2)\Rightarrow$ $Z\left[f(n+1)\right]-Z\left[f(n)\right]=\sum_{n=0}^{\infty} f(n+1)z^{-n}-\sum_{n=0}^{\infty} f(n)z^{-n}$ $\left[zF(z)-zf(0)\right]-F(z)=\sum_{n=0}^{\infty}\left[f(n+1)-f(n)\right]z^{-n}$ $\lim_{z\to 1}\left[(z-1)F(z)-zf(0)\right]=\lim_{z\to 1}\sum_{n=0}^{\infty}\left[f(n+1)-f(n)\right]z^{-n}$ $\lim_{z\to 1}\left[(z-1)F(z)\right]-f(0)=\sum_{n=0}^{\infty}\left[f(n+1)-f(n)\right]$ $\lim_{z\to 1}\left[(z-1)F(z)\right]-f(0)=\left[\cancel{f(1)}-f(0)\right]+\left[\cancel{f(2)}-\cancel{f(1)}\right]+\ldots+\left[f(n+1)-\cancel{f(n)}\right]+\ldots\infty$ $\lim_{z\to 1}\left[(z-1)F(z)\right]\cancel{-f(0)}=\cancel{-f(0)}+f(n+1)+\ldots\infty$ $\lim_{z\to 1}\left[(z-1)F(z)\right]=\lim_{n\to\infty} f(n) \quad \because f(n+1)=f(n) \text{ when } n\to\infty$ Hence proved