

4.5 Trapezoidal rule and Simpson's $\frac{1}{3}$ rule

Trapezoidal rule

$$\int f(x)dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + y_3 \dots y_{n-1})]$$

1. Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ with $h = \frac{1}{6}$ by Trapezoidal Rule

Solution : Let $f(x) = \frac{1}{1+x^2}$ and $h = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = f(x)$	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$

$$f(x) = \frac{1}{1+x^2}$$

$$f(0) = \frac{1}{1+0^2} = \frac{1}{1+0} = 1$$

$$f\left(\frac{1}{6}\right) = \frac{1}{1+\frac{1}{36}} = \frac{1}{\frac{36+1}{36}} = \frac{36}{37}$$

$$f\left(\frac{2}{6}\right) = \frac{1}{1+\frac{4}{36}} = \frac{1}{\frac{36+4}{36}} = \frac{36}{40} = \frac{9}{10}$$

By Trapezoidal Rule

$$\int f(x)dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + y_3 \dots y_{n-1})]$$

$$\begin{aligned}
 \int_0^1 \frac{1}{1+x^2} dx &= \frac{1/6}{2} \left[\left(1 + \frac{1}{2}\right) + 2 \left(\frac{36}{37} + \frac{9}{10} + \frac{4}{5} + \frac{9}{13} + \frac{36}{61} \right) \right] \\
 &= \frac{1}{12} \left[\left(\frac{3}{2}\right) + 2(3.9554) \right] \\
 &= \frac{1}{12} \left[\left(\frac{3}{2}\right) + 7.9108 \right] \\
 &= 0.7842
 \end{aligned}$$

2. Evaluate $\int_1^2 \frac{1}{1+x^2} dx$ with four sub intervals
by Trapezoidal Rule

Solution:

$$\text{Let } y = f(x) = \frac{1}{1+x^2} \text{ and } h = \frac{2-1}{4} = \frac{1}{4} = 0.25$$

x	1	1.25	1.5	1.75	2
$y = f(x)$	0.5	0.3902	0.3077	0.2462	0.2

$$f(x) = \frac{1}{1+x^2}$$

$$f(1) = \frac{1}{1+1^2} = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

$$f(1.25) = \frac{1}{1+(1.25)^2} = 0.3902$$

$$f(1.5) = \frac{1}{1 + (1.5)^2} = 0.3077$$

Trapezoidal Rule

$$\int f(x)dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + y_3 \dots y_{n-1})]$$

$$\begin{aligned} \int_0^1 \frac{1}{1+x^2} dx &= \frac{0.25}{2} [(0.5 + 0.2) + 2(0.3902 + 0.3077 + 0.2462)] \\ &= 0.125[(0.7) + 2(0.9441)] \\ &= 0.125[2.5882] = 0.3235 \end{aligned}$$

Simpson's $\frac{1}{3}$ rule

$$\int f(x)dx = \frac{h}{2} [(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \dots) + 4(y_1 + y_3 + y_5 \dots)]$$

1. Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ with $h = \frac{1}{6}$ by Simpson's $\frac{1}{3}$ rule

Solution :

Let $f(x) = \frac{1}{1+x^2}$ and $h = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
y	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$

$$f(x) = \frac{1}{1+x^2}$$

$$f(0) = \frac{1}{1+x^2} = \frac{1}{1+0} = 1$$

$$f\left(\frac{1}{6}\right) = \frac{1}{1+\frac{1}{36}} = \frac{1}{\frac{36+1}{36}} = \frac{36}{37}$$

$$f\left(\frac{2}{6}\right) = \frac{1}{1+\frac{4}{36}} = \frac{1}{\frac{36+4}{36}} = \frac{36}{40} = \frac{9}{10}$$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
y	1	$\frac{36}{37}$	$\frac{9}{10}$	$\frac{4}{5}$	$\frac{9}{13}$	$\frac{36}{61}$	$\frac{1}{2}$
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{1}{3}$ rule

$$\int f(x)dx = \frac{h}{2} [(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \dots) + 4(y_1 + y_3 + y_5 \dots)]$$

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{1/6}{2} \left[\left(1 + \frac{1}{2}\right) + 2\left(\frac{9}{10} + \frac{9}{13}\right) + 4\left(\frac{36}{37} + \frac{4}{5} + \frac{36}{61}\right) \right]$$

$$= \frac{1}{12} \left[\left(\frac{3}{2}\right) + 2(1.5923) + 4(2.3632) \right]$$

$$= \frac{1}{12} \left[\left(\frac{3}{2}\right) + 3.1846 + 9.4528 \right]$$

$$= \frac{1}{12} \left[\left(\frac{3}{2}\right) + 12.6374 \right]$$

$$= 1.1781$$

2. Evaluate $\int_1^2 \frac{1}{1+x^2} dx$ with Four sub interval by Simpson's $\frac{1}{3}$ rule

Solution :

$$\text{Let } f(x) = \frac{1}{1+x^2} \quad \text{and } h = \frac{2-1}{4} = \frac{1}{4} = 0.25$$

x	1	1.25	1.5	1.75	2
y	0.5	0.3902	0.3077	0.2462	0.2
	y_0	y_1	y_2	y_3	y_4

$$f(x) = \frac{1}{1+x^2}$$

$$f(1) = \frac{1}{1+1^2} = \frac{1}{1+1} = \frac{1}{2} = 0.5$$

$$f(1.25) = \frac{1}{1+(1.25)^2} = 0.3902$$

$$f(1.5) = \frac{1}{1+(1.5)^2} = 0.3077$$

Simpson's $\frac{1}{3}$ rule

$$\int f(x) dx = \frac{h}{2} [(y_0 + y_n) + 2(y_2 + y_4 + y_6 + \dots) + 4(y_1 + y_3 + y_5 \dots)]$$

$$\int_1^2 \frac{1}{1+x^2} dx = \frac{0.25}{2} [(0.5 + 0.2) + 2(0.3077) + 4(0.3902 + 0.2462)]$$

$$= 0.125[(0.7) + 0.6154 + 2.5456]$$

$$= 0.125[3.861]$$

$$= 0.4826$$

Trapezoidal rule for Double Integral

$$I = \frac{hk}{4} [(Sum\ of\ four\ corners) + 2(Sum\ of\ nodes\ on\ boundary) + 4(Sum\ of\ interior\ nodes)]$$

Simpson's $\frac{1}{3}$ rule for Double Integral

$$\begin{aligned} \text{Simpson's } 1/3 \text{ rule} = & \frac{hk}{9} [(Sum\ of\ the\ corner\ of\ the\ boundary) \\ & + 2(sum\ of\ the\ odd\ nodes\ of\ the\ boundary) \\ & + 4(sum\ of\ the\ even\ nodes\ of\ the\ boundary) \\ & + 4(sum\ of\ the\ odd\ nodes\ of\ the\ odd\ rows) \\ & + 8(sum\ of\ the\ even\ nodes\ of\ the\ odd\ rows) \\ & + 8(sum\ of\ the\ odd\ nodes\ of\ the\ even\ rows) \\ & + 16(sum\ of\ the\ even\ nodes\ of\ the\ even\ rows)] \end{aligned}$$

1. Evaluate $\int_1^2 \int_3^4 \frac{1}{(x+y)^2} dx dy$ with $h = k = 0.5$ by **Trapezoidal**

and Simpson's rule

Solution :

$$\text{Let } f(x, y) = \frac{1}{(x+y)^2}$$

(i) Range for x : 3 to 4 and $h = 0.5$

(ii) Range for y : 1 to 2 and $k = 0.5$

$x \backslash y$			
	3	3.5	4
1	0.0625	0.0494	0.04
1.5	0.0494	0.04	0.0331
2	0.04	0.0331	0.0278

$$f(x, y) = \frac{1}{(x + y)^2}$$

$$f(3, 1) = \frac{1}{(3 + 1)^2} = \frac{1}{16} = 0.0625$$

$$f(3.5, 1) = \frac{1}{(3.5 + 1)^2} = \frac{1}{(4.5)^2} = 0.0494$$

$$f(4, 1) = \frac{1}{(4 + 1)^2} = \frac{1}{25} = 0.04$$

$$I = \frac{hk}{4} [(Sum\ of\ four\ corners) + 2(Sum\ of\ nodes\ on\ boundary) + 4(Sum\ of\ interior\ nodes)]$$

$$I = \frac{(0.5)(0.5)}{4} [(0.0625 + 0.04 + 0.04 + 0.0278) + 2(0.0494 + 0.0494 + 0.0331 + 0.0331) + 4(0.04)]$$

$$I = \frac{0.25}{4} [(0.1703) + 0.330 + 0.16]$$

$$I = 0.0413$$

$$I = \frac{(0.5)(0.5)}{9} [(0.0625 + 0.04 + 0.04 + 0.0278) + 4(0.0494 + 0.0494 + 0.0331 + 0.0331) + 16(0.04)]$$

$$I = \frac{0.25}{9} [(0.1703) + 0.660 + 0.64]$$

$I = 0.0408$

