Co-ordinate Systems

In order to describe the spatial variations of the quantities, we require using appropriate co-ordinate system. A point or vector can be represented in a **curvilinear** coordinate system that may be **orthogonal** or **non-orthogonal** .

An orthogonal system is one in which the co-ordinates are mutually perpendicular. Non- orthogonal co-ordinate systems are also possible, but their usage is very limited inpractice .

Let u = constant, v = constant and w = constant represent surfaces in a coordinate system,

the surfaces may be curved surfaces in general. Furthur, let $\hat{a}_u$, $\hat{a}_v$ and $\hat{a}_w$ be the unit vectors in the three coordinate directions(base vectors). In a general right handed orthogonal curvilinear systems, the vectors satisfy the following relations :

$$\hat{a}_u \times \hat{a}_v = \hat{a}_w$$
$$\hat{a}_v \times \hat{a}_w = \hat{a}_u$$
$$\hat{a}_w \times \hat{a}_u = \hat{a}_v$$

.....................................(1.13)

These equations are not independent and specification of one will automatically implythe other two. Furthermore, the following relations hold

$$\hat{a}_u \cdot \hat{a}_v = \hat{a}_v \cdot \hat{a}_w = \hat{a}_w \cdot \hat{a}_u = 0$$
$$\hat{a}_u \cdot \hat{a}_u = \hat{a}_v \cdot \hat{a}_v = \hat{a}_w \cdot \hat{a}_w = 1$$

................(1.14)

A vector can be represented as sum of its orthogonalcomponents,

$$\vec{A} = A_u \hat{a}_u + A_v \hat{a}_v + A_w \hat{a}_w$$

.... (

1.15)

In general u, v and w may not represent length. We multiply u, v and w by conversion factors h_1,h_2 and h_3 respectively to convert differential changes du, dv and dw tocorresponding changes in length dl_1, dl_2, and dl_3. Therefore

$$d\vec{l} = \hat{a}_u\, dl_1 + \hat{a}_v\, dl_2 + \hat{a}_w\, dl_3$$
$$= h_1 du\, \hat{a}_u + h_2 dv\, \hat{a}_v + h_3 dw\, \hat{a}_w \qquad \text{...............(1.16)}$$

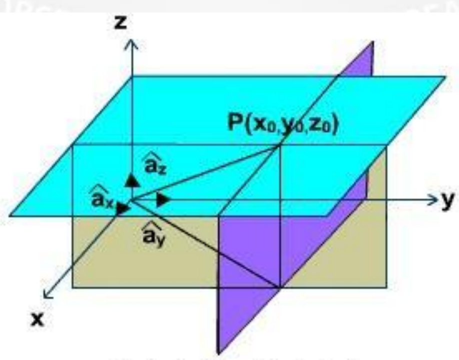
In the same manner, differential volume dv can be written as $dv = h_1 h_2 h_3\, du\, dv\, dw$ anddifferential area ds_1 $\hat{a}_u$ $ds_1 = h_2 h_3\, dv\, dw$ $\hat{a}_w$

In the following sections we discuss three most commonly used orthogonal co-ordinate systems, viz:

1. **Cartesian (or rectangular) co-ordinate system**
2. **Cylindrical co-ordinate system**
3. **Spherical polar co-ordinate system**

Cartesian Co-ordinate System :

In Cartesian co-ordinate system, we have, $(u,v,w) = (x,y,z)$. A point $P(x_0, y_0, z_0)$ in Cartesian co-ordinate system is represented as intersection of three planes $x = x_0$, $y = y_0$ and $z = z_0$. The unit vectors satisfies the following relation as shown in figure 2.1:



Cartesian Coordinate System

Fig 2.1 Intersection of three planes (www.brainkart.com/subject/Electromagnetic-Theory_206/)

$$\hat{a}_x \times \hat{a}_y = \hat{a}_z$$
$$\hat{a}_y \times \hat{a}_z = \hat{a}_x$$
$$\hat{a}_z \times \hat{a}_x = \hat{a}_y$$
$$\hat{a}_x . \hat{a}_y = \hat{a}_y . \hat{a}_z = \hat{a}_z . \hat{a}_x = 0$$
$$\hat{a}_x . \hat{a}_x = \hat{a}_y . \hat{a}_y = \hat{a}_z . \hat{a}_z = 1$$

In cartesian co-ordinate system, a vector $\vec{A}$ can be written as $\vec{A} = \hat{a}_x A_x + \hat{a}_y A_y + \hat{a}_z A_z$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \quad ....(1.19)$$

$$\vec{A} \times \vec{B} = a_x (A_y B_z - A_z B_y) + a_y (A_z B_x - A_x B_z) + \hat{a}_z (A_x B_y - A_y B_x)$$
$$= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \quad ....(1.20)$$

Since x, y and z all represent lengths, $h_1 = h_2 = h_3 = 1$. The differential length, area and volume are defined respectively as

$$d\vec{l} = dx\,\hat{a}_x + dy\,\hat{a}_y + dz\,\hat{a}_z$$

$$d\vec{s}_x = dydz\,\hat{a}_x$$
$$d\vec{s}_y = dxdz\,\hat{a}_y \quad ....(1.21)$$
$$d\vec{s}_z = dxdy\,\hat{a}_z$$

$$d\upsilon = dxdydz \quad ....(1.22)$$

Cylindrical Co-ordinate System :

For cylindrical coordinate systems we have $(u, v, w) = (r, \phi, z)$ a point $P(r_0, \phi_0, z_0)$ is determined as th containing the z-axis and making an angle $\phi = \phi_0$ with the xz plane and a plane parallel to xy plane located at $z=z_0$ as shown in figure 2.2 and 2.3.

In cylindrical coordinate system, the unit vectors satisfy the following relations

A vector $\vec{A}$ can be written as $\vec{A} = A_\rho \hat{a}_\rho + A_\phi \hat{a}_\phi + A_z \hat{a}_z$,.. (1.24)

The differential length is defined as,

$$d\vec{l} = \hat{a}_\rho d\rho + \rho d\phi \hat{a}_\phi + dz \hat{a}_z \qquad h_1 = 1, h_2 = \rho, h_3 = 1 \quad ....(1.25)$$

$$a_\rho \times a_\phi = a_z$$

$$\hat{a}_\phi \times \hat{a}_z = \hat{a}_\rho$$

$$\hat{a}_z \times \hat{a}_\rho = \hat{a}_\phi \quad ....(1.23)$$

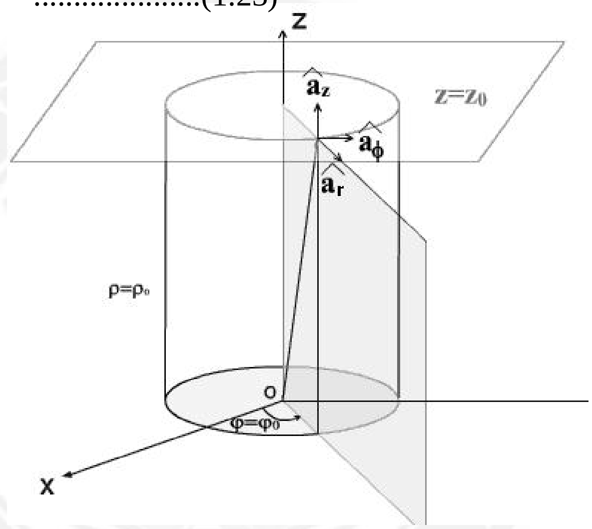
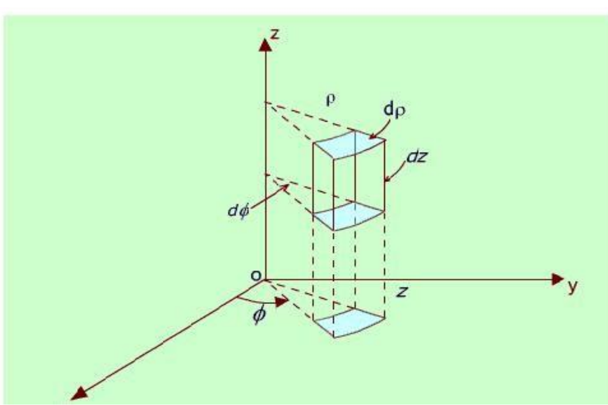


Fig 2.2 cylindrical co-ordinate system
(www.brainkart.com/subject/Electromagnetic-Theory_206/)



Differential areas are:

$$\overrightarrow{ds_\rho} = \rho d\phi dz \hat{a}_\rho$$

$$\overrightarrow{ds_\phi} = d\rho dz \hat{a}_\phi \quad ....(1.26)$$

$$\overrightarrow{ds_z} = \rho d\phi d\rho \hat{a}_z$$

Differential volume,

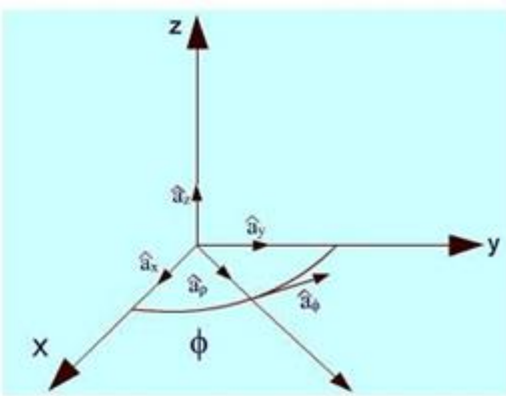
$$d\upsilon = \rho d\rho d\phi dz \quad ....(1.27)$$

Fig 2.3 cylindricalsystem surface

(www.brainkart.com/subject/Electromagnetic-Theory_206/)

Transformation between Cartesian and Cylindrical coordinates:

Let us consider $\vec{A} = \hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z$ is to be expressed in Cartesian co-ordinate as . In doing so we note that $A_x = \vec{A}.\hat{a}_x = \left(\hat{a}_\rho A_\rho + \hat{a}_\phi A_\phi + \hat{a}_z A_z\right).\hat{a}_x$ and it applies for other components as well as shown in figure 2.4.



$$\hat{a}_\rho . \hat{a}_x = \cos\phi$$
$$\hat{a}_\rho . \hat{a}_y = \sin\phi$$
$$\hat{a}_\phi . \hat{a}_x = \cos(\phi + \frac{\pi}{2}) = -\sin\phi \quad \text{.............(1.28)}$$
$$\hat{a}_\phi . \hat{a}_y = \cos\phi$$

Therefore we can write,

$$A_x = \vec{A}\hat{a}_x = A_\rho \cos\phi - A_\phi \sin\phi$$
$$A_y = \vec{A}\hat{a}_y = A_\rho \sin\phi + A_\phi \cos\phi \quad \text{..........(1.29)}$$
$$A_z = \vec{A}\hat{a}_z = A_z$$

Fig 2.4: Unit Vectors in Cartesian and Cylindrical Coordinates

(www.brainkart.com/subject/Electromagnetic-Theory_206/)

These relations can be put conveniently in the matrix form as:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \quad \text{.....................(1.30)}$$

A_ρ, A_ϕ and A_z themselves may be functions of ρ, ϕ and z as:

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z \quad \text{............................(1.31)}$$

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\frac{y}{x}$$
$$z = z$$

The inverse relationships are: .... (1.32)

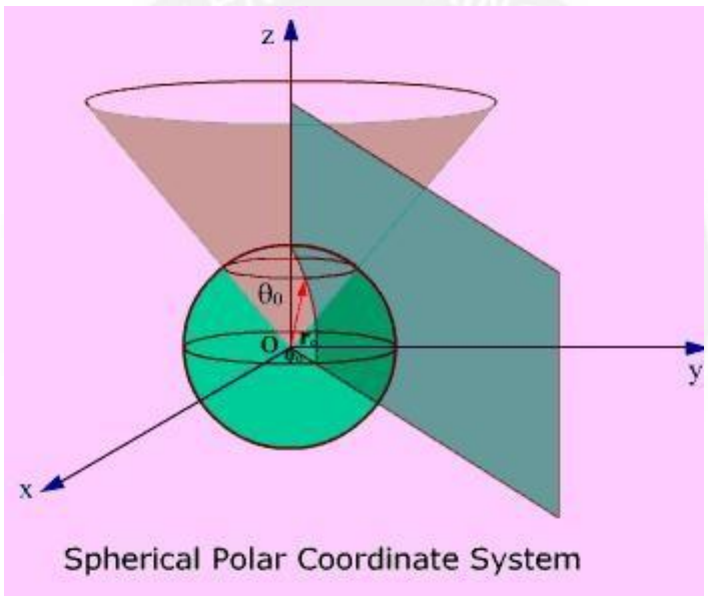


Fig 2.5: Spherical Polar Coordinate System
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

Thus we see that a vector in one coordinate system is transformed to another coordinate system through two-step process: Finding the component vectors and then variable transformation as shown in fig 2.5.

Spherical Polar Coordinates:

For spherical polar coordinate system, we have $(u,v,w) = (r,\theta,\phi)$ $P(r_0,\theta_0,\phi_0)$ is represented as the intersection of

(i) Spherical surface $r=r_0$

(ii) Conical surface $\theta = \theta_0$,and

. A point

(iii) half plane containing z-axis making angle $\phi = \phi_1$ with the *xz* plane as shown in the figure 1.10.

$$\hat{a}_r \times \hat{a}_\theta = \hat{a}_\phi$$

$$\hat{a}_\theta \times \hat{a}_\phi = \hat{a}_r$$

$$\hat{a}_\phi \times \hat{a}_r = \hat{a}_\theta$$

The unit vectors satisfy the following relationships:............................ (1.33)

The orientation of the unit vectors are shown in the figure 2.6.

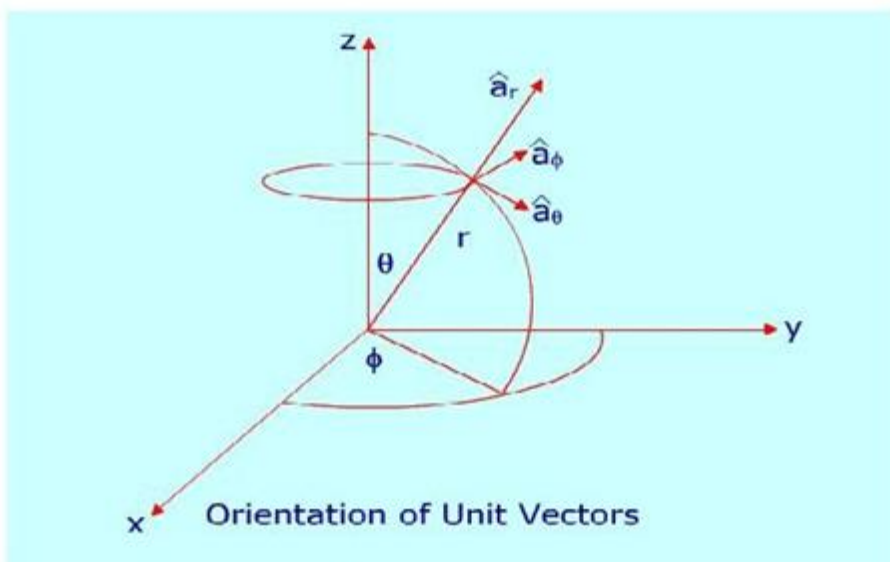


Fig 2.6 : Orientation of Unit Vectors

(www.brainkart.com/subject/Electromagnetic-Theory_206/)

A vector in spherical polar co-ordinates is written as : $\vec{A} = A_r \hat{a}_r + A_\theta \hat{a}_\theta + A_\phi \hat{a}_\phi$

$$d\vec{l} = \hat{a}_r dr + \hat{a}_\theta r d\theta + \hat{a}_\phi r \sin\theta d\phi$$

$r \sin\theta$ an

dFor spherical polar coordinate system we have $h1$=1, $h2$= r and

$h3$= .

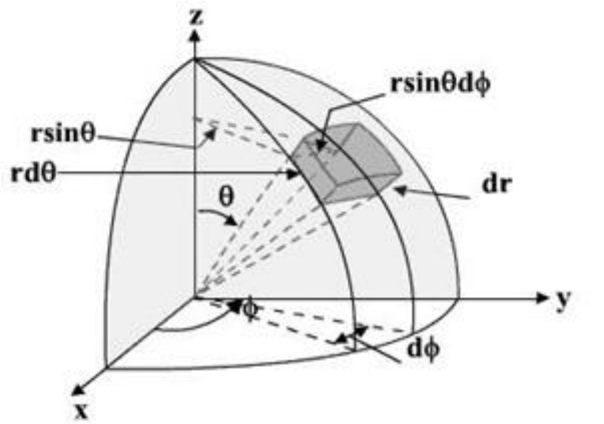


Fig 2.7 : Differential volume in s-p coordinates

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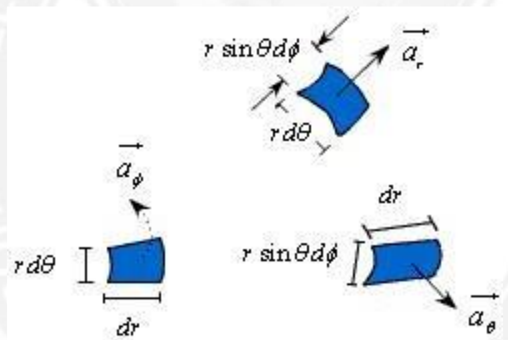

Fig 2.8 : Exploded view

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With reference to the Figure 1.12, the elemental areas are:

$$ds_r = r^2 \sin\theta d\theta d\phi \hat{a}_r$$

$$ds_\theta = r \sin\theta dr d\phi \hat{a}_\theta$$

$$ds_\phi = r dr d\theta \hat{a}_\phi \qquad \text{.......................(1.34)}$$

and elementary volume is given by

$$dv = r^2 \sin\theta dr d\theta d\phi \qquad \text{.........................(1.35)}$$

Coordinate transformation between rectangular and spherical polar: With reference to the figure 2.7 and 2.8 ,we can write the following equations:

$$\hat{a}_r \cdot \hat{a}_x = \sin\theta\cos\phi$$

$$\hat{a}_r \cdot \hat{a}_y = \sin\theta\sin\phi$$

$$\hat{a}_r \cdot \hat{a}_z = \cos\theta$$

$$\hat{a}_\theta \cdot \hat{a}_x = \cos\theta\cos\phi$$

$$\hat{a}_\theta \cdot \hat{a}_y = \cos\theta\sin\phi$$

$$\hat{a}_\theta \cdot \hat{a}_z = \cos(\theta + \frac{\pi}{2}) = -\sin\theta$$

$$\hat{a}_\phi \cdot \hat{a}_x = \cos(\phi + \frac{\pi}{2}) = -\sin\phi$$

$$\hat{a}_\phi \cdot \hat{a}_y = \cos\phi$$

$$\hat{a}_\phi \cdot \hat{a}_z = 0$$

....(1.36)

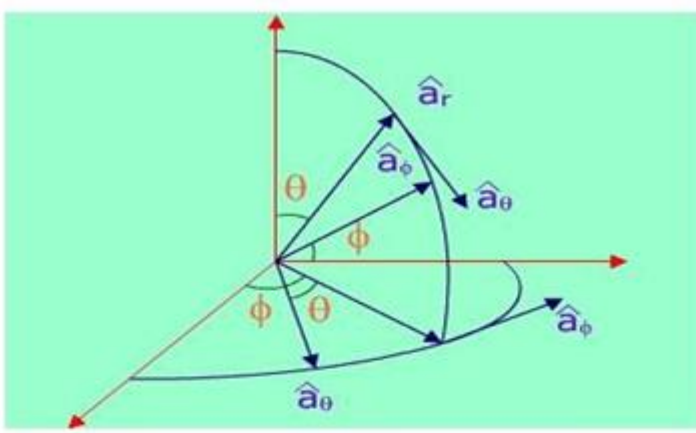

Fig 2.9 : Coordinate transformation

(www.brainkart.com/subject/Electromagnetic-Theory_206/)

Given a vector $\vec{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_\phi\hat{a}_\phi$ in the spherical polar coordinate system as shown in fig 2.9, its component in the cartesian coordinate system can be found out as follows:

$$A_x = \vec{A}.\hat{a}_x = A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi \quad \text{.................................}(1.37)$$

Similarly,

$$A_y = \vec{A}\,\hat{a}_y = A_r \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi \quad \text{.................................(1.38a)}$$

$$A_z = \vec{A}\,\hat{a}_z = A_r \cos\theta - A_\theta \sin\theta \quad \text{.................................(1.38b)}$$

The above equation can be put in a compact form:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

.................................(1.39)

The components $A_r, A_\theta \text{ and } A_\phi$ themselves will be functions of $r, \theta \text{ and } \phi$. $r, \theta \text{ and } \phi$ arerelated to x,y a

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

....................(1.40)

and conversely,

$$r = \sqrt{x^2 + y^2 + z^2}$$

....(1.41a)

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

.................................(1.41b)

$$\phi = \tan^{-1}\frac{y}{x}$$

....(1.41c)

Using the variable transformation listed above, the vector components, which are functions of variables of one coordinate system, can be transformed to functions of variables of other coordinate system and a total transformation can be done.