UNIT-1 INTRODUCTION

Part- A

1. **Distinguish one Dimensional bar element and Beam Element (May/June 2011)**
   1D bar element: Displacement is considered.
   1D beam element: Displacement and slope is considered

2. **What do you mean by Boundary value problem?**

   The solution of differential equation is obtained for physical problems, which satisfies some specified conditions known as boundary conditions.

   The differential equation together with these boundary conditions, subjected to a boundary value problem.

   Examples: Boundary value problem.

   d^2y/dx^2 - a(x) dy/dx – b(x)y –c(x) = 0 with boundary conditions, y(m) = S and y(n) = T.

3. **What do you mean by weak formulation? State its advantages.** ***(April/May 2015), (May/June 2013)***

   A weak form is a weighted integral statement of a differential equation in which the differentiation is distributed among the dependent variable and the weight function and also includes the natural boundary conditions of the problem.

   - A much wider choice of trial functions can be used.
   - The weak form can be developed for any higher order differential equation.
   - Natural boundary conditions are directly applied in the differential equation.
   - The trial solution satisfies the essential boundary conditions.

4. **Why are polynomial types of interpolation functions preferred over trigonometric functions?** ***(May/June 2013)***

   Polynomial functions are preferred over trigonometric functions due to the following reasons:

   1. It is easy to formulate and computerize the finite element equations
   2. It is easy to perform differentiation or integration
   3. The accuracy of the results can be improved by increasing the order of the polynomial.

5. **What do you mean by elements & Nodes?*(May/June 2014)***

   In a continuum, the field variables are infinite. Finite element procedure reduces such unknowns to a finite number by dividing the solution region into small parts called Elements. The common points between two adjacent elements in which the field variables are expressed are called Nodes.

6. **What is Ritz method?*(May/June 2014)***

   It is integral approach method which is useful for solving complex structural problem, encountered in finite element analysis. This method is possible only if a suitable function is available. In Ritz method approximating functions satisfying the boundary conditions are used to get the solutions

7. **Distinguish Natural & Essential boundary condition (May/June 2009)**

There are two types of boundary conditions.

They are:

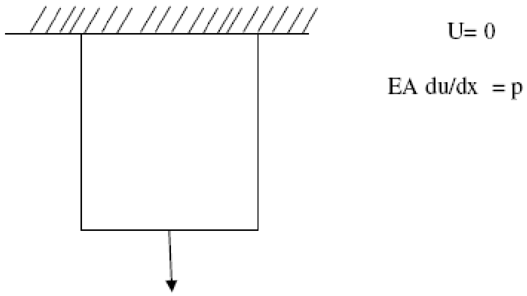
1. **Primary boundary condition (or) Essential boundary condition**

The boundary condition, which in terms of field variable, is known as primary boundary condition.

2. Secondary boundary condition or natural boundary conditions

The boundary conditions, which are in the differential form of field variables, are known as secondary boundary condition.

Example: A bar is subjected to axial load as shown in fig.



In this problem, displacement u at node 1 = 0, that is primary boundary condition. EA du/dx = P, that is secondary boundary condition.

8. **Compare Ritz method with nodal approximation method.*(Nov/Dec 2014), (Nov/Dec 2012)***

<u>Similarity:</u>

(i) Both methods use approximating functions as trial solution

(ii) Both methods take linear combinations of trial functions.

(iii) In both methods completeness condition of the function should be satisfied

(iv) In both methods solution is sought by making a functional stationary.

<u>Difference</u>

(i) Rayleigh-Ritz method assumes trial functions over entire structure, while finite element method uses trial functions only over an element.

(ii) The assumed functions in Rayleigh-Ritz method have to satisfy boundary conditions over entire structure while in finite element analysis, they have to satisfy continuity conditions at nodes and sometimes along the boundaries of the element. However completeness condition should be satisfied in both methods.

9. **What do you mean by elements & Nodes?**

In a continuum, the field variables are infinite. Finite element procedure reduces such unknowns to a finite number by dividing the solution region into small parts called Elements. The common points between two adjacent elements in which the field variables are expressed are called Nodes.

10. **State the discretization error. How it can be reduced? *(April /May 2015)***

Splitting of continuum in to smallest elements is known as discretization. In some context like structure having boundary layer the exact connectivity can’t be achieved. It means that it may not resemble the original structure. Now there is an error developed in calculation. Such type of error is discretization error.

To Reduce Error:

(i) Discretization error can be minimized by reducing the finite element (or) discretization element.

(ii) By introducing finite element it has a curved member.

11. What are the various considerations to be taken in Discretization process?

(i) Types of Elements.
(ii) Size of Elements.
(iii) Location of Nodes.
(iv) Number of Elements.

12. State the principleofminimum potential energy. (Nov/Dec 2010)

Amongallthedisplacementequationsthatsatisfiedinternalcompatibilityandthe boundaryconditionthosethatalsosatisfytheequationofequilibriummakethe potential energya minimum is astable system.

PART-B

1. The following differential equation is available for a physical phenomenon. $AE = \frac{d^2u}{dx^2} + ax = 0$, The boundary conditions are u(0) = 0, $AE = \frac{du}{dx}\Big|_{x=L} = 0$ By using Galerkin's technique, find the solution of the above differential equation.

Given Data:

Differential equ. $AE = \frac{d^2u}{dx^2} + ax = 0$

Boundary Conditions $u(0) = 0$, $AE = \frac{d^2u}{dx^2} + ax = 0$

To Find:

u(x) by using galerkin's technique

Formula used

$$\int_0^L w_i\, R\, dx = 0$$

Solution:

Assume a trial function

Let $u(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ …….. (1)

Apply first boundary condition

i.e) at x=0, u(x) = 0

$(1) \Rightarrow 0 = a_0 + 0 + 0 + 0$

$a_0 = 0$

Apply first boundary condition i.e at $x = L$, $AE = \frac{du}{dx} = 0$

$$\Longrightarrow \frac{du}{dx} = 0 + a_1 + 2a_2x + 3a_3L^2$$

$$\Longrightarrow 0 = a_1 + 2a_2L + 3a_3L^2$$

$$\Longrightarrow a_1 = -(2a_2L + 3a_3L^2)$$

sub a_0 and a_1 in value in equation (1)

$$u(x) = 0 + -(2a_2L + 3a_3L^2)x + a_2x^2 + a_3x^3$$

$$= -2a_2Lx - 3a_3L^2a_2x + a_2x^2 + a_3x^3$$

$$= a_2(x^2 - 2Lx) + a_3(x^3 - 3L^2x) \qquad \text{………} (2)$$

We Know That

$$\text{Residual, } R = AE\frac{d^2u}{dx^2} + ax \qquad \text{……….} (3)$$

$$(2) \Longrightarrow \frac{du}{dx} = a_2(2x - 2L) + a_3(3x^2 - 3L^2)$$

$$\frac{d^2u}{dx^2} = a_2(2) + a_3(6x)$$

$$\frac{d^2u}{dx^2} = 2a_2 + 6a_3x$$

Sub $\frac{d^2u}{dx^2}$ value in equation (3)

$$(3) \Longrightarrow R = AE(2a_2 + 6a_3x) + ax$$

$$\text{Residual, } R = AE(2a_2 + 6a_3x) + ax \qquad \text{………} (4)$$

From Galerkn's technique

$$\int_0^L w_i\, R\, dx = 0 \qquad \text{……….} (5)$$

from equation (2) we know that

$$w_1 = x^2 - 2Lx$$

$$w_2 = x^3 - 3L^2x$$

sub w_1, w_2 and R value in equation (5)

$$(5) \Longrightarrow \int_0^L (x^2 - 2Lx)\, [AE(2a_2 + 6a_3x) + ax]dx = 0 \qquad \text{… … … … …} (6)$$

$$\int_0^L (x^3 - 3L^2x)\, [AE(2a_2 + 6a_3x) + ax]dx = 0 \qquad \text{… … … … …} (7)$$

$$(6) \Longrightarrow \int_0^L (x^2 - 2Lx)\, [AE(2a_2 + 6a_3x) + ax]dx = 0$$

$$\int_0^L (x^2 - 2Lx)\,[2a_2AE + 6a_3AEx + \ ax]dx = 0$$

$$\int_0^L [2a_2AEx^2 + 6a_3AEx^3 + ax^3 - 4a_2AELx - 12a_3AELx^2 - 2aLx^2] = 0$$

$$\Rightarrow [2a_2AE\frac{x^3}{3} + 6a_3AE\ \frac{x^4}{4} + a\frac{x^4}{4} - 4a_2AEL\frac{x^2}{2} - 12a_3AEL\frac{x^3}{3} - 2aL\frac{x^3}{3}]_0^L = 0$$

$$\Rightarrow 2a_2AE\frac{L^3}{3} + 6a_3AE\ \frac{L^4}{4} + a\frac{L^4}{4} - 4a_2AE\frac{L^3}{2} - 12a_3AE\frac{L^4}{3} - 2a\frac{L^4}{3} = 0$$

$$\Rightarrow \tfrac{2}{3}a_2AEL^3 + \tfrac{3}{2}a_3AE\ L^4 + a\tfrac{L^4}{4} - 2a_2AEL^3 - 4a_3AEL^4 - \tfrac{2}{3}aL^4 = 0$$

$$\Rightarrow AEa_2L^3\left[\frac{2}{3} - 2\right] + a_3AE\ L^4\left[\frac{3}{2} - 4\right] + a\frac{L^4}{4} - \frac{2}{3}a_2L^4 = 0$$

$$\Rightarrow \frac{-4}{3}AEL^3a_2 - \frac{5}{2}AEL^4a_3 = \left[\frac{2}{3} - \frac{1}{4}\right]aL^4 - \frac{4}{3}AEL^3a_2 - \frac{5}{2}AEL^4a_3 = \frac{5}{12}aL^4$$

$$\frac{-4}{3}AEL^3a_2 - \frac{5}{2}AEL^4a_3 = -\frac{5}{12}aL^4 \qquad \ldots\ldots\ldots.(8)$$

Equation (7)

$$\Rightarrow \int_0^L (x^3 - 3L^2x)\,[AE(2a_2 + 6a_3x) + ax]dx = 0$$

$$\Rightarrow \int_0^L (x^3 - 3L^2x)\,[2a_2AE + 6a_3AEx + ax]dx = 0$$

$$\Rightarrow \int_0^L [2AEa_2x^3 + 6AEa_3x^4 + ax^4 - 6AEa_2L^2x - 18AEa_3L^2x^2 - 3aL^2x^2]dx = 0$$

$$\Rightarrow \left[2AEa_2\frac{x^4}{4} + 6AEa_3\frac{x^5}{5} + a\frac{x^5}{5} - 6AEa_2L^2\frac{x^2}{2} - 18AEa_3L^2\frac{x^3}{3} - 3aL^2\frac{x^3}{3}\right]_0^L = 0$$

$$\Rightarrow \left[\frac{1}{2}AEa_2x^4 + \frac{6}{5}AEa_3x^5 + \frac{1}{5}ax^5 - 3AEa_2L^2x^2 - 6AEa_3L^2x^3 - aL^2x^3\right]_0^L = 0$$

$$\Rightarrow \frac{1}{2}AEa_2L^4 + \frac{6}{5}AEa_3L^5 + \frac{1}{5}aL^5 - 3AEa_2L^2(L^2) - 6AEa_3L^2(L^3) - aL^2(L^3) = 0$$

$$\Rightarrow \frac{1}{2}AEa_2L^4 + \frac{6}{5}AEa_3L^5 + \frac{1}{5}aL^5 - 3AEa_2L^4 - 6AEa_3L^5 - aL^5 = 0$$

$$\Rightarrow AEa_2L^4\left[\frac{1}{2} - 3\right] + AEa_3L^5\left[\frac{6}{5} - 6\right] + aL^5 + \left[\frac{1}{5} - 1\right] = 0$$

$$\Rightarrow AEa_2L^4\left[\frac{5}{2}\right] - \frac{24}{5}AEa_3L^5 = \frac{4}{5}aL^5$$

$$\left[\frac{5}{2}\right]AEa_2L^4 + \frac{24}{5}AEa_3L^5 = -\frac{4}{5}aL^5 \qquad \ldots\ldots\ldots\ldots.(9)$$

Solving Equation (8) and (9)

Equation (8) $\Longrightarrow \frac{4}{3}AEa_2L^3 + \frac{5}{2}AEa_3L^4 = -\frac{5}{12}aL^4$

Equation (9) $\Longrightarrow \frac{5}{2}AEa_2L^4 + \frac{24}{5}AEa_3L^5 = -\frac{4}{5}aL^5$

Multiplying Equation (8) $\frac{5}{2}L$ and Equation (9) by $\frac{4}{3}$

$$\frac{20}{6}AEa_2L^4 + \frac{25}{4}AEa_3L^5 = -\frac{25}{24}aL^5$$

$$\frac{20}{6}AEa_2L^4 + \frac{25}{4}AEa_3L^5 = -\frac{16}{15}aL^5$$

Subtracting

$$\left[\frac{25}{4} - \frac{96}{15}\right]AEa_3L^5 = \left[\frac{16}{15} - \frac{25}{24}\right]aL^5$$

$$\left[\frac{375-384}{60}\right]AEa_3L^5 = \left[\frac{384-375}{360}\right]aL^5$$

$$\Longrightarrow \frac{-9}{60}AEa_3L^5 = \frac{9}{360}aL^5$$

$$\Longrightarrow -0.15AEa_3 = 0.025a$$

$$a_3 = -0.1666\frac{a}{AE}$$

$$a_3 = -\frac{a}{6AE} \qquad \ldots\ldots\ldots.(10)$$

Substituting a_3 value in Equation (8)

$$\frac{4}{3}AEa_2L^3 + \frac{5}{2}AE\left[\frac{-a}{6AE}\right]L^4 = \frac{-5}{12}aL^4$$

$$\frac{4}{3}AEa_2L^3 = \frac{-5}{12}aL^4 - \frac{5}{2}AEL^4 = \frac{-a}{6AE}$$

$$\frac{4}{3}AEa_2L^3 = \frac{-5}{12}aL^4 + \frac{5}{2}AEL^4$$

$$\frac{4}{3}AEa_2L^3 = 0$$

$$a_2 = 0$$

Sub a_2 and a_3 value in equation (2)

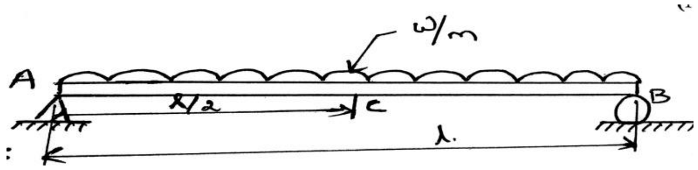
$$\Longrightarrow u(x) = 0x[x_2 - 2Lx] + \left(\frac{-a}{6AE}\right)[x^3 - 3L^2x] = 0$$

$$\Longrightarrow u(x) = \frac{a}{6AE}[3L^2x - x^3]$$

Result:

$$u(x) = \frac{a}{6AE}[3L^2x - x^3]$$

2. **Find the deflection at the centre of a simply supported beam of span length "l" subjected to uniformly distributed load throughout its length as shown in figure using (a) point collocation method, (b) sub-domain method, (c) Least squares method, and (d) Galerkin's method. (Nov/Dec 2014)**



Given data

Length (L) = l

UDL = $\omega\ N/m$

To find

Deflection

Formula used

$$EI\frac{d^4y}{dx^4} - \omega = 0, \quad 0 \le x \le l$$

Point Collocation Method R = 0

Sub-domain collocation method = $\int_0^l Rdx = 0$

Least Square Method $I = \int_0^l R^2 dx\ is\ minimum$

Solution:

The differential equation governing the deflection of beam subjected to uniformly distributed load is given by

$$EI\frac{d^4y}{dx^4} - \omega = 0, \quad 0 \le x \le l \qquad \dots\dots\dots (1)$$

The boundary conditions are Y=0 at x=0 and x = l, where y is the deflection.

$$EI\frac{d^4y}{dx^4} = 0, \qquad at\ x = 0\ and\ x = l$$

Where

$EI\frac{d^4y}{dx^4} = M,$ (Bending moment)

E → Young's Modules

I → Moment of Inertia of the Beam.

Let us select the trial function for deflection as $Y = asin\frac{\pi x}{l}$ ……. (2)

Hence it satisfies the boundary conditions

$$\Rightarrow \frac{dy}{dx} = a\frac{\pi}{l}.\cos\frac{\pi x}{l}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -a\frac{\pi^2}{l^2}.\sin\frac{\pi x}{l}$$

$$\Rightarrow \frac{d^3y}{dx^3} = -a\frac{\pi^3}{l^3}.\cos\frac{\pi x}{l}$$

$$\Rightarrow \frac{d^4y}{dx^4} = a\frac{\pi^4}{l^4}.\sin\frac{\pi x}{l}$$

Substituting the Equation (3) in the governing Equation (1)

$$EI\left[a\frac{\pi^4}{l^4}.\sin\frac{\pi x}{l}\right] - \omega = 0$$

Take, Residual $R = EIa\frac{\pi^4}{l^4}.\sin\frac{\pi x}{l} - \omega$

a) Point Collocation Method:

In this method, the residuals are set to zero.

$$\Rightarrow R = EIa\frac{\pi^4}{l^4}.\sin\frac{\pi x}{l} - \omega = 0$$

$$EIa\frac{\pi^4}{l^4}.\sin\frac{\pi x}{l} = \omega$$

To get maximum deflection, take $k = \frac{l}{2}\,(i.e. at\ can\ be\ of\ beam)$

$$EIa\frac{\pi^4}{l^4}.\sin\frac{\pi}{l}\left[\frac{l}{2}\right] = \omega$$

$$EIa\frac{\pi^4}{l^4} = \omega \qquad [\because \sin\frac{\pi}{l} = 1]$$

$$a = \frac{\omega l^4}{\pi^4 EI}$$

Sub "a" value in trial function equation (2)

$$Y = \frac{\omega l^4}{\pi^4 EI}.\sin\frac{\pi x}{l}$$

$$At\ x = \frac{l}{2} \Rightarrow Y_{\max} = \frac{\omega l^4}{\pi^4 EI}.\sin\frac{\pi}{2}\left[\frac{l}{2}\right]$$

$$Y_{\max} = \frac{\omega l^4}{\pi^4 EI} \qquad [\because \sin\frac{\pi}{2} = 1]$$

$$Y_{\max} = \frac{\omega l^4}{97.4 EI}$$

b) Sub-domain collocation method:

In this method, the integral of the residual over the sub-domain is set to zero.

$$\int_0^l R dx = 0$$

Sub R value

$$\Rightarrow \left[aEI\frac{\pi^4}{l^4}\sin\frac{\pi x}{l} - \omega\right]dx = 0$$

$$\Rightarrow \left[aEI\frac{\pi^4}{l^4}\left[\frac{-\cos\frac{\pi x}{l}}{\frac{\pi}{l}}\right] - \omega\, x\right]_0^l = 0$$

$$\Rightarrow \left[aEI\frac{\pi^4}{l^4}\left[-\cos\frac{\pi x}{l}\right]\left[\frac{l}{u}\right] - \omega\, x\right]_0^l = 0$$

$$\Rightarrow -aEI\frac{\pi^3}{l^3}[\cos\pi - cos0]\,\omega\, l = 0 \qquad \left[\because \begin{matrix}\cos\pi = -1\\ cos0 = 1\end{matrix}\right],$$

$$-aEI\frac{\pi^3}{l^3}[-1-1] = \omega\, l$$

$$\Rightarrow -a = \frac{\omega l^4}{2\pi^3 EI} = \frac{\omega l^4}{62EI}$$

Sub "a" value in the trial function equation (2)

$$Y = \frac{\omega l^4}{62EI}.\sin\frac{\pi x}{l}$$

$$At\ x = \frac{l}{2},\ \ Y_{max} = \frac{\omega l^4}{62EI}.\sin\frac{\pi}{l}(\frac{l}{2})$$

$$Y_{max} = \frac{\omega l^4}{62EI}$$

c) Least Square Method:

In this method the functional

$$I = \int_0^l R^2 dx\ is\ minimum$$

$$I = \int_0^l (aEI\frac{\pi^4}{l^4}.sin\frac{\pi x}{l} - \omega)^2 dx$$

$$= \int_0^l [a^2E^2I^2\frac{\pi^8}{l^8}.sin^2\frac{\pi x}{l} - \omega^2 - 2aEI\omega\frac{\pi^4}{l^4}.sin\frac{\pi x}{l}]dx$$

$$= [a^2E^2I^2\frac{\pi^8}{l^8}\left[\frac{1}{2}x\ sin\frac{2\pi x}{l}\right]\left(\frac{l}{2\pi}\right) + \omega^2 - 2aEI\omega\frac{\pi^4}{l^4}.[-cos\frac{\pi x}{l}\left(\frac{l}{\pi}\right)]]_0^l$$

$$= a^2E^2I^2\frac{\pi^8}{l^8}\left[\frac{1}{2}l - \frac{l}{2\pi}[(sin2\pi - sin0)]\right] + \omega^2 l + 2aEI\omega\frac{\pi^4}{l^4}.\frac{l}{\pi}[-cos\pi - cos0]$$

$$[\because sin2\pi = 0; sin0 = 0; cos\pi = 0; cos0 = 1]$$

$$I = a^2E^2I^2\frac{\pi^8}{l^2}\frac{l}{2} + \omega^2 l + 2aEI\omega\frac{\pi^3}{l^3}.(-1-1)$$

$$I = \frac{a^2E^2I^2\pi^8}{2l^7} + \omega^2 l - 4aEI\omega\frac{\pi^3}{l^3}$$

Now, $\frac{\partial\pi}{\partial a} = 0$

$$\Longrightarrow \frac{a^2E^2I^2\pi^8}{2l^7} = 4EI\omega\frac{\pi^3}{l^3}$$

$$\frac{a^2E^2I^2\pi^8}{l^7} = 4EI\omega\frac{\pi^3}{l^3}$$

$$a = \frac{4EI\omega l^5}{\pi^5 EI}$$

Hence the trial Function

$$Y = \frac{4\omega l^4}{\pi^5 EI}.\sin\frac{\pi x}{l}$$

$At\ x = \frac{l}{2}, \max deflection \quad [\because sin\frac{\pi}{2} = 1]$

$$Y_{max} = \frac{4\omega l^4}{\pi^5 EI}\ sin\frac{\pi}{2}(\frac{l}{2})$$

$$Y_{max} = \frac{\omega l^4}{76.5\ EI}$$

d) Galerkin's Method:

In this method

$$\int_0^l Y.R\ dx = 0$$

$$\Longrightarrow \int_0^l \left[\left(asin\frac{\pi x}{l}\right)\left(aEI\frac{\pi^4}{l^4}sin\frac{\pi x}{l} - \omega\right)\right]dx = 0$$

$$\Longrightarrow \int_0^l \left[a^2EI\frac{\pi^4}{l^4}sin^2\frac{\pi x}{l} - a\omega sin\frac{\pi x}{l}\right]dx = 0$$

$$\Longrightarrow \int_0^l \left[a^2EI\frac{\pi^4}{l^4}[\frac{1}{2}(1 - cos\frac{2\pi x}{l}) - a\omega sin\frac{\pi x}{l}\right]dx = 0$$

$$\Longrightarrow \left[a^2EI\frac{\pi^4}{l^4}[\frac{1}{2}\left(1 - \left\{\left(x - \frac{1}{2\pi}\right)sin^2\frac{2\pi x}{l}\right\}\right] + a\omega\ \frac{l}{\pi}cos\frac{\pi x}{l}\right]_0^l = 0$$

$$a^2EI\frac{\pi^4}{l^4}\left[\frac{l}{2}\right] - 2a\omega\left(\frac{l}{\pi}\right) = 0$$

$$\therefore a = \frac{2\omega l}{\pi}.\frac{2l^3}{EI\pi^4}$$

$$a = \frac{4\omega l^3}{\pi^5 EI}$$

Hence the trial Function

$$Y = \frac{4\omega l^4}{\pi^5 EI}.\sin\frac{\pi x}{l}$$

$At\ x = \frac{l}{2}$, max $deflection$ $[\because sin\frac{\pi}{2} = 1]$

$$Y_{max} = \frac{4\omega l^4}{\pi^5 EI}\ sin\frac{\pi}{2}(\frac{l}{2})$$

$$Y_{max} = \frac{4\omega l^4}{\pi^5\ EI}$$

$$Y_{max} = \frac{\omega l^4}{76.5\ EI}$$

Verification,

We know that simply supported beam is subjected to uniformly distributed load, maximum deflection is,

$$Y_{max} = \frac{5}{384}\frac{\omega l^4}{EI}$$

$$= 0.01\frac{\omega l^4}{EI}$$

3) i) What is constitutive relationship? Express the constitutive relations for a linear elastic isotropic material including initial stress and strain. (4)

[Nov/Dec 2009]

Solution:

It is the relationship between components of stresses in the members of a structure or in a solid body and components of strains. The structure or solids bodies under consideration are made of elastic material that obeys Hooke's law.

$$\{\sigma\} = [D]\{e\}$$

Where

[D] is a stress – strain relationship matrix or constitute matrix.

The constitutive relations for a linear elastic isotropic material is

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \delta_{xy}\\ \delta_{yz}\\ \delta_{zx}\end{Bmatrix} = \frac{E}{(1+v)(1-2v)}\begin{bmatrix}(1-v) & 0 & 0 & 0 & 0 & 0\\ v & (1-v) & 0 & 0 & 0 & 0\\ v & v & (1-v) & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2}\end{bmatrix}\begin{bmatrix}e_x\\ e_y\\ e_z\\ v_{xy}\\ v_{yz}\\ v_{zx}\end{bmatrix}$$

ii) Consider the differential equation $\frac{d^2y}{dx^2} + 400x^2 = 0$ for $0 \le x \le 1$ subject to boundary conditions Y(0) = 0, Y(1) = 0. The functions corresponding to this problem, to be eternized is given by $I = \int_0^l \left\{-0.5\left[\frac{dy}{dx}\right]^2 + 400x^2Y\right\}$. Find the solution of the problem using Ray Light Ritz method by considering a two term solution as $Y(x) = c_1x(1-x) + c_2x^2(1-x)$ (12)

Given data

Differential equation = $\frac{d^2y}{dx^2} + 400x^2 = 0$ for $0 \leq x \leq 1$

Boundary conditions Y(0) = 0, Y(1) = 0

$I = \int_0^l \left\{-0.5\left[\frac{dy}{dx}\right]^2 + 400x^2Y\right\}$

$Y(x) = c_1x(1-x) + c_2x^2(1-x)$

To find:

Rayleigh- Ritz method

Formula used

$\frac{\partial I}{\partial c_1} = 0$

$$\frac{\partial I}{\partial c_2} = 0$$

Solution:

$$Y(x) = c_1x(1-x) + c_2x^2(1-x)$$

$$Y(x) = c_1x(x-x^2) + c_2(x^2-x^3)$$

$$\frac{dy}{dx} = c_1(1-2x) + c_2(2x-3x^2)$$

$$= c_1(1-2x) + c_2x(2-3x)$$

$$\left[\frac{dy}{dx}\right]^2 = [c_1(1-2x) + c_2x(2-3x)^2]^2$$

$$= c_1^2(1-4x+4x^2) + c_2^2x^2(4-12x+9x^2) + 2c_1c_2x(1-2x)(2-3x)$$

$$= c_1^2(1-4x+4x^2) + c_2^2x^2(4-12x+9x^2) + 2c_1c_2x(2-3x-4x+6x^2)$$

$$\left[\frac{dy}{dx}\right]^2 = c_1^2(1-4x+4x^2) + c_2^2x^2(4-12x+9x^2) + 2c_1c_2x(2-7x+6x^2)$$

We know that

$$I = \int_0^l [-0.5\left(\frac{dy}{dx}\right)^2 + 400x^2y] = \frac{-1}{2}\int_0^l \left[\frac{dy}{dx}\right]^2 + 400\int_0^l x^2y$$

$$= \int_0^l c_1^2(1-4x+4x^2) + c_2^2x^2(4-12x+9x^2) + 2c_1c_2x(2-7x+6x^2)$$

$$+ 400[\int_0^l x^2[c_1x(1-x) + c_2x^2(1-x)]$$

By Solving

$$I = \frac{-1}{2}\left[\frac{c_1^2}{3} + \frac{2}{15}c_2^2 + \frac{1}{3}c_1c_2\right] + 400\left[\frac{c_1}{20} + \frac{c_2}{30}\right]$$

$$I = \frac{-1}{6}c_1^2 - \frac{1}{15}c_2^2 - \frac{1}{6}c_1c_2 + 20c_1 + \frac{40}{3}c_2$$

$$\frac{\partial I}{\partial c_1} = 0$$

$$\Rightarrow \frac{-1}{6} \times 2c_1 - \frac{1}{6}c_2 + 20 = 0$$

$$\Rightarrow \frac{-1}{3} \times c_1 - \frac{1}{6}c_2 + 20 = 0 \qquad \ldots\ldots\ldots..(1)$$

Similarly,

$$\frac{\partial I}{\partial c_2} = 0$$

$$\Rightarrow \frac{-2}{15}c_2 - \frac{1}{6}c_1 + \frac{40}{3} = 0 \qquad \ldots\ldots\ldots..(2)$$

By Solving (1) and (2)

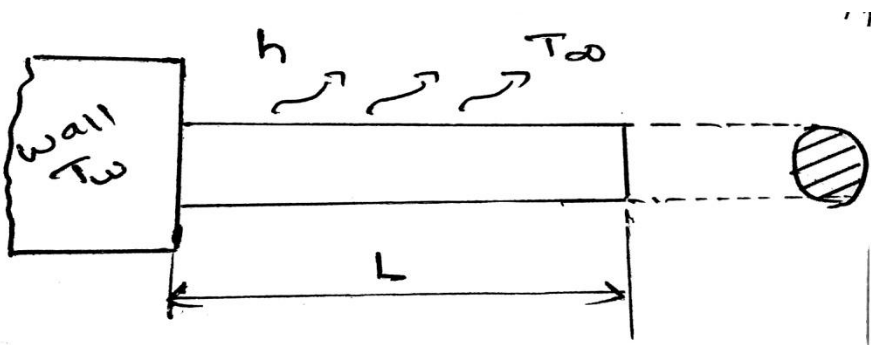
$$c_1 = \frac{80}{3};\ c_1 = \frac{200}{3}$$

We know that

$$Y = c_1x(1-x) + c_2x^2(1-x)$$

$$Y = \frac{80}{3}x(1-x) + \frac{200}{3}x^2(1-x)$$

4) Consider a 1mm diameter, 50m long aluminum pin-fin as shown in figure used to enhance the heat transfer from a surface wall maintained at 300°C. Calculate the temperature distribution in a pin-fin by using Rayleigh – Ritz method. Take, $k = 200 w/_{m°C}$ for aluminum h= $200 w/_{m^2°C}$, $T_\infty = 30°C$.



$$k\frac{d^2T}{dx^2} = \frac{Ph}{A}(T - T_\infty)\ ,\ T(0) = T_w = 300°C,\ q_L = KA\frac{dT}{dx}(L) = 0$$ **(insulated tip)**

Given Data:

The governing differential equation

$$k\frac{d^2T}{dx^2} = \frac{Ph}{A}(T - T_\infty)$$

Diameter d = 1mm = 1x10^{-3}m

Length L = 50mm = 50x10^{-3}m

Thermal K = $200w/_{m°C}$

Conductivity Heat transfer co-efficient h = $200w/_{m°C}$

Fluid Temp $T_\infty = 30°C$.

Boundary Conditions $T(0) = T_w = 300°C$

$q_L = KA\frac{dT}{dx}(L) = 0$

To Find:

Ritz Parameters

Formula used

$$\pi = strain\ energy - work\ done$$

Solution:

The equivalent functional representation is given by,

$\pi = strain\ energy - work\ done$

$\pi = u - v$

$$\pi = \int_0^L \frac{1}{2}K\left[\frac{dT}{dx}\right]^2 dx + \int_0^L \frac{1}{2}\frac{Ph}{A}[T - T_\infty]^2 dx - q_L T_L \qquad \dots\dots\dots\dots(1)$$

$$\pi = \int_0^L \frac{1}{2}K\left[\frac{dT}{dx}\right]^2 dx + \int_0^L \frac{1}{2}\frac{Ph}{A}[T - T_\infty]^2 dx \qquad \dots\dots\dots\dots..(2)$$

$\because q_L = 0$

Assume a trial function

Let

$$T(x) = a_0 + a_1x + a_2x^2 \qquad \dots\dots\dots\dots\dots..(3)$$

Apply boundary condition

at x = 0, T(x) = 300

$300 = a_0 + a_1(0) + a_2(0)^2$

$a_0 = 300$

Substituting a_0 value in equation (3)

$$T(x) = 300 + a_1x + a_2x^2 \qquad \dots\dots\dots\dots\dots..(4)$$

$$\Rightarrow \frac{dT}{dx} = a_1 + 2a_2 x \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Substitute the equation (4), (5) in (2)

$$\pi = \int_0^l \frac{1}{2} k\,(a_1 + 2a_2 x)^2 dx + \int_0^l \frac{1}{2}\frac{Ph}{A}[270 + a_1 + a_2 x^2]^2 dx.$$

$[\because (a+b)^2 = a^2 + b^2 + 2ab; (a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$$\pi = \frac{k}{2}\int_0^l (a_1^2 + 4a_2^2 x^2 + 4a_1\, a_2 x) + \frac{Ph}{2A}\left[\int_0^l 270^2 + {a_1}^2 x^2 + {a_2}^2 x^4 + 540 a_1 x + 2a_1 x^3 + 540 a_2 x^2\right] dx$$

$$\pi = \frac{k}{2}\left[(a_1^2 x + \frac{4a_2^2 x^3}{3} + \frac{4a_1 a_2 x^2}{2}\right]_0^{50x10^{-3}}$$

$$+ \frac{Ph}{2A}\left[72900k + \frac{{a_1}^2 x^3}{3} + \frac{{a_2}^2 x^5}{5} + \frac{540 a_1 x^2}{2} + \frac{2a_1 a_2 x^4}{4} + \frac{540 a_2 x^3}{3}\right]_0^{50x10^{-3}}$$

$$[\because l = 50x10^{-3}]$$

$$\pi = \frac{k}{2}(50\times10^{-3})a_1^2 + \frac{4a_2^2(50\times10^{-3})^3}{3} + \frac{4a_1 a_2(50\times10^{-3})^2}{2}$$

$$+ \frac{Ph}{2A}\left[72900k + \frac{{a_1}^2(50\times10^{-3})^3}{3} + \frac{{a_2}^2(50\times10^{-3})^5}{5}\right]$$

$$\pi = \frac{200}{2}(50\times10^{-3}a_1^2 + 1.666\times10^{-4}a_2^2 + 50\times10^{-3}a_1 a_2) + \frac{\pi\times10^{-3}\times20}{2\times\frac{\pi}{2}\times(10^{-3})^2}$$

$$= 364.5 + 4.166\times10^{-5}a_1^2 + 6.25\times10^{-8}a_2^2 + 0.675a_1 + 3.125\times10^{-6}a_1 a_2 + 0.0225a_2$$

$$\pi = [5a_1^2 + 0.0166a_2^2 + 0.5a_1 a_2] + [14.58\times10^{-7} + 1.6691^2 + 2.5\times10^{-3}a_2^2 + 2700]a_1$$

$$+ 0.125\, a_1 a_2 + 900a_2]$$

$$\pi = [6.66a_1^2 + 0.0191a_2^2 + 0.625a_1 a_2 + 2700a_1 + 900a_2 + 14.58\times10^7]$$

Apply $\frac{\partial\pi}{\partial a_2} = 0$

$$\Rightarrow 13.32a_1 + 0.625a_2 + 27000 = 0$$

$$13.32a_1 + 0.625a_2 = -+27000 \qquad \ldots\ldots\ldots\ldots\ldots (6)$$

$$\Rightarrow 0.625a_1 + 0.382a_2 + 900 = 0$$

$$0.625a_1 + 0.382a_2 = -900 \qquad \ldots\ldots\ldots\ldots.. (7)$$

Solve the equation (6) and (7)

$$13.32a_1 + 0.625a_2 = -+27000 \qquad \ldots\ldots\ldots\ldots\ldots (6)$$

$$0.625a_1 + 0.382a_2 = -900 \qquad \ldots\ldots\ldots\ldots.. (7)$$

(6) x 0.625

$$8.325a_1 + 0.3906a_2 = -16875 \qquad \dots\dots\dots\dots..(8)$$

(7) x -13.32

$$-8.325a_1 - 0.5088a_2 = 11988 \qquad \dots\dots\dots\dots..(9)$$

$$-0.1182a_2 = -4887$$

$$a_2 = 41345$$

Sub $a_2$value in equation (6)

$$13.32a_1 + 0.625(41345) = - + 27000$$

$$a_1 = -3967.01$$

Sub $a_0, a_1$and $a_2$values in equation (3)

$$T = 300 - 3697.01x + 41345x^2$$

5) Explain briefly about General steps of the finite element analysis.

[Nov/Dec 2014]

Step: 1

Discretization of structure

The art of sub dividing a structure into a convenient number of smaller element is known as discretization.

Smaller elements are classified as

i) One dimensional element

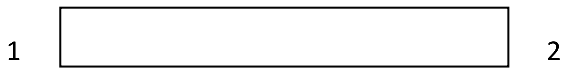
ii) Two dimensional element

iii) Three dimensional element

iv) Axisymmetric element

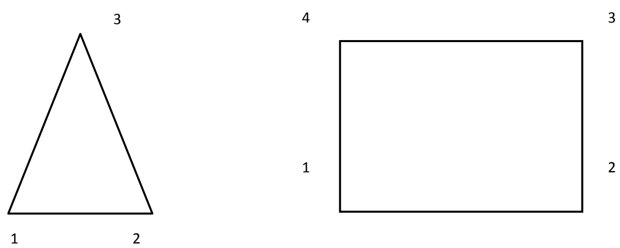
(i) One dimensional element:-

a. A bar and beam elements are considered as one dimensional element has two nodes, one at each end as shown.



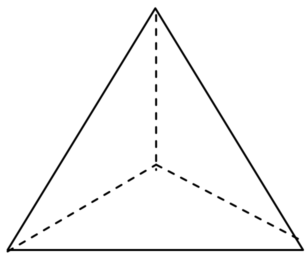
(ii) Two Dimensional element:-

Triangular and Rectangular elements are considered as 2D element. These elements are loaded by forces in their own plane.



iii) Three dimensional element:-

The most common 3D elements are tetrahedral and lexahendral (Brick) elements. These elements are used for three dimensional stress analysis problems.



iv) Axisymmetric element:-

The axisymmetric element is developed by relating a triangle or quadrilateral about a fixed axis located in the plane of the element through 360^0. When the geometry and loading of the problems are axisymmetric these elements are used.

The stress-strain relationship is given by,

$\sigma = Ee$

Where, $\sigma =$ Stress in x direction

E = Modulus of elasticity

Step 2:- Numbering of nodes and Elements:-

The nodes and elements should be numbered after discretization process. The numbering process is most important since if decide the size of the stiffness matrix and it leads the reduction of memory requirement . While numbering the nodes, the following condition should be satisfied.

{Maximum number node} – {Minimum number node} = minimum

Longer Side Numbering Process:

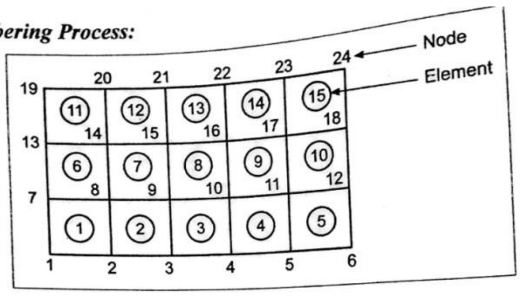
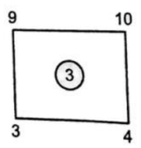


Fig. 1.6. (a)

[*Note:* Number with circle denotes element.
Number without circle denotes node]

Considering element (3),



Maximum node number = 10

Minimum node number = 3

Difference = 7

Shorter Side Numbering Process:

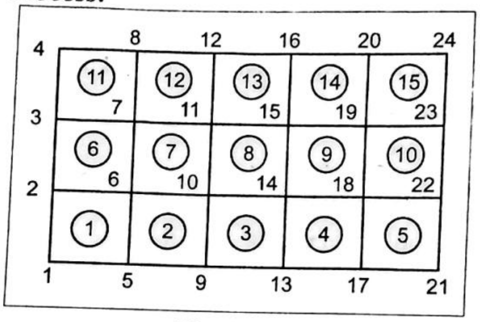
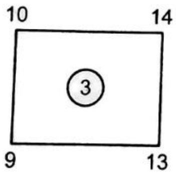


Fig. 1.6. (b)

Considering the same element (3).



Maximum node number = 14

Minimum node number = 9

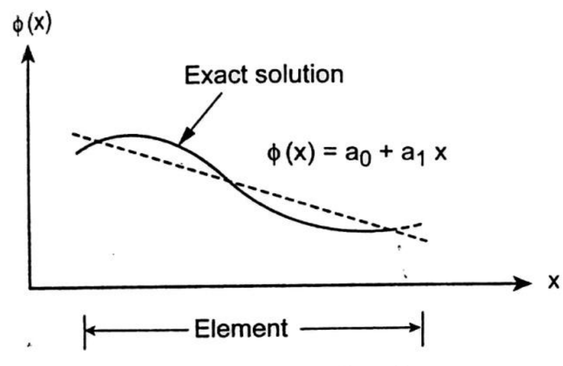
Difference = 5 ... (1.2)

From equation (1.1) and (1.2), we came to know, shorter side numbering process is followed in the finite element analysis and it reduces the memory requirements.

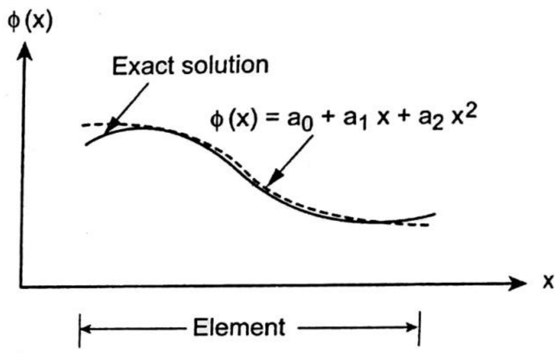
Step 3:

Selection of a displacement function or a Interpolation function:-

It involves choosing a displacement function within each element. Polynomial of linear, quadratic and cubic form are frequently used as displacement Function because they are simple to work within finite element formulation. $d(x)$.



(a) Linear approximation



(b) Quadratic approximation

The polynomial type of interpolation functions are mostly used due to the following reasons.

1. It is easy to formulate and computerize the finite element equations.
2. It is easy to perform differentiation or Intigration.
3. The accuracy of the result can be improved by increasing the order of the polynomial.

Step – 4:-

Define the material behavior by using strain – Displacemnt and stress. Strain relationship:

Strain – displacement and stress – strain relationship and necessary for deriving the equatins for each finite element.

In case of the dimensional deformation, the strain – displacement relationship is given by,

$e = \frac{du}{dx}$

Where, u → displacement field variable x direction e → strain.

Step – 5

Deviation of equation is in matrix form as

$$\begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_4 \end{Bmatrix} \begin{bmatrix} k_{11}, & k_{12}, & k_{13} \ldots\ldots & k_{1n} & u_1 \\ k_{21}, & k_{22}, & k_{23} \ldots\ldots & k_{2n} & u_2 \\ k_{31}, & k_{32}, & k_{33} \ldots\ldots & k_{3n} & u_3 \\ & & & & \vdots \\ k_{n1}, & k_{42}, & k_{43} \ldots\ldots & k_{4n} & u_n \end{bmatrix} \begin{Bmatrix} \\ \vdots \\ \\ \end{Bmatrix}$$

In compact matrix form as.

Where,

e is a element, $\{F\}$ is the vector of element modal forces, $[k]$ is the element stiffness matrix and the equation can be derived by any one of the following methods.

(i) Direct equilibrium method.

(ii) Variational method.

(iii) Weighted Residual method.

Step (6):-

Assemble the element equations to obtain the global or total equations.

The individual element equations obtained in step s are added together by using a method of super position i.e. direction stiffness method. The final assembled or global equation which is in the form of

$$\{f\} = [k]\{u\}$$

Where, $[F]$ → Global Force Vector

$[K]$ → Global Stiffness matrix

$\{u\}$ → Global displacement vector.

Step (7):-

Applying boundary conditions:

The global stiffness matrix $[k]$ is a singular matrix because its determinant is equal to zero. In order to remove the singularity problem certain boundary conditions are applied so that the structure remains in place instead of moving as a rigid body.

Step (8):-

Solution for the unknown displacement formed in step (6) simultaneous algebraic equations matrix form as follows.

Deviation of equation is in matrix form as

$$\begin{array}{c} f_1 \\ f_2 \\ f_3 \\ f_3 \\ \vdots \\ f_4 \end{array} \quad \begin{array}{cccc} k_{11}, & k_{12}, & k_{13} \ldots.. & k_{1n} \\ k_{21}, & k_{22}, & k_{23} \ldots.. & k_{2n} \\ k_{31}, & k_{32}, & k_{33} \ldots.. & k_{3n} \\ k_{41}, & k_{42}, & k_{43} \ldots.. & k_{4n} \\ & & & \\ k_{n1}, & k_{42}, & k_{43} \ldots.. & k_{4n} \end{array} \quad \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \\ u_n \end{array} \quad \begin{array}{c} \\ \\ \\ \\ \vdots \\ \\ \end{array}$$

These equation can be solved and unknown displacement $\{u\}$ calculated by using Gauss elimination.

Step (9):-

Computation of the element strains and stresses from the modal displacements $\{\boldsymbol{u}\}$:

In structural stress analysis problem. Stress and strain are important factors from the solution of displacement vector $\{u\}$, stress and strain value can be calculated. In case of 1D the strain displacement can strain.

$$e = \frac{d}{u}$$

$$= u_2 - u_1$$

Where, u_1 and u_2 are displacement at model 1 and 2

$x_1 - x_2$ = Actual length of the element from that we can find the strain value,

By knowing the strain, stress value can be calculated by using the relation.

Stress $\sigma = Ee$

Where, E → young's modulus

e → strain

Step – 10

Interpret the result (Post processing)

Analysis and Evaluation of the solution result is referred to as post-processing. Post processor computer programs help the user to interpret the results by displaying them in graphical form.

6) Explain in detail about Boundary value, Initial Value problems.

The objective of most analysis is to determine unknown functions called dependent variables, that are governed by a set of differential equations posed in a given domain. Ω and some conditions on the boundary Γ of the domain. Often, a domin not including its boundary is called an open domain. A domain boundary is called an open domain. A domain Ω with its boundary Γ is called a closed domain.

Boundary value problems:- Steady state heat transfer : In a fin and axial deformation of a bar shown in fig. Find $u(x)$ that satisfies the second – order differential equation and boundary conditions.

$$\frac{-d}{dx}\left(a\,\frac{du}{dx}\right) + \; cu = f \;\text{ for }\; 0 < x < L$$

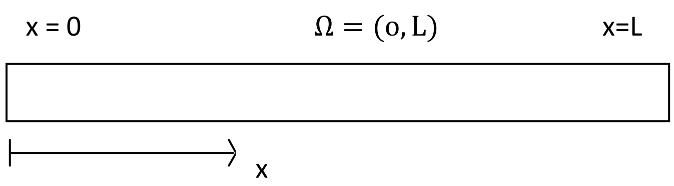
$$u(o) = \; u_0, \left(a\,\frac{du}{dx}\right)_{x=L} = \; q_0$$

i) Bending of elastic beams under Transverse load : find $u(x)$ that satisfies the fourth order differential equation and boundary conditions.

$$\frac{d^2}{dx^2} \;\; b\,\left(\frac{d^2u}{dx^2}\right) + \; cu = F \;\text{ for } 0 < x < -L$$

$$u(o) = u_0, \left(\frac{du}{dx}\right)_{x=0} = d_0$$

$$\left[\frac{d}{dx}\left(b\,\frac{d^2u}{dx^2}\right)\right]_{x=L} = m_0 \,.\, \left[\left(b\,\frac{d^2u}{dx^2}\right)\right] \;\; = v_0$$



Initial value problems:-

i) A general first order equation:-

Find $u(t)$ that satisfies the first-order differential equation and initial condition.

Equation and initial condition:-

$$a\,\frac{du}{dt} + \; cu = F \text{ for } 0 < t \;\leq T$$

$$u(0) = \; u_0.$$

ii) A general second order equation:-

Find $u(t)$ that satisfies the second – order differential equation and initial conditions:-

$$a\,\frac{du}{dt} + b\,\frac{d^2u}{dt^2} + cu = F \text{ for } 0 < t \le T$$

$$u(o) = u_0, \left[b\frac{du}{dt}\right]_{t=0} = v_0$$

Eigen value problems:-

(i) Axial vibration of a bar:

Find $u(x)$ and l that satisfy the differential equation and boundary conditions.

$$\frac{-d}{dx}\left[a\frac{du}{dx}\right] - \lambda u = 0 \text{ for } o < x < L$$

$$u(o) = 0, \left[a\frac{du}{dx}\right]_{x=L} = 0$$

(ii) Transverse vibration of a membrane:-

Find $u\,(x, y)$ and λ that satisfy the partial differential equation and boundary condition.

$$-\frac{d}{dx}\left[a_1\frac{du}{dx}\right] + \frac{d}{dy}\left[a_2\frac{du}{dy}\right] - \lambda u = 0 \text{ in } \Omega$$

$$u = 0 \text{ on } \Gamma_q$$

The values of λ are called cigen values and the associated functions u are called cigen functions.

b) A simple pendulum consists of a bob of mass $m(kg)$attached to one end of a rod of length $l(m)$ and the other end is pivoted to fixed point 0.

Soln:-

$$F = \frac{d}{dt}\,(mv) = ma$$

$$Fx = m.\frac{dv_x}{dt}$$

$$-mg\sin\theta = ml\,\frac{d^2Q}{dt^2}$$

or

$$\frac{d^2Q}{dt^2} + \frac{g}{l}\sin Q = 0$$

$$\frac{d^2Q}{dt^2} + \frac{s}{l}Q = 0$$

$$\frac{dQ}{dt} + (o) = U_{0.}$$

$$Q(t) = As\ in\ \lambda t + B \cos \lambda\ t.$$

Where,

$\lambda = \sqrt{\frac{s}{l}}$ and A and B are constant to be determined using the initial condition we obtain.

$A - \frac{v_0}{\lambda}, B = \theta_0$

the solution to be linear problem is

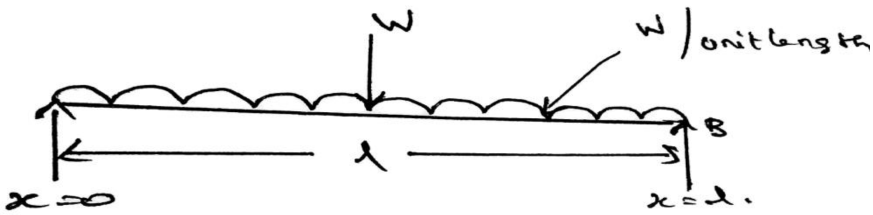
$\theta(t) = \frac{v_0}{\lambda}\ Sin\ \wedge t + 0.Cos\ \lambda t$

for zero initial velocity and non zero initial position θ_0 , we have.

$\theta(t) = \theta_0 \cos \lambda t.$

7) A simply supported beam subjected to uniformly distributed load over entire span and it is subject to a point load at the centre of the span. Calculate the bending moment and deflection at imdspan by using Rayleish – Ritz method. (Nov/Dec 2008).

Given data:-



To Find:

1. Deflection and Bending moment at mid span.
2. Compare with exact solutions.

Formula used

$\pi = strain\ energy - work\ done$

Solution:

We know that,

Deflection, $y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$ ------------------------ (1)

Total potential energy of the beam is given by,

$$\pi = U - H \quad \text{-------------------------} \quad (2)$$

Where, U – Strain Energy.

H – Work done by external force.

The strain energy, U of the beam due to bending is given by,

$$U = \frac{EI}{2}\int_0^1 \left(\frac{d^2y}{dx^2}\right)^2 \, dx \quad \text{-------------------------} \quad (3)$$

$$\frac{dy}{dx} = a_1 \cos\frac{\pi x}{l} \times \left(\frac{\pi}{l}\right) + a_2 \cos\frac{3\pi x}{l} \times \left(\frac{3\pi}{l}\right)$$

$$\frac{dy}{dx} = \frac{a_1 \pi x}{l}\cos\frac{\pi x}{l} + \frac{a_2 3\pi x}{l}\cos\frac{3\pi x}{l}$$

$$\frac{d^2y}{dx^2} = -\frac{a_1\pi}{l}\sin\frac{\pi x}{l} \times \frac{\pi}{l} - \frac{a_2 3\pi}{l}\sin\frac{3\pi x}{l} \times \frac{3\pi}{l}$$

$$\frac{d^2y}{dx^2} = -\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - 9\frac{a_2\pi^2}{l^2}\sin\frac{3\pi x}{l} \quad \text{-------------------------} \quad (4)$$

Substituting $\frac{d^2y}{dx^2}$ value in equation (3),

$$U = \frac{EI}{2}\int_0^l \left(-\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} - 9\frac{a_2\pi^2}{l^2}\sin\frac{3\pi x}{l}\right)^2 dx$$

$$= \frac{EI}{2}\int_0^l \left(\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} + 9\frac{a_2\pi^2}{l^2}\sin\frac{3\pi x}{l}\right)^2 dx$$

$$= \frac{EI}{2}\frac{\pi^4}{l^4}\int_0^l \left[a_1^2\sin^2\frac{\pi x}{l} + 81a_2^2\sin^2\frac{3\pi x}{l} + 2\,a_1\sin\frac{\pi x}{l}\,.9\,a_2\sin\frac{3\pi x}{l}\right] dx$$

$$[\therefore\ (a+b)^2 = a^2 + b^2 + 2ab]$$

$$U = \frac{EI}{2}\frac{\pi^4}{l^4}\int_0^l \left[a_1^2\sin^2\frac{\pi x}{l} + 81a_2^2\sin^2\frac{3\pi x}{l} + 18\,a_1a_2\sin\frac{\pi x}{l}\,.\sin\frac{3\pi x}{l}\right] dx \quad \text{--------} \quad (5)$$

$$\int_0^l a_1^2\sin^2\frac{\pi x}{l}dx = a_1^2\int_0^l \frac{1}{2}\left(1-\cos\frac{2\pi x}{l}\right)dx \qquad \left[\therefore\ \sin^2 x = \frac{1-\cos 2x}{2}\right]$$

$$= a_1^2\frac{1}{2}\int_0^l\left(1-\cos\frac{2\pi x}{l}\right)dx$$

$$= \frac{a_1^2}{2}\left[\int_0^l dx - \int_0^1 \cos\frac{2\pi x}{l}\,dx\right]$$

$$= \frac{a_1^2}{2}\left[(x)_0^l - \left(\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right)_0^l\right]$$

$$= \frac{a_1^2}{2}\left[l - 0 - \frac{1}{2\pi}\left(\sin\frac{2\pi l}{l} - \sin 0\right)\right]$$

$$= \frac{a_1^2}{2}\left[l - \frac{1}{2\pi}(0-0)\right] = \frac{a_1^2\, l}{2} \quad [\therefore\ \sin 2\pi = 0;\ \sin 0 = 0]$$

$$\boxed{\int_0^l a_1^2 \sin^2\frac{\pi x}{l}\, dx = \frac{a_1^2\, l}{2}} \quad \text{------------------------} \quad (6)$$

Similarly,

$$\int_0^l 81\, a_2^2 \sin^2\frac{3\pi x}{l}\, dx = 81 a_2^2 \int_0^l \frac{1}{2}\left(1 - \cos\frac{6\pi x}{l}\right) dx \quad \left[\therefore\ \sin^2 x = \frac{1-\cos 2x}{2}\right]$$

$$= 81 a_2^2\, \frac{1}{2}\int_0^l \left(1 - \cos\frac{6\pi x}{l}\right) dx$$

$$= \frac{81 a_2^2}{2}\left[\int_0^l dx - \int_0^l \cos\frac{6\pi x}{l}\, dx\right]$$

$$= \frac{81 a_2^2}{2}\left[(x)_0^l - \left(\frac{\sin\frac{6\pi x}{l}}{\frac{6\pi}{l}}\right)_0^l\right]$$

$$= \frac{81 a_2^2}{2}\left[l - 0 - \frac{1}{6\pi}\left(\sin\frac{6\pi l}{l} - \sin 0\right)\right]$$

$$= \frac{81 a_2^2}{2}\left[l - \frac{1}{6\pi}(0-0)\right] = \frac{a_1^2\, l}{2} \qquad [\therefore\ \sin 6\pi = 0;\ \sin 0 = 0]$$

$$\boxed{\int_0^l 81 a_2^2 \sin^2\frac{3\pi x}{l}\, dx = \frac{81 a_2^2\, l}{2}} \quad \text{------------------------} \quad (7)$$

$$\int_0^l 18\, a_1 a_2 \sin\frac{\pi x}{l} \,.\sin\frac{3\pi x}{l}\, dx = 18\, a_1 a_2 \int_0^l \sin\frac{\pi x}{l} \,.\sin\frac{3\pi x}{l}\, dx$$

$$= 18\, a_1 a_2 \int_0^l \sin\frac{3\pi x}{l} \,.\sin\frac{\pi x}{l}\, dx$$

$$= 18\, a_1 a_2 \int_0^l \frac{1}{2}\left(\cos\frac{2\pi x}{l} - \cos\frac{4\pi x}{l}\right) dx$$

$$\left[\therefore\ \sin A \sin B = \frac{\cos(A-B) - \cos(A+B)}{2}\right]$$

$$= \frac{18\, a_1 a_2}{2}\left[\int_0^l \cos\frac{2\pi x}{l}\, dx - \int_0^l \cos\frac{4\pi x}{l}\, dx\right]$$

$$= \frac{18\, a_1 a_2}{2}\left[\left(\frac{\sin\frac{2\pi x}{l}}{\frac{2\pi}{l}}\right)_0^l - \left(\frac{\sin\frac{4\pi x}{l}}{\frac{4\pi}{l}}\right)_0^l\right]$$

$$= 9\, a_1 a_2 [0 - 0] = 0 \qquad [\therefore\ \sin 2\pi = 0; \sin 4\pi = 0;\ \sin 0 = 0]$$

$$\int_0^l 18\, a_1 a_2 \sin\frac{\pi x}{l} . \sin\frac{3\pi x}{l} dx = 0 \qquad \text{--------- (8)}$$

Substitute (6), (7) and (8) in equation (5),

$$U = \frac{EI}{2}\frac{\pi^4}{l^4}\left[\frac{a_1^2\, l}{2} + \frac{81 a_2^2\, l}{2} + 0\right]$$

$$U = \frac{EI}{4}\frac{\pi^4\, l}{l^4}[a_1^2 \ + 81 a_2^2]$$

$$\text{Strain Energy, } U = \frac{EI\pi^4}{4l^3}[a_1^2 + 81 a_2^2] \qquad \text{--------- (9)}$$

Work done by external forces,

$$H = \int_0^l \omega\, y\, dx + W\, y_{max} \qquad \text{--------- (10)}$$

$$\int_0^l \omega\, y\, dx = \frac{2\omega l}{\pi}\left(a_1 + \frac{a_2}{3}\right) \qquad \text{--------- (11)}$$

We know that, $y = a_1 \sin\frac{\pi x}{l} + a_2 \sin\frac{3\pi x}{l}$

In the span, deflection is maximum at $x = \frac{1}{2}$

$$y_{max} = a_1 \sin\frac{\pi \times \frac{1}{2}}{l} + a_2 \sin\frac{3\pi \times \frac{1}{2}}{l}$$

$$= a_1 \sin\frac{\pi}{2} + a_2 \sin\frac{3\pi}{2} \qquad \left[\therefore \sin\frac{\pi}{2} = 1;\ \sin\frac{3\pi}{2} = -1\right]$$

$$y_{max} = a_1 - a_2 \qquad \text{--------- (12)}$$

Substitute (11) and (12) values in equation (8),

$$H = \frac{2\omega l}{\pi}\left(a_1 + \frac{a_2}{3}\right) + W\,(a_1 - a_2) \qquad \text{--------- (13)}$$

Substituting U and H values in equation (2), we get

$$\pi = \frac{EI\pi^4}{4l^3}[a_1^2 + 81a_2^2] - \left[\frac{2\omega l}{\pi}\left(a_1 + \frac{a_2}{3}\right) + W\ (a_1 - a_2)\right]$$

$$\pi = \frac{EI\pi^4}{4l^3}[a_1^2 + 81a_2^2] - \frac{2\omega l}{\pi}\left(a_1 + \frac{a_2}{3}\right) - W\ (a_1 - a_2) \quad \text{-------------------------} \quad (14)$$

For stationary value of π, the following conditions must be satisfied.

$$\frac{\partial \pi}{\partial a_1} = 0 \text{and} \frac{\partial \pi}{\partial a_2} = 0$$

$$\frac{\partial \pi}{\partial a_1} = \frac{EI\pi^4}{4l^3}(2a_1) - \frac{2\omega l}{\pi} - W = 0$$

$$\frac{EI\pi^4}{2l^3}a_1 - \frac{2\omega l}{\pi} - W = 0$$

$$\frac{EI\pi^4}{2l^3}a_1 = \frac{2\omega l}{\pi} + W$$

$$\boxed{a_1 = \frac{2l^3}{EI\pi^4}\left(\frac{2\omega l}{\pi} + W\right)} \quad \text{-------------------------} \quad (15)$$

$$\frac{\partial \pi}{\partial a_2} = \frac{EI\pi^4}{4l^3}(162a_2) - \frac{2\omega l}{\pi}\left(\frac{1}{3}\right) + W = 0$$

Similarly,

$$\frac{EI\pi^4}{4l^3}(162a_1) - \frac{2\omega l}{\pi} + W = 0$$

$$\frac{EI\pi^4}{2l^3}(162a_1) = \frac{2\omega l}{\pi} - W$$

$$\boxed{a_2 = \frac{2l^3}{81EI\pi^4}\left(\frac{2\omega l}{3\pi} - W\right)} \quad \text{-------------------------} \quad (16)$$

From equation (12), we know that,

Maximum deflection, $y_{max} = a_1 - a_2$

$$y_{max} = \frac{2l^3}{EI\pi^4}\left(\frac{2\omega l}{\pi} + W\right) - \frac{2l^3}{81EI\pi^4}\left(\frac{2\omega l}{3\pi} - W\right)$$

$$y_{max} = \frac{4\omega l^4}{EI\pi^5} + \frac{2Wl^3}{EI\pi^4} - \frac{4\omega l^4}{243EI\pi^5} + \frac{2Wl^3}{81EI\pi^4}$$

$$y_{max} = 0.0130\ \frac{\omega l^4}{EI} + 0.0207\ \frac{Wl^3}{EI} \quad \text{----------------------} \quad (17)$$

We know that, simply supported beam subjected to uniformly distributed load, maximum deflection is,

$$y_{max} = \frac{5}{384}\frac{\omega l^4}{EI}$$

Simply supported beam subjected to point load at centre, maximum deflection is,

$$y_{max} = \frac{\omega l^3}{48EI}$$

So, total deflection,

$$y_{max} = \frac{5}{384}\frac{\omega l^4}{EI} + \frac{\omega l^3}{48EI}$$

$$\boxed{y_{max} = 0.0130\ \frac{\omega l^4}{EI} + 0.0208\ \frac{Wl^3}{EI}} \quad \text{----------------------} \quad (18)$$

From equations (17) and (18), we know that, exact solution and solution obtained by using Rayleigh-Ritz method are same.

Bending Moment at Mid span

We know that,

$$\text{Bending moment, M = EI}\ \frac{d^2y}{dx^2} \quad \text{----------------------} \quad (19)$$

From equation (9), we know that,

$$\frac{d^2y}{dx^2} = -\left[\frac{a_1\pi^2}{l^2}\sin\frac{\pi x}{l} + \frac{a_2\,9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]$$

Substitute a_1 and a_2 values from equation (15) and (16),

$$\frac{d^2y}{dx^2} = -\left[\frac{2l^3}{EI\pi^4}\left(\frac{2\omega l}{\pi} + W\right) \times \frac{\pi^2}{l^2}\sin\frac{\pi x}{l} + \frac{2l^3}{81EI\pi^4}\left(\frac{2\omega l}{3\pi} - W\right) \times \frac{9\pi^2}{l^2}\sin\frac{3\pi x}{l}\right]$$

Maximum bending occurs at $x = \frac{l}{2}$

$$= -\left[\frac{2l^3}{EI\pi^4}\left(\frac{2\omega l}{\pi} + W\right) \times \frac{\pi^2}{l^2}\sin\frac{\pi \times \frac{1}{2}}{l} + \frac{2l^3}{81EI\pi^4}\left(\frac{2\omega l}{3\pi} - W\right) \times \frac{9\pi^2}{l^2}\sin\frac{3\pi \times \frac{1}{2}}{l}\right]$$

$$= -\left[\frac{2l^3}{EI\pi^4}\left(\frac{2\omega l}{\pi} + W\right) \times \frac{\pi^2}{l^2}\ (1) + \frac{2l^3}{81EI\pi^4}\left(\frac{2\omega l}{3\pi} - W\right) \times \frac{9\pi^2}{l^2}\ (-1)\right]$$

$$\left[\therefore\ \sin\frac{\pi}{2} = 1;\ \sin\frac{3\pi}{2} = -1\right]$$

$$= -\left[\frac{2l}{EI\pi^2}\left(\frac{2\omega l}{\pi} + W\right) - \frac{2l}{9EI\pi^2}\left(\frac{2\omega l}{3\pi} - W\right)\right]$$

$$= -\left[\frac{4\omega l^2}{EI\pi^3} + \frac{2Wl}{EI\pi^2} - \frac{4\omega l^2}{27EI\pi^3} + \frac{2Wl}{9EI\pi^2}\right]$$

$$= -\left[\frac{3.8518\omega l^2}{EI\pi^3} + \frac{2.222Wl}{EI\pi^2}\right]$$

$$\frac{d^2y}{dx^2} = -\left[0.124\frac{\omega l^2}{EI} + 0.225\ \frac{Wl}{EI}\right]$$

Substitute $\frac{d^2y}{dx^2}$ value in bending moment equation,

$$M_{centre} = EI\frac{d^2y}{dx^2} = -EI\left[0.124\frac{\omega l^2}{EI} + 0.225\ \frac{Wl}{EI}\right]$$

$$M_{centre} = -(0.124\ \omega l^2 + 0.225\ Wl\ ) \quad \text{-----} \quad (20)$$

(∴Negative sign indicates downward deflection)

We know that, simply supported beam subjected to uniformly distributed load, maximum bending moment is,

$$M_{centre} = \frac{\omega l^2}{8}$$

Simply supported beam subjected to point load at centre, maximum bending moment is,

$$M_{centre} = \frac{Wl}{4}$$

Total bending moment, $M_{centre} = \frac{\omega l^2}{8} + \frac{Wl}{4}$

$$M_{centre} = 0.125\ \omega l^2 + 0.25\ Wl \quad \text{-----} \quad (21)$$

From equation (20) and (21), we know that, exact solution and solution obtained by using Rayleigh-Ritz method are almost same. In order to get accurate results, more terms in Fourier series should be taken.