4.2 DISCRETE TIME FOURIER TRANSFORM AND ITS PROPERTIES

The DTFT is a transformation that maps Discrete-time (DT) signal $x[n]$ into a complex valued function of the real variable namely:

$$F[x(n)] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

INVERSE DISCRETE FOURIER TRANSFORM

IDTFT is given by

$$x(n) = F^{-1}[X(e^{j\omega})] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega \text{ , for } n = -\infty \text{ to } \infty$$

Example 1: Find the DTFT of $x(n) = a^n u(n)$.

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^{-n} = \frac{1}{1-ae^{-j\omega}}.$$
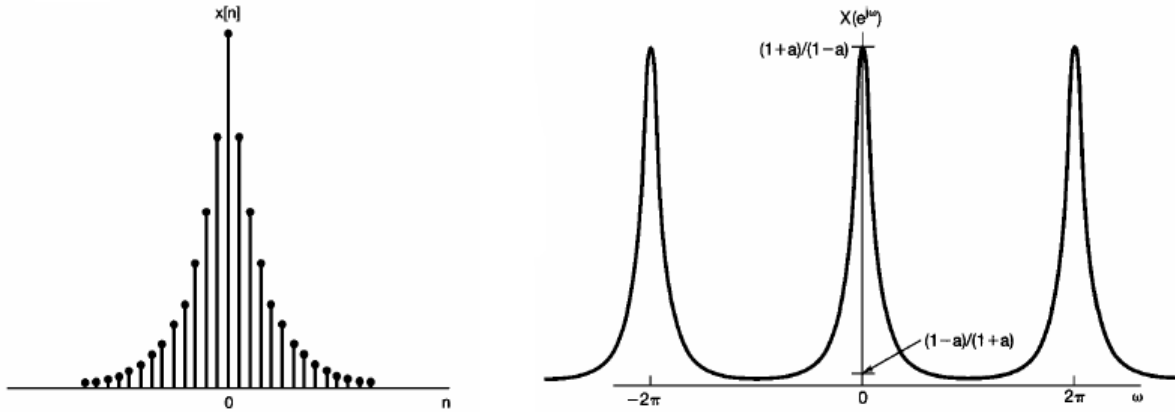
Example 2: Find the DTFT of $x(n) = a^{|n|}$, $|a| < 1$.

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n]e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

Let $m = -n$ in the first summation, we obtain

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n]e^{-j\omega n} = \sum_{m=1}^{\infty} a^m e^{j\omega m} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} = \frac{1-a^2}{1-2a\cos\omega + a^2}$$

PROPERTIES OF DTFT:

Linearity:

$$\mathcal{F}_{DT}\{x_1(n) + x_2(n)\} = \mathcal{F}_{DT}\{x_1(n)\} + \mathcal{F}_{DT}\{x_2(n) = X_1(e^{j\omega}) + X_2(e^{j\omega})$$

Time Shifting:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(n - n_0)\} = e^{-j\omega n_0} X(e^{j\omega})$$

Proof:

$$\sum_{n=-\infty}^{\infty} x(n-n_0)e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x(m)e^{-j\omega(m+n_0)} = e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} x(m)e^{-j\omega m}$$

Time Reversal:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(-n)\} = X(e^{-j\omega})$$

Convolution:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$G(e^{j\omega}) = \mathcal{F}_{DT}\{x(n) * y(n)\} = X(e^{j\omega}) Y(e^{j\omega})$$

Proof:

$$G\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x(m)\,y(n-m)\,e^{-jn\omega} = \sum_{n=-\infty}^{\infty} x(m)\sum_{m=-\infty}^{\infty} y(n-m)\,e^{-jn\omega}$$

$$= \sum_{m=-\infty}^{\infty} x(m)\sum_{r=-\infty}^{\infty} y(r)\,e^{-j(m+r)\omega} = \sum_{m=-\infty}^{\infty} x(m)\,e^{-jm\omega}\sum_{r=-\infty}^{\infty} y(r)\,e^{-jr\omega}$$

Frequency Shifting:

If $\mathscr{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathscr{F}_{DT}\left\{e^{j\omega_0 n}x(n)\right\} = X\left(e^{j(\omega-\omega_0)}\right),$$

Time Multiplication:

If $\mathscr{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathscr{F}_{DT}\left\{nx(n)\right\} = -z\frac{dX(z)}{dz}\bigg|_{z=e^{j\omega}}$$

Parseval's Theorem:

If $\mathscr{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathscr{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)Y^*\left(e^{j\omega}\right)d\omega$$

Proof:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty}x(n)e^{-jn\omega}\right]Y^*\left(e^{j\omega}\right)d\omega = \sum_{n=-\infty}^{\infty}x(n)\frac{1}{2\pi}\int_{-\pi}^{\pi}Y^*\left(e^{j\omega}\right)e^{-jn\omega}d\omega$$

For the case $x(n) = y(n)$, then:

$$\sum_{n=-\infty}^{\infty}\left|x(n)\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X\left(e^{j\omega}\right)\right|^2 d\omega$$

Multiplication of Sequences:

If $\mathscr{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathscr{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$\mathscr{F}_{DT}\left\{x(n)y(n)\right\} = \sum_{n=-\infty}^{\infty}x(n)y(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty}x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}Y\left(e^{j\lambda}\right)e^{j\lambda n}d\lambda\right]e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}Y\left(e^{j\lambda}\right)d\lambda\sum_{n=-\infty}^{\infty}x(n)e^{-j(\omega-\lambda)n} = \frac{1}{2\pi}\int_{-\pi}^{\pi}Y\left(e^{j\lambda}\right)X\left(e^{j(\omega-\lambda)}\right)d\lambda$$

$$= \frac{1}{2\pi}Y\left(e^{j\omega}\right) * X\left(e^{j\omega}\right)$$

Differentiation in the Frequency Domain:

If $\mathscr{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\frac{dX\left(e^{j\omega}\right)}{d\omega} = \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}\right] = -j\sum_{n=-\infty}^{\infty}nx(n)e^{-j\omega n} = -j\mathscr{F}_{DT}\left\{nx(n)\right\}$$