UNIT-V

Z – Transforms AND DIFFERENCE EQUATIONS

Introduction

The Z-transform plays a vital role in the field of communication Engineering and control Engineering, especially in digital signal processing. Laplace transform and Fourier transform are the most effective tools in the study of continuous time signals, where as Z – transform is used in discrete time signal analysis. The application of Z – transform in discrete analysis is similar to that of the Laplace transform in continuous systems. Moreover, Z-transform has many properties similar to those of the Laplace transform. But, the main difference is Z-transform operates only on sequences of the discrete integer-valued arguments. This chapter gives concrete ideas about Z-transforms and their properties. The last section applies Z-transforms to the solution of difference equations.

Difference Equations

Difference equations arise naturally in all situations in which sequential relation exists at various discrete values of the independent variables. These equations may be thought of as the discrete counterparts of the differential equations. Z-transform is a very useful tool to solve these equations.

A **difference equation** is a relation between the independent variable, the dependent variable and the successive differences of the dependent variable.

For example, $\Delta^2 y_n + 7\Delta y_n + 12 y_n = n^2$ -- (i)

and $\Delta^3 y_n - 3\Delta y_n - 2y_n = \cos n$ ----------------------------------- (ii)

are difference equations.

The differences Δy_n, $\Delta^2 y_n$, etc can also be expressed as.

$\Delta y_n = y_{n+1} - y_n$,
$\Delta^2 y_n = y_{n+2} - 2y_{n+1} + y_n$.
$\Delta^3 y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n$ and so on.

Substituting these in (i) and (ii), the equations take the form

$y_{n+2} + 5y_{n+1} + 6y_n = n^2$ ----------- (iii)

and $y_{n+3} - 3y_{n+2} = \cos n$ -------------------------- (iv)

Note that the above equations are free of Δ's.

If a difference equation is written in the form free of Δ's, then the **order** of the difference equation is the difference between the highest and lowest subscripts of y‟s occurring in it. For example, the order of equation (iii) is 2 and equation (iv) is 1.

The highest power of the y's in a difference equation is defined as its **degree** whenit is written in a form free of Δ's. For example, the degree of the equations

$y_{n+3} + 5y_{n+2} + y_n = n^2 + n + 1$ is 3 and $y^3_{n+3} + 2y_{n+1} y_n = 5$ is 2.

Linear Difference Equations

A **linear difference equation with constant coefficients** is of the form

$a_0 y_{n+r} + a_1 y_{n+r-1} + a_2 y_{n+r-2} + ... + a_r y_n = \phi(n)$.

i.e., $(a_0 E^r + a_1 E^{r-1} + a_2 E^{r-2} + ... + a_r) y_n = \phi(n)$ ------------------- (1)

where $a_0, a_1, a_2, ... a_r$ are constants and $\phi(n)$ are known functions of n.

The equation (1) can be expressed in symbolic form as

$f(E)\, y_n = \phi(n)$ -- (2)

If $\phi(n)$ is zero, then equation (2) reduces to

$$f(E)\, y_n = 0 \text{ -- } (3)$$

which is known as the **homogeneous difference equation** corresponding to (2).Thesolution
of (2) consists of two parts, namely, the complementary function and the particular integral.
The solution of equation (3) which involves as many arbitrary constants as the order ofthe equation is called the **complementary function**. The **particular integral** is a particular solution of equation(1) and it is a function of „n‟ without any arbitrary constants.

Thus the complete solution of (1) is given by y_n = C.F + P.I.

Example 1

Form the difference equation for the Fibonacci sequence .

The integers 0,1,1,2,3,5,8,13,21, ... are said to form a Fibonacci sequence.

If y_n be the n^{th} term of this sequence, then

$$y_n = y_{n-1} + y_{n-2} \text{ for } n > 2$$

$$\text{or } \; y_{n+2} - y_{n+1} - y_n = 0 \text{ for } n > 0$$

Z - Transforms and its Properties

Definition

Let $\{f_n\}$ be a sequence defined for n = 0,1,2,…….,then its Z-transform F(z) is defined as

$$F(z) = Z\{f_n\} = \sum_{n=0}^{\infty} f_n z^{-n} ,$$

whenever the series converges and it depends on the sequence $\{f_n\}$.The

inverse Z-transform of F(z) is given by $Z^{-1}\{F(z)\} = \{f_n\}$.

Note: If $\{f_n\}$ is defined for n = 0, ± 1, ± 2, ……., then

$$F(z) = Z\{f_n\} = \sum_{n=-\infty}^{\infty} f_n z^{-n} ,$$ which is known as the two – sided Z- transform.

Properties of Z-Transforms

1. The Z-transform is linear.

i.e, if $F(z) = Z\{f_n\}$ and $G(z) = Z\{g_n\}$, then

$$Z\{af_n + bg_n\} = aF(z) + bG(z).$$

Proof:

$$Z\{ af_n + bg_n\} = \sum_{n=0}^{\infty} \{ af_n + bg_n\} z^{-n} \quad \text{(by definition)}$$

$$= a \sum_{n=0}^{\infty} f_n z^{-n} + b\sum_{n=0}^{\infty} g_n z^{-n}$$

$$= aF(z) + b\, G(z)$$

2. If $Z\{f_n\} = F(z)$, then $Z\{a^n f_n\} = F(z/a)$

Proof: By definition, we have

$$Z\{ a^n f_n\} = \sum_{n=0}^{\infty} a^n f_n z^{-n}$$

$$= \sum_{n=0}^{\infty} f_n (z/a)^{-n} = F(z/a)$$

Corollary:

If $Z\{f_n\} = F(z)$, then $Z\{ a^{-n} f_n\} = F(az)$.

3. $Z\{nf_n\} = -z \dfrac{dF(z)}{dz}$

Proof

We have $F(z)= \sum_{n=0}^{\infty} f_n z^{-n}$

Differentiating, we get

$$\frac{dF(z)}{dz} = \sum_{n=0}^{\infty} f_n (-n) z^{-n-1}$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty} nf_n z^{-n}$$

$$= -\frac{1}{z} Z\{nf_n\}$$

Hence, $Z\{nf_n\} = -z \dfrac{dF(z)}{dz}$

4. If $Z\{f_n\} = F(z)$, then

$$Z\{f_{n+k}\} = z^k\{ F(z) - f_0 - (f_1 / z) - \ldots - (f_{k-1} / z^{k-1}) \} \ (k > 0)$$

Proof

$$Z\{ f_{n+k}\} = \sum_{n=0}^{\infty} f_{n+k} z^{-n} \text{, by definition.}$$

$$= \sum_{n=0}^{\infty} f_{n+k} z^{-n} z^k z^{-k}$$

$$= z^k \sum_{n=0}^{\infty} f_{n+k} z^{-(n+k)}$$

$$= z^k \sum_{m=k}^{\infty} f_m z^{-m}, \text{ where } m = n+k .$$

$$= z^k \{F(z) - f_0 - (f_1/z) - \ldots\ldots\ldots (f_{k-1} / z^{k-1}) \}$$

In Particular,

(i) $Z\{f_{n+1}\} = z \{F(z) - f_0\}$

(ii) $Z\{f_{n+2}\} = z^2 \{ F(z) - f_0 - (f_1/z) \}$

Corollary

If $Z\{f_n\} = F(z)$, then $Z\{f_{n-k}\} = z^{-k} F(z)$.

(1) Initial value Theorem

If $Z \{f_n\} = F(z)$, then $f_o = \lim_{z \to \infty} F(z)$

Proof

We know that $F(z) = f_0 + f_1 z^{-1} + f_2 z^{-2} + \ldots$

Taking limits as $z \to \infty$ on both sides, we get

$$\lim_{z \to \infty} F(z) = f_0$$

Similarly, we can find

$$f_1 = \lim_{z \to \infty} \{ z [F(z) - f_0]\}; \quad f_2 = \lim_{z \to \infty} \{ z^2 [F(z) - f_0 - f_1 z^{-1}]\} \text{ and so on.}$$

(2) Final value Theorem

If $Z\{f_n\} = F(z)$, then $\lim_{n \to \infty} f_n = \lim_{z \to 1} (z-1) F(z)$

Proof

By definition, we have

$$Z \{f_{n+1} - f_n\} = \sum_{n=0}^{\infty} \{f_{n+1} - f_n\} z^{-n}$$

$$Z\{f_{n+1}\} - Z\{f_n\} = \sum_{n=0}^{\infty} \{f_{n+1} - f_n\} z^{-n}$$

$$\text{ie, } z \{F(z) - f_0\} - F(z) = \sum_{n=0}^{\infty} \{f_{n+1} - f_n\} z^{-n}$$

$$(z-1) F(z) - f_0 z = \sum_{n=0}^{\infty} \{f_{n+1} - f_n\} z^{-n}$$

Taking, limits as $z \to 1$ on both sides, we get

$$\lim_{z \to 1} \{(z-1) F(z)\} - f_0 = \lim_{z \to 1} \sum_{n=0}^{\infty} \{f_{n+1} - f_n\} z^{-n}$$

$$= \sum_{n=0}^{\infty} (f_{n+1} - f_n) = (f_1 - f_0) + (f_2 - f_1) + \dots + (f_{n+1} - f_n)$$

$$= \lim_{n \to \infty} f_{n+1} - f_0$$

$$\text{i.e, } \lim_{z \to 1} \{(z-1) F(z)\} - f_0 = f_\infty - f_0$$

$$\text{Hence, } f_\infty = \lim_{z \to 1} [(z-1) F(z)]$$

$$\text{i.e, } \lim_{n \to \infty} f_n = \lim_{z \to 1} [(z-1) F(z)]$$

SOME STANDARD RESULTS

1. $Z\{a^n\} = z / (z-a)$, for $|z| > |a|$.

Proof

By definition, we have

$$Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a/z)^n$$

$$= \frac{1}{1-(a/z)}$$

$$= z / (z-a), \text{ for } |z| > |a|$$

In particular, we have

$$Z\{1\} = z / (z-1), \text{ (taking } a = 1).$$

and $Z\{(-1)^n\} = z / (z+1)$, (taking $a = -1$).

2. $Z\{na^n\} = az/(z-a)^2$

Proof: By property, we have

$$Z\{nf_n\} = -z \frac{dF(z)}{dz}$$

$$= -z \frac{d}{dz} Z\{a^n\}$$

$$\therefore Z\{na^n\} = -z \frac{d}{dz}\frac{z}{z-a} = \frac{az}{(z-a)^2}$$

Similarly, we can prove

$$Z\{n^2a^n\} = \{az(z+a)\}/ (z-a)^3$$

(3) $Z\{n^m\} = -z \dfrac{d}{dz} Z\{n^{m-1}\}$, where m is a positive integer.

Proof

$$Z\{n^m\} = \sum_{n=0}^{\infty} n^m z^{-n}$$

$$= z \sum_{n=0}^{\infty} n^{m-1} \; n \, z^{-(n+1)} \quad \text{------------------------} \quad (1)$$

Replacing m by m-1, we get

$$Z\{n^{m-1}\} = z \sum_{n=0}^{\infty} n^{m-2} \; n \, z^{-(n+1)}$$

i.e, $$Z\{n^{m-1}\} = \sum_{n=0}^{\infty} n^{m-1} \; z^{-n}.$$

Differentiating with respect to z, we obtain

$$\frac{d}{dz} Z\{n^{m-1}\} = \sum_{n=0}^{\infty} n^{m-1} (-n) \, z^{-(n+1)} \quad \text{----------------} \quad (2)$$

Using (2) in (1), we get

$Z\{n^m\} = -z \dfrac{d}{dz} Z\{n^{m-1}\}$, which is the recurrence formula.

In particular, we have

$$Z\{n\} = -z \frac{d}{dz} Z\{1\}$$

$$= -z \frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{z}{(z-1)^2}$$

Similarly,

$$Z\{n^2\} = -z \frac{d}{dz} Z\{n\}$$

$$= -z \frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= \frac{z(z+1)}{(z-1)^3}.$$

4. $Z\{\cos n\theta\} = \dfrac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1z}$ and

$Z\{\sin n\theta\} = \dfrac{\sin\theta}{z^2 - 2z\cos\theta + 1}$

We know that

$Z\{a^n\} = z/(z-a)$, if $|z| > |a|$

Letting $a = e^{i\theta}$, we have

$$Z\{e^{in\theta}\} = \frac{z}{z-e^{i\theta}} = \frac{z}{z-(\cos\theta + i\sin\theta)z}$$

$$Z\{\cos n\theta + i\sin n\theta\} = \frac{}{(z-\cos\theta) - i\sin\theta}$$

$$= \frac{z\{(z-\cos\theta) + i\sin\theta\}}{\{(z-\cos\theta) - i\sin\theta\}\{(z-\cos\theta) + i\sin\theta\}}$$

$$= \frac{z(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating the real & imaginary parts, we get

$Z(\cos n\theta) = \dfrac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1z}$ and

$Z(\sin n\theta) = \dfrac{\sin\theta}{z^2 - 2z\cos\theta + 1}$

5 . $Z\{r^n \cos n\theta\} = \dfrac{z(z - r\cos\theta)}{z^2 - 2rz\cos\theta + r^2}$ and

$$Z\{r^n \sin n\theta\} = \frac{zr\sin\theta}{z^2 - 2rz\cos\theta + r^2} \text{ if } |z|>|r|$$

We know that

$$Z\{a^n\} = z/(z-a), \text{ if } |z|>|a|$$

Letting $a = re^{i\theta}$, we have

$$Z\{r^n e^{in\theta}\} = z/(z - re^{i\theta}).$$

i.e, $Z\{r^n (\cos n\theta + i\sin n\theta)\} = \dfrac{z}{z - re^{i\theta}}$

$$= \frac{z}{z - r(\cos\theta + i\sin\theta)}$$

$$= \frac{z\{(z - r\cos\theta) + i\, r\sin\theta\}}{\{(z - r\cos\theta) - i\, r\sin\theta\}\{(z - r\cos\theta) + i\, r\sin\theta\}}$$

$$= \frac{z(z - r\cos\theta) + i\, rz\sin\theta}{(z - r\cos\theta)^2 + r^2\sin^2\theta}$$

$$= \frac{z(z - r\cos\theta) + i\, rz\sin\theta}{z^2 - 2rz\cos\theta + r^2}$$

Equating the Real and Imaginary parts, we get

$$Z\{r^n \cos n\theta\} = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2} \text{ and}$$

$$Z\{r^n \sin n\theta\} = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2}; \text{ if } |z| > |r|$$

Table of Z – Transforms

	f_n	$F(z)$
1.	1	$\dfrac{z}{z-1}$

2.	$(-1)^n$	$\frac{z}{z+1}$
3.	a^n	$\frac{z}{z-a}$
4.	n	$\frac{z}{(z-1)^2}$
5.	n^2	$\frac{z^2+z}{(z-1)^3}$
6.	$n(n-1)$	$\frac{2z}{(z-1)^3}$
7.	$n^{(k)}$	$\frac{k!z}{(z-1)^{k+1}}$
8.	na^n	$\frac{az}{(z-1)^2}$
9.	$\cos n\theta$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
10.	$\sin n\theta$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$
11.	$r^n\cos n\theta$	$\frac{z(z-r\cos\theta)}{z^2-2rz\cos\theta+r^2}$
12.	$r^n\sin n\theta$	$\frac{rz\sin\theta}{z^2-2rz\cos\theta+r^2}$
13.	$\cos(n\pi/2)$	$\frac{z^2}{z^2+1}$

14.	$\sin(n\pi/2)$	$\dfrac{z}{z^2 + 1}$
15	t	$\dfrac{Tz}{(z-1)^2}$
16	t^2	$\dfrac{T^2 z(z + 1)}{(z-1)^3}$
17	e^{at}	$\dfrac{z}{z - e^{aT}}$
18	e^{-at}	$\dfrac{z}{z - e^{-aT}}$
19	$Z\{\cos\omega t\}$	$\dfrac{z (z - \cos\omega T)}{z^2 - 2z \cos\omega T +1}$
20	$Z\{\sin\omega t \}$	$\dfrac{z \sin\omega T}{z^2 - 2z \cos\omega T +1}$
21	$Z\{e^{-at} \cos bt\}$	$\dfrac{ze^{aT} (ze^{aT} - \cos bT)}{z^2e^{2aT} - 2ze^{aT} \cos bT +1}$
22	$Z\{e^{-at} \sin bt\}$	$\dfrac{ze^{aT} \sin bT}{z^2e^{2aT} - 2ze^{aT} \cos bT +1}$

$2\,(z-1)^3$

$2\,(z-1)^3$

Example 2

Find the Z– transform of

(i) n(n-1)

(ii) $n^2 + 7n + 4$

(iii) (1/2)(n+1)(n+2)

(i) $Z\{ n(n-1)\} = Z\{n^2\} - Z\{n\}$

$$= \frac{z\,(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2}$$

$$= \frac{z\,(z+1) - z\,(z-1)}{(z-1)^3}$$

$$= \frac{2z}{(z-1)^3}$$

(iii) $Z\{ n^2 + 7n + 4\} = Z\{n^2\} + 7\,Z\{n\} + 4\,Z\{1\}$

$$= \frac{z\,(z+1)}{(z-1)^3} + 7\frac{z}{(z-1)^2} + 4\frac{z}{z-1}$$

$$= \frac{z\,\{(z+1) + 7(z-1) + 4(z-1)^2\}}{(z-1)^3}$$

$$= \frac{2z(z^2-2)}{(z-1)^3}$$

(iii) $Z\left\{\frac{(n+1)(n+2)}{2}\right\} = \frac{1}{2}\{ Z\{n^2\} + 3Z\{n\} + 2Z\{1\}\}$

$$= \frac{1}{2}\left\{\frac{z(z+1)}{(z-1)^3} + \frac{3z}{(z-1)^2} + 2\frac{2z}{(z-1)}\right\} \text{ if } |z| > 1$$

$$= \frac{z^3}{(z-1)^3}$$

Example 3

Find the Z- transforms of 1/n and 1/n(n+1)

.

$$\text{(i) } Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \ldots$$

$= -\log(1 - 1/z)$ if $|1/z| < 1$

$= -\log(z-1/z)$

$= \log(z/z-1)$, if $|z| > 1$.

$$\text{(ii) } Z\left\{\frac{1}{n(n+1)}\right\} = Z\left\{\frac{1}{n} - \frac{1}{n+1}\right\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} z^{-n} - \sum_{n=0}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= \log\frac{z}{z-1} - \left\{1 + \frac{1}{2z} + \frac{1}{3z^2} + \ldots\right\}$$

$$= \log\left(\frac{z}{z-1}\right) - z\left\{\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \ldots\right\}$$

$$= \log\frac{z}{z-1} - z\{-\log(1 - 1/z)\}$$

$$= \log\frac{z}{z-1} - z\log(z/z-1)z-1$$

$= (1- z)\log\{z/(z-1)\}$

Example 4

Find the Z- transforms of

(i) $\cos n\pi/2$

(ii) $\sin n\pi/2$

(i) $$Z\{\cos n\pi/2\} = \sum_{n=0}^{\infty} \cos \frac{n\pi}{2} z^{-n}$$

$$= 1 - \frac{1}{z^2} + \frac{1}{z^4} - \ldots\ldots\ldots\ldots$$

$$= \left\{1 + \frac{1}{z^2}\right\}^{-1}$$

$$= \left\{\frac{z^2 + 1}{z^2}\right\}^{-1}$$

$$= \frac{z^2}{z^2 + 1}, \text{ if } |z| > |z^2$$

(ii) $$Z\{\sin n\pi/2\} = \sum_{n=0}^{\infty} \sin \frac{n\pi}{2} z^{-n}$$

$$= \frac{1}{z} - \frac{1}{z^3} + \frac{1}{z^5} - \ldots\ldots \ldots$$

$$= \frac{1}{z}\left\{1 - \frac{1}{z^2} + \frac{1}{z^4} - \ldots\right.$$

$$= \frac{1}{z}\left\{1 + \frac{1}{z^2}\right\}^{-1}$$

$$= \frac{1}{z}\left\{\frac{z^2 + 1}{z^2}\right\}^{-1}$$

$$= \frac{1}{z} \cdot \frac{z^2}{z^2+1} = \frac{z}{z^2+1}$$

Example 5

Show that Z{1/ n!} = $e^{1/z}$ and hence find Z{1/ (n+1)!} and Z{1/ (n+2)!}

$$Z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{(z^{-1})^n}{n!}$$

$$= 1 + \frac{z^{-1}}{1!} + \frac{(z^{-1})^2}{2!} + \ldots$$

$$= e^{z^{-1}} = e^{1/z}$$

To find $Z\left\{\frac{1}{(n+1)!}\right\}$

We know that $Z\{f_{n+1}\} = z\{F(z) - f_0\}$

Therefore,

$$Z\left\{\frac{1}{(n+1)!}\right\} = z\left\{Z\left\{\frac{1}{n!}\right\} - 1\right\}$$

$$= z\{e^{1/z} - 1\}$$

Similarly,

$$Z\left\{\frac{1}{(n+2)!}\right\} = z^2\{e^{1/z} - 1 - (1/z)\}.$$

Example 6

Find the Z- transforms of the following

(i) $f(n) = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$

(ii) $f(n) = \begin{cases} 0, & \text{if } n > 0 \\ 1, & \text{if } n \leq 0 \end{cases}$

(iii)$f(n) = a^n / n!, n \geq 0$
0, otherwise

(i) $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$

$= \sum_{n=0}^{\infty} n z^{-n}$

$= (1/z) + (2/z^2) + (3/z^3) + \ldots$

$= (1/z)\{1 + (2/z) + (3/z^2) + \ldots\}$

$= (1/z)\{1 - (1/z)\}^{-2} z\text{-}1$

$= 1/z \; \text{------}^{-2}$
z

$= z/(z-1)^2$, if $|z| > |$

(ii) $Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$

$= \sum_{n=-\infty}^{\infty} z^{-n}$

$= \sum_{n=0}^{\infty} z^{n}$

$= (1/1 - z)$, if $|z| < 1$.

(iii) $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$

$= \sum_{n=0}^{\infty} \frac{(az^{-1})^n}{n!}$

$= e^{az^{-1}} = e^{a/z}$

Example 7

If $F(z) = \dfrac{2z^2 + 3z + 12}{(z-1)^4}$, find the value of „$f_2$" and „$f_3$".

$2z^2 + 3z + 12$

Given that $F(z) = \dfrac{\quad}{(z-1)^4}$.

This can be expressed as

$$F(z) = \frac{1}{z^2}\,\frac{2 + 3z^{-1} + 12z^{-2}}{(1- z^{-1})^4}.$$

By the initial value theorem, we have

$$f_o = \lim_{z\to\infty} F(z) = 0.$$

Also, $$f_1 = \lim_{z\to\infty} \{z[F(z) - f_o]\} = 0.$$

Now, $$f_2 = \lim_{z\to\infty} \{z^2 [F(z) - f_o - (f_1 /z)]\}$$

$$= \lim_{z\to\infty} \frac{2 + 3z^{-1} + 12z^{-2}}{(1- z^{-1})^4} - 0 - 0.$$

$$= 2.$$

and $$f_3 = \lim_{z\to\infty} \{z^3 [F(z) - f_o - (f_1 /z) - (f_2/ z^2)]\}$$

$$= \lim_{z\to\infty} z^3 \left(\frac{2 + 3z^{-1} + 12z^{-2}}{(1- z^{-1})^4} - \frac{2}{z^2} \right.$$

$11z^3 + 8z - 2$

Given that $= \lim_{z \to \infty} \frac{z^3}{z^2 (z-1)^4} = 11.$