CONVOLUTION THEOREM

Definition: Convolution of two functions

The convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$ and defined by

$f(t) * g(t) = \int_0^t f(u)g(t-u)du.$

State and prove Convolution theorem

Statement: If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$, then $L[f(t)] * L[g(t)] = F(s)G(s)$

Proof:

We have $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

$$L[f(t) * g(t)] = \int_0^\infty [f(t) * g(t)]\, e^{-st} dt$$

$$= \int_0^\infty \int_0^t f(u)g(t-u)du e^{-st} dt$$

$$= \int_0^\infty \int_0^t f(u)g(t-u)e^{-st} dudt \cdots (1)$$

Now we have no change the order of integration.

$u = 0, u = t; t = 0, t = \infty$

Change of order is . Draw horizontal strip PQ

At P, $t = u$, At A $u = \infty$

$$L[f(t) * g(t)] = \int_0^\infty \int_u^\infty f(u)g(t-u)e^{-st} dtdu$$

$$= \int_0^\infty f(u) \left[\int_u^\infty g(t-u)e^{-st} dt\right] du \cdots (2)$$

Put $t - u = x \cdots (3)$

$t = u + x \Rightarrow dt = dx$

When $t = u; (3) \Rightarrow x = 0$

When $t = \infty; (3) \Rightarrow x = \infty$

$$(2) \Rightarrow L[f(t) * g(t)] = \int_0^\infty f(u) \left[\int_0^\infty g(x)e^{-s(u+x)} dx\right] du$$

$$= \int_0^\infty f(u) \left[\int_0^\infty g(x)e^{-su}e^{-sx} dx\right] du$$

$$= \int_0^\infty f(u)e^{-su} du \int_0^\infty g(x)e^{-sx} dx$$

$$= L[f(u)]L[g(x)]$$

$\therefore L[f(t) * g(t)] = F(s)G(s)$

Note: Convolution theorem is very useful to compute inverse Laplace transform of product of two terms

Convolution theorem is $L[f(t) * g(t)] = F(s)G(s)$

$$L^{-1}[F(s)G(s)] = f(t) * g(t)$$

$$L^{-1}[F(s)G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

Example: Find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$ using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{(s+a)}\right] * L^{-1}\left[\frac{1}{(s+b)}\right]$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au} e^{-b(t-u)}\, du$$

$$= e^{-bt} \int_0^t e^{-au} e^{bu} du$$

$$= e^{-bt} \int_0^t e^{(b-a)u}\, du$$

$$= e^{-bt} \left[\frac{e^{(b-a)u}}{b-a}\right]_0^t$$

$$= \frac{e^{-bt}}{b-a}\left[e^{(b-a)t} - 1\right]$$

$$= \frac{e^{-bt}}{b-a}\left[e^{bt-at} - 1\right]$$

$$= \frac{1}{b-a}\left[e^{-bt+bt-at} - e^{-bt}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]$$

Example: Find the inverse Laplace transform $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\ \frac{s}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+b^2)}\right]$$

$$= cosat * cosbt$$

$$= \int_0^t cosau\ cosb(t-u)du$$

$$= \int_0^t \frac{\cos(au+bt-bu)+\cos(au-bt+bu)}{2} du$$

$$= \frac{1}{2}\int_0^t (\cos(au+bt-bu) + \cos(au-bt+bu))\, du$$

$$= \frac{1}{2}\int_0^t [\cos(a-b)\, u + bt + \cos(a+b)\, u - bt] du$$

$$= \frac{1}{2}\left[\frac{sin[(a-b)u+bt]}{a-b} + \frac{sin[(a+b)u+bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b} + \frac{\sin(at-bt+bt)}{a+b} - \frac{sinbt}{a-b} + \frac{sinbt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{sinbt}{a-b} + \frac{sinbt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{(a+b)sinat+(a-b)sinat-(a+b)sinbt+(a-b)sinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2asinat-2bsinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2(asinat-bsinbt)}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{asinat-bsinbt}{a^2-b^2}$$

Example: Find the inverse Laplace transform $\frac{1}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\ \frac{1}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+b^2)}\right]$$

$$= \frac{1}{a}sinat * \frac{1}{b}sinbt$$

$$= \frac{1}{ab}\int_0^t sinau\ sinb(t-u)du$$

$$= \frac{1}{ab}\int_0^t \frac{\cos(au-bt+bu)-\cos(au+bt-bu)}{2}du$$

$$= \frac{1}{2ab}\int_0^t (\cos(au-bt+bu) - \cos(au+bt-bu))\,du$$

$$= \frac{1}{2}\int_0^t [\cos[(a+b)u - bt] - \cos[(a-b)u + bt]]du$$

$$= \frac{1}{2ab}\left[\frac{sin[(a+b)u-bt]}{a+b} - \frac{sin[(a-b)u+bt]}{a-b}\right]_0^t$$

$$= \frac{1}{2ab}\left[\frac{\sin(at+bt-bt)}{a+b} - \frac{\sin(at-bt+bt)}{a-b} + \frac{sinbt}{a+b} + \frac{sinbt}{a-b}\right]$$

$$= \frac{1}{2ab}\left[\frac{\sin at}{a+b} - \frac{\sin at}{a-b} - \frac{sinbt}{a+b} + \frac{sinbt}{a-b}\right]$$

$$= \frac{1}{2ab}\left[\frac{(a-b)sinat-(a+b)sinat+(a-b)sinbt+(a+b)sinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2ab}\left[\frac{-2bsinat+2asinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2ab}\left[\frac{2(asinbt-bsinat)}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = \frac{asinbt-bsinat}{ab(a^2-b^2)}$$

Example: Find the inverse Laplace transform $\frac{s}{(s^2+4)(s^2+9)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = L^{-1}\left[\frac{1}{(s^2+4)}\ \frac{s}{(s^2+9)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+4)}\right] * L^{-1}\left[\frac{s}{(s^2+9)}\right]$$

$$= \frac{1}{2} sin2t * cos3t$$

$$= \frac{1}{2}\int_0^t sin2u\, cos3(t-u)du$$

$$= \frac{1}{2}\int_0^t \frac{\sin(2u+3t-3u)+\sin(2u-3t+3u)}{2} du$$

$$= \frac{1}{4}\int_0^t [\sin(3\mathrm{t}-u)+\sin(5\mathrm{u}-3\mathrm{t})]du$$

$$= \frac{1}{4}\left[\frac{-\cos(3t-u)}{-1} - \frac{\cos(5u-3t)}{5}\right]_0^t$$

$$= \frac{1}{4}\left[\frac{\cos(3\mathrm{t}-\mathrm{t})}{1} - \frac{\cos(5\mathrm{t}-3t)}{5} - \frac{cos3t}{1} + \frac{cos3t}{5}\right]$$

$$= \frac{1}{4}\left[cos2t - \frac{\cos 2t}{5} - cos3t + \frac{cos3t}{5}\right]$$

$$= \frac{1}{4}\left[\frac{5cos2t - cos2t - 5cos3t + cos3t}{5}\right]$$

$$= \frac{1}{20}[4cos2t - 4cos3t]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = \frac{cos2t - cos3t}{5}$$

Example: Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\frac{s}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+a^2)}\right]$$

$$= \frac{1}{a} sinat * cosat$$

$$= \frac{1}{a}\int_0^t sinau\, cosa(t-u)du$$

$$= \frac{1}{a}\int_0^t \frac{\sin(au+at-au)+\sin(au-at+au)}{2} du$$

$$= \frac{1}{2a}\int_0^t [sinat + \sin(2au-at)]du$$

$$= \frac{1}{2a}\left[\int_0^t sinatdu + \int_0^t \sin(2au-at)\, du\right]$$

$$= \frac{1}{2a}\left[sinat\int_0^t du + \int_0^t \sin(2au-at)\, du\right]$$

$$= \frac{1}{2a}\left[sinat(u)_0^t - \left(\frac{\cos(2au-at)}{2a}\right)_0^t\right]$$

$$= \frac{1}{2a}\left[tsinat - \frac{\cos(2at-at)}{2a} + \frac{cosat}{2a}\right]$$

$$= \frac{1}{2a}\left[tsinat - \frac{\cos at}{2a} + \frac{cosat}{2a}\right]$$

$$= \frac{1}{2a}\ tsinat$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{tsinat}{2a}$$