Surface Integral

The integral of the normal component of $\vec{F}$ is denoted by $\iint_S \vec{F} \cdot \vec{n}\, ds$ and is called the surface integral.

Evaluation of surface integral

Let R_1 be the projection of S on the $xy-$plane, $\vec{k}$ is the unit vector normal to the $xy-$plane then $ds = \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_{R1} \vec{F} \cdot \vec{n} \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$$

If R_2 be the projection of s on $yz-$plane

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_{R2} \vec{F} \cdot \vec{n} \frac{dx\,dy}{|\vec{n}\cdot\vec{\imath}|}$$

If R_3 be the projection of s on $xz-$plane

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_{R3} \vec{F} \cdot \vec{n} \frac{dx\,dy}{|\vec{n}\cdot\vec{\jmath}|}$$

Example: Evaluate $\iint_S \vec{F} \cdot \vec{n}\, ds$ if $\vec{F} = (x+y^2)\vec{\imath} - 2x\vec{\jmath} + 2yz\vec{k}$ and s is the surface of the plane $2x+y+2z=6$ in the first octant.

Solution:

Given $\vec{F} = (x+y^2)\vec{\imath} - 2x\vec{\jmath} + 2yz\vec{k}$

Let $\varphi = 2x + y + 2z - 6$

Then $\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

$= 2\vec{\imath} + 1\vec{\jmath} + 2\vec{k}$

$|\nabla\varphi| = \sqrt{4+1+4} = \sqrt{9} = 3$

$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\vec{\imath} + 1\vec{\jmath} + 2\vec{k}}{3}$

$\vec{F}\cdot\hat{n} = \left[(x+y^2)\vec{\imath} - 2x\vec{\jmath} + 2yz\,\vec{k}\right] \cdot \left(\frac{2\vec{\imath} + 1\vec{\jmath} + 2\vec{k}}{3}\right)$

$= \frac{1}{3}\left[2(x+y^2) - 2x + 4yz\right]$

$= \frac{2}{3}\left[y^2 + 2yz\right]$

$= \frac{2}{3}\, y[y + 2z]$

$= \frac{2}{3}\, y[y + 6 - 2x - y]$ $\qquad [\because 2z = 6 - 2x - y]$

$= \frac{2}{3}\, y[6 - 2x]$

$= \frac{4}{3}\, y[3 - x]$

Let R be the projection of S on the $xy -$ plane

$$\therefore ds = \frac{dx\, dy}{|\hat{n} \cdot \vec{k}|}$$

$$\hat{n} \cdot \vec{k} = \left(\frac{2\vec{i} + 1\vec{j} + 2\vec{k}}{3}\right) \cdot \vec{k} = \frac{2}{3}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_R \vec{F} \cdot \hat{n} \frac{dx\, dy}{|\hat{n} \cdot \vec{k}|}$$

$$= \iint_R \frac{4}{3}\, y(3 - x) \frac{dx\, dy}{\left(\frac{2}{3}\right)}$$

$$= 2 \int\int (3 - x) y\, dxdy$$

In $R_1 (2x + y = 6)$, x varies from 0 to $\frac{6-y}{2}$

y varies from 0 to 6

$$= 2 \int_0^6 \int_0^{\frac{6-y}{2}} y\,(3 - x)\, dxdy$$

$$= 2 \int_0^6 y \left[3x - \frac{x^2}{2}\right]_0^{\frac{6-y}{2}} dy$$

$$= 2 \int_0^6 y \left[3\left(\frac{6-y}{2}\right) - \frac{1}{2}\left(\frac{6-y}{2}\right)^2\right] dy$$

$$= 2 \int_0^6 \frac{1}{2}(18y - 3y^2) - \frac{1}{8}(6 - y)^2\, dy$$

$$= \frac{2}{2}\left[18\, \frac{y^2}{2} - \frac{3y^3}{3} - \frac{1}{8}\frac{(6-y)^3}{3(-1)}\right]$$

$$= \left[9(6)^2 - (6)^3 + \frac{1}{12}(0)\right] - \left[0 - 0 + \frac{1}{12}\,(6)^3\right]$$

$$= 81 \text{ units}$$

Example: Show that $\iint_S \left(yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}\right) \cdot \hat{n}\, ds = \frac{3}{8}$ **where s is the surface of the sphere** $x^2 + y^2 + z^2 = 1$ **in the first octant.**

Solution:

Given $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$

Let $\varphi = x^2 + y^2 + z^2 - 1$

$$\nabla\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2(1)$$

$\therefore$ The unit outward normal is $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2}$

$$\vec{F}\cdot\hat{n} = \left[yz\vec{i} + zx\vec{j} + xy\,\vec{k}\right]\cdot\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$= 3xyz$$

Let R be the projection of S on xy −plane

$$\therefore ds = \frac{dx\,dy}{\left|\hat{n}\cdot\vec{k}\right|}$$

$$\left|\hat{n}\cdot\vec{k}\right| = \left(x\vec{i} + y\vec{j} + z\vec{k}\right)\cdot\vec{k} \quad = z$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{\left|\hat{n}\cdot\vec{k}\right|}$$

$$= \int\int 3xyz\,\frac{dxdy}{z}$$

$$= \int\int 3xy\;dxdy$$

In $R_1(x^2 + y^2 = 1)$, x varies from 0 to $\sqrt{1 - y^2}$

y varies from 0 to 1

$$= \int_0^1\int_0^{\sqrt{1-y^2}} 3xy\,dxdy$$

$$= 3\int_0^6 \left[y\,\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \frac{3}{2}\int_0^1 y(1 - y^2)dy$$

$$= \frac{3}{2}\int_0^1 y - y^3 dy$$

$$= \frac{3}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8}$$