Laplace transform of elementary functions

Result: 1 Prove that $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[t^n] = \int_0^\infty e^{-st}\, t^n\, dt$$

$$L[t^n] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \int_0^\infty e^{-u} \frac{u^n}{s^{n+1}}\, du$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n\, du$$

$$\therefore \boldsymbol{L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}}$$

$\because \int_0^\infty e^{-u} u^n\, du$

Let $st = u \cdots\cdots (1)$

$t = \frac{u}{s}$

$dt = \frac{du}{s}$

When $t \to 0 (1) => u \to 0$

,

$t \to \infty, (1) => u \to \infty$

Note: If n is an integer, then $\Gamma(n+1) = n!$

$\therefore L[t^n] = \frac{n!}{s^{n+1}}$ if n is an integer

If $n = 0$, then $L[1] = \frac{1}{s}$

If $n = 1$, then $L[t] = \frac{1}{s^2}$

Similarly $L[t^2] = \frac{2!}{s^3}$

$L[t^3] = \frac{3!}{s^4}$

Result: 2 Prove that $L(e^{at}) = \frac{1}{s-a}, s > a$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore L(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-t(s-a)}\, f(t)\, dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= -\left[0 - \left(\frac{1}{s-a}\right)\right]$$

$$\therefore L(e^{at}) = \frac{1}{s-a}$$

Result: 3 Prove that $L(e^{-at}) = \frac{1}{s+a}, s > a$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore L(e^{-at}) = \int_0^\infty e^{-st} e^{-at}\, dt$$

$$= \int_0^\infty e^{-t(s+a)} f(t)\, dt$$

$$= \left[\frac{e^{-t(s+a)}}{-(s+a)}\right]_0^\infty$$

$$= -\left[0 - \left(\frac{1}{s+a}\right)\right]$$

$$\therefore L(e^{at}) = \frac{1}{s+a}$$

Result: 4 Prove that $L[sinat] = \frac{a}{s^2+a^2}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[sinat] = \int_0^\infty e^{-st}\, sinat\, dt$$

$$\therefore L[sinat] = \frac{a}{s^2+a^2}, s > |a| \qquad \left[\because \int_0^\infty e^{-at}\, sinbt\, dt = \frac{b}{a^2+b^2}\right]$$

Result: 5 Prove that $L[cosat] = \frac{s}{s^2+a^2}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[cosat] = \int_0^\infty e^{-st}\, cosat\, dt$$

$$\therefore L[cosat] = \frac{s}{s^2+a^2}, s > |a| \qquad \because \int_0^\infty e^{-at}\, cosbt\, dt = \frac{a}{a^2+b^2}$$

Result: 6 Prove that $L[sinhat] = \frac{a}{s^2-a^2}\ , s > |a|$

Proof:

We have $L[sinhat] = L\left[\frac{e^{at}-e^{-at}}{2}\right]$

$$= \frac{1}{2}[L(e^{at}) - L(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

$$\therefore L[sinhat] = \frac{a}{s^2-a^2}\ , s > |a|$$

Result: 7 Prove that $L[coshat] = \frac{s}{s^2-a^2}\ , s > |a|$

Proo

We have $L[coshat] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$

$= \frac{1}{2}[L(e^{at}) + L(e^{-at})]$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$

$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$

$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$

$\therefore L[coshat] = \frac{s}{s^2-a^2}$, $s > |a|$

Example: Find $L\left[t^{\frac{1}{2}}\right]$

Solution:

We have $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Put $n = \frac{1}{2}$

$\therefore L\left[t^{\frac{1}{2}}\right] = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$ $\quad \because \Gamma(n+1) = n\Gamma n$

$= \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}+1}}$ $\quad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$

$\therefore L\left[t^{\frac{1}{2}}\right] = \frac{\sqrt{\pi}}{2s\sqrt{s}}$

Example: Find the Laplace transform of $t^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{t}}$

Solution:

We have $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Put $n = -\frac{1}{2}$

$\therefore L\left[t^{-\frac{1}{2}}\right] = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}$ $\quad \because \Gamma(n+1) = n\Gamma n$

$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}$ $\quad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$= \frac{\sqrt{\pi}}{\sqrt{s}}$

$\therefore L\left[\frac{1}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}$

FORMULA

$L[f(t)] = F(s)$	$L[f(t)] = F(s)$
$L[1] = \frac{1}{s}$ $L[t] = \frac{1}{s^2}$ $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ **if n is not an integer** $L[t^n] = \frac{n!}{s^{n+1}}$ **if n is an integer** $L(e^{at}) = \frac{1}{s-a}$ $L(e^{at}) = \frac{1}{s+a}$	$L[sinat] = \frac{a}{s^2+a^2}$ $L[cosat] = \frac{s}{s^2+a^2}$ $L[coshat] = \frac{s}{s^2-a^2}$ $L[sinhat] = \frac{a}{s^2-a^2}$

Problems using Linear property

Example:Find the Laplace transform for the following

i. $3t^2+2t+1$	**v.** $sin\sqrt{2}\,t$	**ix.** sin^2t
ii. $(t+2)^3$	**vi.** $sin(at+b)$	**x.** cos^22t
iii. a^t	**vii.** cos^32t	**xi.** $cos5tcos4t$
iv. e^{2t+3}	**viii.** sin^3t	

Solution:

(i) Given $f(t) = 3t^2 + 2t + 1$

$$L[f(t)] = L[3t^2 + 2t + 1]$$
$$= L[3t^2] + L[2t] + L[1]$$
$$= L[3t^2] + L[2t] + L[1]$$
$$= 3L[t^2] + 2L[t] + L[1]$$
$$= 3\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}$$
$$\therefore L[3t^2 + 2t + 1] = \frac{6}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

(ii) Given $f(t) = (t+2)^3 = t^3 + 3t^2(2) + 3t2^2 + 2^3$

$$L[f(t)] = L[t^3 + 3t^2(2) + 3t2^2 + 2^3]$$
$$= L[t^3] + L[6t^2] + L[12t] + L[8]$$
$$= L[t^3] + 6L[t^2] + 12L[t] + 8L[1]$$

$$= \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{12}{s}$$

(iii) Given $f(t) = a^t$

$$L[f(t)] = L[a^t] = L[e^{tloga}]$$

$$L[a^t] = \frac{1}{s-loga}$$

(iv)Given $f(t) = e^{2t+3}$

$$L[f(t)] = L[e^{2t+3}] = L[e^{2t}.e^3]$$

$$= e^3 L[e^{2t}]$$

$$= e^3 \left[\frac{1}{s-2}\right]$$

$$\therefore L[e^{2t+3}] = e^3 \left[\frac{1}{s-2}\right]$$

(v) $L\left[sin\sqrt{2}t\right] = \frac{\sqrt{2}}{s^2+2}$

(vi)Given $f(t) = \sin(at+b) = sinatcosb + cosatsinb$

$$L[f(t)] = L[\sin(at+b)]$$

$$= L[sinatcosb + cosatsinb]$$

$$= cosb\, L[\sin at] + sinb\, L[\cos at]$$

$$L[\sin(at+b)] = cosb \frac{s}{s^2+a^2} + sinb\, \frac{s}{s^2+a^2}$$

(vii) Given $f(t) = \cos^3 2t = \frac{1}{4}[3cos2t + cos6t]$

$$\because \cos^3\theta = \frac{3cos\theta + cos3\theta}{4}$$

$$L[f(t)] = \frac{1}{4} L[3cos2t + cos6t]$$

$$= \frac{1}{4}[3L(cos2t) + L(cos6t)]$$

$$= \frac{1}{4}\left[3\frac{s}{s^2+4} + \frac{s}{s^2+36}\right]$$

$$L[\cos^3 2t] = \frac{1}{4}\left[3\frac{s}{s^2+4} + \frac{s}{s^2+36}\right]$$

(viii) Given $f(t) = \sin^3 t = \frac{1}{4}[3sint - sin3t]$

$$L[f(t)] = \frac{1}{4} L[3sint - sin3t]$$

$$= \frac{1}{4}[3L(sint) - L(sin3t)]$$

$$= \frac{1}{4}\left[3\frac{1}{s^2+1} - \frac{3}{s^2+9}\right]$$

$$L[\sin^3 t] = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

(ix) Given $f(t) = \sin^2 t = \frac{1-cos2t}{2}$

$$L[f(t)] = L\left[\frac{1-cos2t}{2}\right]$$

$= \frac{1}{2}[L(1) - L(cos2t)]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$L[\cos^2 2t] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

(x) Given $f(t) = \cos^2 2t = \frac{1+cos4t}{2}$

$L[f(t)] = L\left[\frac{1+cos4t}{2}\right]$

$= \frac{1}{2}[L(1) + L(cos4t)]$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$

$L[\cos^2 2t] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16}\right]$

(xi) Given $f(t) = cos5tcos4t$

$L[f(t)] = L[cos5tcos4t]$

$= \frac{1}{2}[L(cos9t) + L(cost)]$

$= \frac{1}{2}\left[\frac{s}{s^2+81} + \frac{s}{s^2+1}\right]$