INITIAL AND FINAL VALUE THEOREMS

Initial value theorem

Statement: If $L[f(t)] = F(s)$, then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof:

We know that $L[f'(t)] = s\,L[f(t)] - f(0)$

$$= sF(s) - f(0)$$

$\therefore sF(s) = L[f'(t)] + f(0)$

$$= \int_0^\infty e^{-st} f'(t)dt + f(0)$$

Taking limit as $s \to \infty$ on both sides, we have

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\int_0^\infty e^{-st} f'(t)dt + f(0)\right]$$

$$= \lim_{s\to\infty}\left[\int_0^\infty e^{-st} f'(t)dt\right] + f(0)$$

$$= \int_0^\infty \lim_{s\to\infty}[e^{-st} f'(t)]dt + f(0)$$

$$= 0 + f(0) \qquad \because e^{-\infty} = 0$$

$$= f(0)$$

$$= \lim_{t\to 0} f(t)$$

$$\therefore \lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t)$$

Final value theorem

Statement: If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$, then $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Proof:

We know that $L[f'(t)] = s\,L[f(t)] - f(0)$

$$= sF(s) - f(0)$$

$\therefore sF(s) = L[f'(t)] + f(0)$

$$= \int_0^\infty e^{-st} f'(t)dt + f(0)$$

Taking limit as $s \to 0$ on both sides, we have

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\int_0^\infty e^{-st} f'(t)dt + f(0)\right]$$

$$= \lim_{s\to 0}\left[\int_0^\infty e^{-st} f'(t)dt\right] + f(0)$$

$$= \int_0^\infty \lim_{s\to 0}[e^{-st} f'(t)]dt + f(0)$$

$$= \int_0^\infty f'(t)dt + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$
$$= f(\infty) - f(0) + f(0)$$
$$= f(\infty)$$
$$= \lim_{t\to\infty} f(t)$$

$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Example: Verify the initial value theorem for the function $f(t) = ae^{-bt}$

Solution:

Given $f(t) = ae^{-bt}$

$F(s) = L[f(t)]$

$= L[ae^{-bt}]$

$= a\ \frac{1}{s+b}$

$sF(s) = \frac{as}{s+b}$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} ae^{-bt}$$
$$= a \cdots\cdots\cdots (1)$$
$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{as}{s+b}\right]$$
$$= \lim_{s\to\infty}\left[\frac{as}{s\left(1+\frac{b}{s}\right)}\right] == \lim_{s\to\infty}\left[\frac{a}{\left(1+\frac{b}{s}\right)}\right]$$
$$= a \cdots\cdots\cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$\therefore$ Initial value theorem is verified

Example: Verify the initial value theorem and Final value theorem for the function $f(t) = 1 + e^{-t}[sint + cost]$.

Solution:

Given $f(t) = 1 + e^{-t}[sint + cost]$

$F(s) = L[f(t)]$

$= L\big[1 + e^{-t}[sint + cost]\big]$

$= L[1] + L\big[e^{-t}[sint + cost]\big]$

$= L[1] + L[sint + cost]_{s\to s+1}$

$= \frac{1}{s} + \left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]_{s\to s+1}$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$F(s) = \frac{1}{s} + \frac{1}{s^2+2s+2} + \frac{s+1}{s^2+2s+2}$$

$$sF(s) = 1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}$$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}\left[1 + e^{-t}[sint + cost]\right]$$

$$= 1 + 0 + 1 = 2 \cdots\cdots\cdots (1)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}\right]$$

$$= 1 + \lim_{s\to\infty}\left[\frac{1}{s\left(1+\frac{2}{s}+\frac{2}{s^2}\right)} + \frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}\right]$$

$$= 1 + 0 + 1 = 2 \cdots\cdots\cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left(1 + e^{-t}[sint + cost]\right)$$

$$= 1 + 0 = 1 \cdots\cdots\cdots (3)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}\right]$$

$$= 1 + 0 + 0 = 1 \cdots\cdots\cdots (4)$$

From (3) and (4), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

∴ Final value theorem is verified.

Example: Verify the initial value theorem and Final value theorem for the function $\boldsymbol{f(t) = L^{-1}\left[\frac{1}{s(s+2)^2}\right]}$

Solution:

Given $f(t) = L^{-1}\left[\frac{1}{s(s+2)^2}\right] \cdots (1)$

$$= \int_0^t L^{-1}\left[\frac{1}{(s+2)^2}\right] dt = \int_0^t e^{-2t} L^{-1}\left[\frac{1}{s^2}\right] dt$$

$$= \int_0^t e^{-2t} t\, dt$$

$$= \int_0^t t e^{-2t}\, dt$$

$$= \left[t\left(\frac{e^{-2t}}{-2}\right) - \frac{(1)e^{-2t}}{(-2)^2}\right]_0^t$$

$$= -t\frac{e^{-2t}}{2} - \frac{e^{-2t}}{4} - 0 + \frac{1}{4}$$

$$\therefore f(t) = \frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}$$

From (1), $F(s) = \frac{1}{s(s+2)^2}$

$$sF(s) = \frac{1}{(s+2)^2}$$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}\left[\frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\right]$$

$$= \frac{1}{4} - 0 - \frac{1}{4} = 0$$

$$\therefore \lim_{t\to 0} f(t) = 0 \cdots (2)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{1}{(s+2)^2} = 0$$

$$\therefore \lim_{s\to\infty} sF(s) = 0 \cdots (3)$$

From (2) and (3), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$\therefore$ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left[\frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\right]$$

$$= \frac{1}{4} - 0 - 0 = \frac{1}{4} \cdots (4)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{4} \cdots (5)$$

From (4) and (5), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$\therefore$ Final value theorem is verified

Example: Verify the initial value theorem and Final value theorem for the function $\boldsymbol{f(t) = e^{-t}(t+2)^2}$

Solution:

Given $f(t) = e^{-t}(t+2)^2$

$$= e^{-t}(t^2 + 4t + 4)$$

$$F(s) = L[f(t)]$$

$$= L[e^{-t}(t^2 + 4t + 4)]$$

$$= L[t^2 + 4t + 4]_{s\to s+1}$$

$$= [L(t^2) + 4L(t) + 4L(1)]_{s\to s+1}$$

$$= \left[\frac{2!}{s^3} + 4\frac{1}{s^2} + 4\frac{1}{s}\right]_{s\to s+1}$$

$$= \frac{2}{(s+1)^3} + 4\frac{1}{(s+1)^2} + 4\frac{1}{s+1}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}$$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}[e^{-t}(t^2 + 4t + 4)]$$

$$= 4 \cdots (1)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}\right]$$

$$= \lim_{s\to\infty}\left[\frac{2s}{s^3\left(1+\frac{1}{s}\right)^3} + \frac{4s}{s^2\left(1+\frac{1}{s}\right)^2} + \frac{4s}{s\left(1+\frac{1}{s}\right)}\right]$$

$$= \lim_{s\to\infty}\left[\frac{2}{s^2\left(1+\frac{1}{s}\right)^3} + \frac{4}{s\left(1+\frac{1}{s}\right)^2} + \frac{4}{\left(1+\frac{1}{s}\right)}\right]$$

$$= 0 + 0 + 4$$

$$= 4 \cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}[e^{-t}(t^2 + 4t + 4)]$$

$$= 0 \cdots (3)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}\right]$$

$$= 0 \cdots (4)$$

From (3) and (4), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

∴ Final value theorem is verified.